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Using the method of undetermined coefficients for solving general term formula of fractional linear recursive sequence

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Abstract: In this paper, using the method of undetermined coefficients, by constructing auxiliary series, fractional linear recursive sequence is derived for the general term formula of under various circumstances. .
Keywords: Method of undetermined coefficients, Fractional linear recursive sequence, Auxiliary sequence.

INTRODUCTION

The general form of fractional linear recursive sequence has the following form: $a_{n+1}=\dfrac{Aa_n+B}{Ca_n+D}$,where $AD-BC\neq 0, C\neq 0$ .It is one of the most primary study objects in mathematical analysis. The main ways to solve general term formula of fractional linear recursive sequence are the eigenvalue method , the matrix method and the fixed point method [1-4] and so on for the present. But there is no related articles about the method of undetermined coefficients to solve general term formula of fractional linear recursive sequence.

Its advantages are less mathematical theory required which are difficult to be understood and simpler and easier to calculate.

GENERAL TERM FORMULA OF FRACTIONAL LINEAR RECURSIVE SEQUENCE

For general fractional linear recursive sequence $a_{n+1}=\dfrac{Aa_n+B}{Ca_n+D}$, Where $AD-BC\neq 0, C\neq 0$.

The method of calculating general term formula is divided into the following two cases.

when $B=0$, we have $a_{n+1}=\dfrac{Aa_n}{Ca_n+D}$.

Taking the reciprocal of a_{n+1}, we obtain

$$\frac{1}{a_{n+1}}=\frac{C}{A}+\frac{D}{A}\frac{1}{a_n}.$$

(1) If $A=D$, we get $\dfrac{1}{a_{n+1}}=\dfrac{C}{A}+\dfrac{D}{A}\dfrac{1}{a_n}=\dfrac{C}{A}+\dfrac{1}{a_n}$ by simplifying the above-mentioned formula.

So $\left\{\dfrac{1}{a_n}\right\}$ is an arithmetic progression and its general term formula is

$$\frac{1}{a_n}=\frac{1}{a_1}+\frac{C}{A}(n-1)$$

That is, the general term formula of the sequence $\{a_n\}$ is

$$a_n = \frac{1}{\frac{1}{a_1} + \frac{C}{A}(n-1)} .$$

example 1

Let $a_{n+1} = \frac{2a_n}{3a_n + 2}$, $a_1 = 1$,find the general term formula for $\{a_n\}$.

Solution

We take the reciprocal of a_{n+1} and get

$$\frac{1}{a_{n+1}} = \frac{3a_n + 2}{2a_n} = \frac{3}{2} + \frac{1}{a_n} .$$

and then

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{3}{2}.$$

So the sequence $\left\{\frac{1}{a_n}\right\}$ is an arithmetic sequence.

That is

$$\frac{1}{a_n} = 1 + \frac{3}{2}(n-1) .$$

Therefore

$$a_n = \frac{2}{3n-1} .$$

(2) If $A \neq D$, let $b_n = \frac{1}{a_n}$, we get $b_{n+1} = \frac{C}{A} + \frac{D}{A} b_n$.

We construct auxiliary sequence as follows

$$b_{n+1} + k = \frac{D}{A}(b_n + k) .$$

Thus,

$$(\frac{D}{A} - 1)k = \frac{C}{A}.$$

and hence

$$k = \frac{C}{D-A} .$$

The above-mentioned formula turns to be

$$b_{n+1} + \frac{C}{D-A} = \frac{D}{A}(b_n + \frac{C}{D-A}) .$$

So the sequence $\{b_n + \frac{C}{D-A}\}$ is a geometric sequence and its general term formula is

$$b_n + \frac{C}{D-A} = (b_1 + \frac{C}{D-A})(\frac{D}{A})^{n-1} = (\frac{1}{a_1} + \frac{C}{D-A})(\frac{D}{A})^{n-1}$$

By simplifying the above-mentioned formula, we get

$$b_n = (\frac{1}{a_1} + \frac{C}{D-A})(\frac{D}{A})^{n-1} + \frac{C}{D-A}$$

We take the reciprocal of the above-mentioned formula, we get

$$a_n = \frac{1}{b_n} = \frac{1}{(\frac{1}{a_1} + \frac{C}{D-A})(\frac{D}{A})^{n-1} + \frac{C}{D-A}}$$

example 2

Let $a_{n+1} = \frac{2a_n}{3a_n + 4}$, $a_1 = 1$,find the general term formula for $\{a_n\}$.

Solution

We take the reciprocal of a_{n+1} and get

$$\frac{1}{a_{n+1}} = \frac{3a_n + 4}{2a_n} = \frac{3}{2} + \frac{2}{a_n}.$$

Let $b_n = \frac{1}{a_n}$ and then we have

$$b_{n+1} = \frac{1}{a_{n+1}} = 2b_n + \frac{3}{2}$$

We use the undetermined coefficient method for the following formula

$$b_{n+1} + k = 2(b_n + k)$$

Thus, we obtain

$$k = \frac{3}{2} \text{ and } b_{n+1} + \frac{3}{2} = 2(b_n + \frac{3}{2})$$

So the sequence $\{b_n + \frac{3}{2}\}$ is a geometric sequence.

and hence

$$b_n + \frac{3}{2} = (b_1 + \frac{3}{2})2^{n-1} = \frac{5}{2} \cdot 2^{n-1}.$$

That is

$$b_n = \frac{5}{2} \cdot 2^{n-1} - \frac{3}{2}.$$

Therefore $$a_n = \frac{1}{b_n} = \frac{2}{5 \cdot 2^{n-1} - 3}.$$

When $B \neq 0$, suppose that $a_{n+1} + k = \frac{\alpha(a_n + k)}{Ca_n + D}$, then we have

$$a_{n+1} = \frac{(\alpha - Ck)a_n + \alpha k - kD}{Ca_n + D}.$$

comparing with the old one,we obtain

$$\begin{cases} \alpha - Ck = A \\ (\alpha - D)k = B \end{cases}.$$

We can determine the value of α, k.Thus,the case 2.2 can be turned into this case.

example 3

Let $a_{n+1} = \frac{4a_n - 2}{a_n + 7}$, $a_1 = 2$,find the general term formula for $\{a_n\}$.

Solution

$$a_{n+1} + k = \frac{\alpha(a_n + k)}{a_n + 7},$$

Through the collation, we have

$$a_{n+1}=\frac{(\alpha-4k)a_n+\alpha k-5k}{4a_n+5},$$

So we obtain

$$\begin{cases}\alpha-4k=-3\\(\alpha-5)k=12\end{cases},$$

Through solving the equation, we get

$$\begin{cases}\alpha=-7\\k=-1\end{cases}\text{ or }\begin{cases}\alpha=9\\k=3\end{cases}$$

(1) $a_{n+1}-1=\dfrac{-7(a_n-1)}{4(a_n-1)+9}$.Taking the reciprocal of $a_{n+1}-1$, we obtain

$$\frac{1}{a_{n+1}-1}=-\frac{9}{7}\cdot\frac{1}{a_n-1}-\frac{4}{7},$$

Let $b_n=\dfrac{1}{a_n-1}$, We have

$$b_{n+1}=-\frac{9}{7}b_n-\frac{4}{7},b_{n+1}+\frac{1}{4}=-\frac{9}{7}(b_n+\frac{1}{4}),b_n+\frac{1}{4}=(b_n+\frac{1}{4})(-\frac{9}{7})^{n-1}=\frac{5}{4}\cdot(-\frac{9}{7})^{n-1},$$

$$b_n=\frac{5}{4}\cdot(-\frac{9}{7})^{n-1}-\frac{1}{4},a_n-1=\frac{1}{b_n}=\frac{1}{\frac{5}{4}\cdot(-\frac{9}{7})^{n-1}-\frac{1}{4}}.$$

Therefore

$$a_n=\frac{1}{\frac{5}{4}\cdot(-\frac{9}{7})^{n-1}-\frac{1}{4}}+1.$$

(2) $a_{n+1}+3=\dfrac{9(a_n+3)}{4(a_n+3)-7}$.Taking the reciprocal of $a_{n+1}+3$, we obtain

$$\frac{1}{a_{n+1}+3}=-\frac{7}{9}\cdot\frac{1}{a_n+3}+\frac{4}{9}.$$

Let $b_n=\dfrac{1}{a_n+3}$,we have

$$b_{n+1}=-\frac{7}{9}b_n+\frac{4}{9},b_{n+1}-\frac{1}{4}=-\frac{7}{9}(b_n-\frac{1}{4}),b_n-\frac{1}{4}=(b_1-\frac{1}{4})(-\frac{7}{9})^{n-1}=-\frac{1}{20}\cdot(-\frac{7}{9})^{n-1},$$

$$b_n=-\frac{1}{20}\cdot(-\frac{7}{9})^{n-1}+\frac{1}{4},a_n+3=\frac{1}{b_n}=\frac{1}{-\frac{1}{20}\cdot(-\frac{7}{9})^{n-1}+\frac{1}{4}}.$$

Therefore

$$a_n=\frac{1}{-\frac{1}{20}\cdot(-\frac{7}{9})^{n-1}+\frac{1}{4}}-3.$$

CONCLUSIONS

In this paper, we solve general term formula of fractional linear recursive sequence by transforming them into arithmetic progression or geometric sequence and using the method of undetermined coefficients and constructing auxiliary sequence ingeniously. Furthermore, we derive the general term formula under various circumstances

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REFERENCES

1. Ying LI, Zhou M, Guyan NI; Obtaining the limit of the fractional linear recursive sequence based on eigenvalue theory. College Mathematics, 2014; 30(5):74-77.
2. Xiaoguo LU; Matrix computation for the generality of Fractional formula, linear, recurrent progressions. Journal of Nantong Institute of Technology, 1998; 14(4):6-9.
3. Sheng-xian S, Bing-sen Y. Substitution Method for the Limit of Linear Fractional Recurrences Studies in College Mathematics, 2011; 14(4):72-74.
4. Zhang Ling; Applications for the Contraction Mapping Principle. Bulletin of science and technology, 2011; 27(2):474-478.