

## On The Negative Pell Equation $y^2=15x^2-6$

M. A Gopalan<sup>1\*</sup>, S. Vidhyalakshmi<sup>2</sup>, J. Shanthi<sup>3</sup>, D. Kanaka<sup>4</sup>

<sup>1,2</sup>Professor, <sup>3</sup>Lecturer, <sup>4</sup>PG Student, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India

### \*Corresponding Author:

M. A Gopalan

Email: [mayilgopalan@gmail.com](mailto:mayilgopalan@gmail.com)

**Abstract:** The negative Pell equation represented by the binary quadratic equation  $y^2 = 15x^2 - 6$  is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration, the integer solutions for a few choices of hyperbola and parabola are obtained.

**Keywords:** Binary quadratic, Hyperbola, parabola, Integral solutions, Pell equation.

2010 Mathematics subject classification:11D09

### INTRODUCTION:

Diophantine equation of the form  $y^2 = Dx^2 + 1$  where D is a given positive square-free integer, is known as Pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L.lagrange proved that the positive Pell equation  $y^2 = Dx^2 + 1$  has infinitely many distinct integer solutions where as the negative pell equation  $y^2 = Dx^2 - 1$  does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation  $x^2 - Dy^2 = -1$  where D is any positive non-square integer has been presented. For examples the equations  $y^2 = 3x^2 - 1$ ,  $y^2 = 7x^2 - 4$  have no integer solutions, where as  $y^2 = 65x^2 - 1$ ,  $y^2 = 202x^2 - 1$  have integer solutions. In this context, one may refer [2- 9].More specifically, one may refer " The On-line Encyclopedia of integer sequences " (A031396,A130226,A031398) for values of D for which the negative pell equation  $y^2 = Dx^2 - 1$  is solvable or not. In this communication, the negative Pell equation given by  $y^2 = 15x^2 - 6$  is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

### METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 15x^2 - 6 \quad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 3$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 15x^2 + 1$$

whose general solution is given by

$$\tilde{x}_s = \frac{1}{2\sqrt{15}} g_s, \tilde{y}_s = \frac{1}{2} f_s$$

where

$$f_s = (4 + \sqrt{15})^{s+1} + (4 - \sqrt{15})^{s+1} \quad g_s = (4 + \sqrt{15})^{s+1} - (4 - \sqrt{15})^{s+1}, \quad s=0,1,2,3,\dots$$

Applying Brahmagupta lemma between  $(x_0, y_0)$  and  $(\tilde{x}_s, \tilde{y}_s)$ , the other integer solutions of (1) are given by

$$2x_{s+1} = f_s + \frac{3}{\sqrt{15}} g_s$$

$$2y_{s+1} = 3f_s + \sqrt{15}g_s$$

The recurrence relations satisfied by x and y are given by

$$y_{s+3} - 8y_{s+2} + y_{s+1} = 0, y_0 = 3, y_1 = 27$$

$$x_{s+3} - 8x_{s+2} + x_{s+1} = 0, x_0 = 1, x_1 = 7$$

Some numerical examples of x and y satisfying (1) are given in the following table:

Table: Numerical Examples

s	$x_s$	$y_s$
0	1	3
1	7	27
2	55	213
3	433	1677
4	3409	13203
5	26839	103947
6	211303	818373
7	1663585	6443037

From the above table, we observe some interesting relations among the solutions which are presented below:

1.  $x_s$  is always odd.
2.  $y_s$  is always odd.
3.  $x_{3s-2} \equiv 0 \pmod{7}$
4.  $y_s \equiv 0 \pmod{3}$

A few interesting properties between the solutions and special numbers are given below:

1.  $6(5x_{2s+2} - y_{2s+2} + 2)$  is a nasty number.
2.  $5x_{3s+3} - y_{3s+3} + 3[(4 + \sqrt{15})^{s+1} + (4 - \sqrt{15})^{s+1}]$  is a cubical integer.

3. Each of the following properties represents a perfect square:

- $5x_{2s+2} - y_{2s+2} + 2$
- $9x_{2s+2} - x_{2s+3} + 2$
- $\frac{1}{8}[71x_{2s+2} - x_{2s+4}] + 2$
- $\frac{\sqrt{15}}{3} \left[ \sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+2} \right] + 2$
- $\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+3} \right] + 2$
- $\frac{\sqrt{15}}{93} \left[ 55\sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+4} \right] + 2$
- $71x_{2s+3} - 9x_{2s+4} + 2$

- $\frac{\sqrt{15}}{12} \left[ \sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}}y_{2s+2} \right] + 2$
- $\frac{\sqrt{15}}{3} \left[ 7\sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}}y_{2s+3} \right] + 2$
- $\frac{\sqrt{15}}{12} \left[ 55\sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}}y_{2s+4} \right] + 2$
- $\frac{\sqrt{15}}{93} \left[ \sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}}y_{2s+2} \right] + 2$
- $\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}}y_{2s+3} \right] + 2$
- $\frac{\sqrt{15}}{3} \left[ 55\sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}}y_{2s+4} \right] + 2$
- 4.  $2y_{s+3} = 16y_{s+2} - 2y_{s+1}$
- 5.  $30x_{s+1} = 2y_{s+2} - 8y_{s+1}$
- 6.  $30x_{s+2} = 8y_{s+2} - 2y_{s+1}$
- 7.  $30x_{s+3} = 62y_{s+2} - 8y_{s+1}$
- 8.  $2y_{s+3} - 240x_{s+1} = 62y_{s+1}$
- 9.  $8y_{s+3} - 30x_{s+1} = 62y_{s+2}$
- 10.  $2y_{s+1}y_{s+3} + 60x_{s+1}y_{s+2} = 4y_{s+2}^2 - 2y_{s+1}^2$
- 11.  $2y_{s+2}y_{s+3} + 30x_{s+1}y_{s+1} = 16y_{s+2}^2 - 8y_{s+1}^2$
- 12.  $2y_{s+3} - 60x_{s+2} = 2y_{s+1}$
- 13.  $2y_{s+3} - 30x_{s+2} = 8y_{s+2}$
- 14.  $2y_{s+1}y_{s+3} + 240x_{s+2}y_{s+2} = 64y_{s+2}^2 - 2y_{s+1}^2$
- 15.  $8y_{s+2}y_{s+3} + 30x_{s+2}y_{s+1} = 64y_{s+2}^2 - 2y_{s+1}^2$
- 16.  $62y_{s+3} - 240x_{s+3} = 2y_{s+1}$
- 17.  $8y_{s+3} - 30x_{s+3} = 2y_{s+2}$
- 18.  $2y_{s+1}y_{s+3} + 60x_{s+3}y_{s+2} = 124y_{s+2}^2 - 2y_{s+1}^2$
- 19.  $62y_{s+2}y_{s+3} + 30x_{s+3}y_{s+1} = 496y_{s+2}^2 - 8y_{s+1}^2$
- 20.  $120x_{s+1} - 30x_{s+2} = 30y_{s+1}$
- 21.  $120x_{s+2} - 30x_{s+1} = 30y_{s+2}$
- 22.  $30x_{s+1}y_{s+1} + 30x_{s+2}y_{s+2} = 8y_{s+2}^2 - 8y_{s+1}^2$
- 23.  $30x_{s+1}y_{s+2} + 30x_{s+2}y_{s+1} = 2y_{s+2}^2 - 2y_{s+1}^2$
- 24.  $30x_{s+3} - 930x_{s+1} = 240y_{s+1}$
- 25.  $30x_{s+3} - 30x_{s+1} = 60y_{s+2}$
- 26.  $120x_{s+1}y_{s+1} + 30x_{s+3}y_{s+2} = 62y_{s+2}^2 - 32y_{s+1}^2$
- 27.  $930x_{s+1}y_{s+2} + 120x_{s+3}y_{s+1} = 62y_{s+2}^2 - 32y_{s+1}^2$

- 28.  $120x_{s+3} - 930x_{s+2} = 30y_{s+1}$
- 29.  $30x_{s+3} - 120x_{s+2} = 30y_{s+2}$
- 30.  $30x_{s+2}y_{s+1} + 30x_{s+3}y_{s+2} = 62y_{s+2}^2 - 2y_{s+1}^2$
- 31.  $930x_{s+2}y_{s+2} + 30x_{s+3}y_{s+1} = 248y_{s+2}^2 - 8y_{s+1}^2$

**REMARKABLE OBSERVATIONS:**

1. Let N be any non-zero positive integer such that

$$N = \frac{x_n - 1}{2}$$

Then it is observed that

$$8t_{3,N} + 1 = x_n^2$$

Similarly,

$$8t_{3,M} + 1 = y_n^2$$

2. Let p and q be two non-zero distinct positive integers such that

$$p = x_n + 2y_n \text{ and } q = x_n$$

Note that  $p > q > 0$ . Treat p,q as the generators of the Pythagorean triangle  $T(\alpha, \beta, \gamma)$ .

where  $\alpha = 2pq, \beta = p^2 - q^2$  and  $\gamma = p^2 + q^2$

Let A,P represent the area and perimeter of  $T(\alpha, \beta, \gamma)$ . Then the following interesting relations are observed:

(a)  $\alpha - 30\beta + 29\gamma - 24 = 0$

(b)  $31\alpha - \gamma - 24 = 120\left(\frac{A}{P}\right)$

(c)  $31\beta - 30\gamma + 24 = 4\left(\frac{A}{P}\right)$

**Each of the following represents a hyperbola:**

S.No.	Hyperbola	$f_s (= X), g_s (= Y)$
1.	$9X^2 - 15Y^2 = 36$	$5x_{s+1} - y_{s+1}, \frac{\sqrt{15}}{3} [y_{s+1} - 3x_{s+1}]$
2.	$9X^2 - 15Y^2 = 36$	$9x_{s+1} - x_{s+2}, \frac{\sqrt{15}}{3} [x_{s+2} - 7x_{s+1}]$
3.	$9X^2 - 15Y^2 = 2304$	$\frac{1}{8} [71x_{s+1} - x_{s+3}], \frac{\sqrt{15}}{24} [x_{s+3} - 55x_{s+1}]$
4.	$15X^2 - 15Y^2 = 36$	$\frac{\sqrt{15}}{3} \left[ \sqrt{15}x_{s+1} - \frac{3}{\sqrt{15}}y_{s+1} \right], \frac{\sqrt{15}}{3} [y_{s+1} - 3x_{s+1}]$
5.	$15X^2 - 15Y^2 = 576$	$\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{s+1} - \frac{3}{\sqrt{15}}y_{s+2} \right], \frac{\sqrt{15}}{12} [y_{s+2} - 27x_{s+1}]$
6.	$15X^2 - 15Y^2 = 34596$	$\frac{\sqrt{15}}{93} \left[ 55\sqrt{15}x_{s+1} - \frac{3}{\sqrt{15}}y_{s+3} \right], \frac{\sqrt{15}}{93} [y_{s+3} - 213x_{s+1}]$
7.	$36X^2 - 15Y^2 = 144$	$71x_{s+2} - 9x_{s+3}, \frac{\sqrt{15}}{6} [14x_{s+3} - 110x_{s+2}]$
8.	$15X^2 - 15Y^2 = 576$	$\frac{\sqrt{15}}{12} \left[ \sqrt{15}x_{s+2} - \frac{27}{\sqrt{15}}y_{s+1} \right], \frac{\sqrt{15}}{12} [7y_{s+1} - 3x_{s+2}]$

9.	$15X^2 - 15Y^2 = 36$	$\frac{\sqrt{15}}{3} \left[ 7\sqrt{15}x_{s+2} - \frac{27}{\sqrt{15}} y_{s+2} \right], \frac{\sqrt{15}}{3} [7y_{s+2} - 27x_{s+2}]$
10.	$15X^2 - 15Y^2 = 576$	$\frac{\sqrt{15}}{12} \left[ 55\sqrt{15}x_{s+2} - \frac{27}{\sqrt{15}} y_{s+3} \right], \frac{\sqrt{15}}{12} [7y_{s+3} - 213x_{s+2}]$
11.	$15X^2 - 15Y^2 = 34596$	$\frac{\sqrt{15}}{93} \left[ \sqrt{15}x_{s+3} - \frac{213}{\sqrt{15}} y_{s+1} \right], \frac{\sqrt{15}}{93} [55y_{s+1} - 3x_{s+3}]$
12.	$15X^2 - 15Y^2 = 576$	$\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{s+2} - \frac{213}{\sqrt{15}} y_{s+2} \right], \frac{\sqrt{15}}{12} [55y_{s+2} - 27x_{s+3}]$
13.	$15X^2 - 15Y^2 = 36$	$\frac{\sqrt{15}}{3} \left[ 55\sqrt{15}x_{s+3} - \frac{213}{\sqrt{15}} y_{s+3} \right], \frac{\sqrt{15}}{3} [55y_{s+3} - 213x_{s+3}]$
14.	$15X^2 - 9Y^2 = 540$	$\frac{1}{3} [y_{s+2} - 7y_{s+1}], \frac{1}{\sqrt{15}} [9y_{s+1} - y_{s+3}]$
15.	$5X^2 - 3Y^2 = 11520$	$\frac{1}{24} [y_{s+3} - 55y_{s+1}], \frac{1}{8\sqrt{15}} [71y_{s+1} - y_{s+3}]$
16.	$15X^2 - 9Y^2 = 540$	$\frac{1}{3} [7y_{s+3} - 55y_{s+2}], \frac{1}{\sqrt{15}} [71y_{s+2} - 9y_{s+3}]$

Each of the following represents a parabola:

S.NO	Parabola	$f_s^2 (= X), g_s (= Y)$
1.	$15Y^2 = 9X - 36$	$5x_{2s+2} - y_{2s+2} + 2, \frac{\sqrt{15}}{3} [y_{s+1} - 3x_{s+1}]$
2.	$15Y^2 = 9X - 36$	$9x_{2s+2} - x_{2s+3} + 2, \frac{\sqrt{15}}{3} [x_{s+2} - 7x_{s+1}]$
3.	$15Y^2 = 72X - 2304$	$\frac{1}{8} [71x_{2s+2} - x_{2s+4}] + 2, \frac{\sqrt{15}}{24} [x_{s+3} - 55x_{s+1}]$
4.	$15Y^2 = 3X - 36$	$\frac{\sqrt{15}}{3} \left[ \sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+2} \right] + 2, \frac{\sqrt{15}}{3} [y_{s+1} - 3x_{s+1}]$
5.	$15Y^2 = 12X - 576$	$\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+3} \right] + 2, \frac{\sqrt{15}}{12} [y_{s+2} - 27x_{s+1}]$
6.	$15Y^2 = 93X - 34596$	$\frac{\sqrt{15}}{93} \left[ 55\sqrt{15}x_{2s+2} - \frac{3}{\sqrt{15}} y_{2s+4} \right] + 2, \frac{\sqrt{15}}{93} [y_{s+3} - 213x_{s+1}]$
7.	$15Y^2 = 36X - 144$	$71x_{2s+3} - 9x_{2s+4} + 2, \frac{\sqrt{15}}{6} [14x_{s+3} - 110x_{s+2}]$
8.	$15Y^2 = 12X - 576$	$\frac{\sqrt{15}}{12} \left[ \sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}} y_{2s+2} \right] + 2, \frac{\sqrt{15}}{12} [7y_{s+1} - 3x_{s+2}]$
9.	$15Y^2 = 3X - 36$	$\frac{\sqrt{15}}{3} \left[ 7\sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}} y_{2s+3} \right] + 2, \frac{\sqrt{15}}{3} [7y_{s+2} - 27x_{s+2}]$

10.	$15Y^2 = 12X - 576$	$\frac{\sqrt{15}}{12} \left[ 55\sqrt{15}x_{2s+3} - \frac{27}{\sqrt{15}} y_{2s+4} \right] + 2, \frac{\sqrt{15}}{12} [7y_{s+3} - 213x_{s+2}]$
11.	$15Y^2 = 93X - 34596$	$\frac{\sqrt{15}}{93} \left[ \sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}} y_{2s+2} \right] + 2, \frac{\sqrt{15}}{93} [55y_{s+1} - 3x_{s+3}]$
12.	$15Y^2 = 12X - 576$	$\frac{\sqrt{15}}{12} \left[ 7\sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}} y_{2s+3} \right] + 2, \frac{\sqrt{15}}{12} [55y_{s+2} - 27x_{s+3}]$
13.	$15Y^2 = 3X - 36$	$\frac{\sqrt{15}}{3} \left[ 55\sqrt{15}x_{2s+4} - \frac{213}{\sqrt{15}} y_{2s+4} \right] + 2, \frac{\sqrt{15}}{3} [55y_{s+3} - 213x_{s+3}]$
14.	$Y^2 = 5X - 60$	$\frac{1}{3} [y_{2s+4} - 7y_{2s+2}] + 2, \frac{1}{\sqrt{15}} [9y_{s+1} - y_{s+2}]$
15.	$Y^2 = 40X - 3840$	$\frac{1}{24} [y_{2s+4} - 55y_{2s+2}] + 2, \frac{1}{8\sqrt{15}} [7y_{s+1} - y_{s+3}]$
16.	$Y^2 = 5X - 60$	$\frac{1}{3} [7y_{2s+4} - 55y_{2s+3}] + 2, \frac{1}{\sqrt{15}} [71y_{s+2} - 9y_{s+3}]$

#### CONCLUSION:

In this paper , We have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation  $y^2 = 15x^2 - 6$  . As the binary quadratic Diophantine equation are rich in variety , one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

#### REFERENCES

1. Mollin RA, Srinivasan A; A Note on the negative pell equation. International Journal of Algebra, 2010; 4(19):919-922.
2. Whitford EE; Some solutions of the pellian Equations  $x^2 - Ay^2 \pm 4$  . JSTOR: Annals of Mathematics, Second series, 1913-1914; 15(1/4):157-160.
3. Ahmet T, Betw G, Osman B; On the integer solutions of the pell Equation  $x^2 - dy^2 = 2^t$  . World Academy of science, Engineering and Technology 2007;1:522-526.
4. Ahmet T; The pell equation  $x^2 - (k^2 - k)y^2 = 2^t$  . World Academy of science, Engineering and Technology, 2008;19:697-701.
5. Merve G; Solutions of the pell equations ,  $x^2 - (a^2b^2 + 2b)y^2 = 2^t$  when  $N \in (\pm 1, \pm 4)$  . Mathematica Aeterna , 2012; 2(7):629-638.
6. Gopalan MA, Vidhyalakshmi S, Thiruniraiselvi N; A study on the hyperbola  $y^2 = 8x^2 - 31$  . International Journal of latest Research in science and technology, 2013; 2(1):454-456.
7. Sangeetha V; Gopalan MA, Somanath M; On the integral solutions of the pell equation  $x^2 = 13y^2 - 3^t$  . International journal of appield Mathematical research, 2014;3(1):58-61.
8. Gopalan MA, Sumathi G, Vidhyalakshmi S; Observations on the hyperbola  $x^2 = 19y^2 - 3^t$  . Scholars Journal of the Engineering and Technology, 2014;:2(2A):152-155.
9. Gopalan MA, Vidhyalakshmi S, Kavitha A; On the integral solution of the Binary quadratic equation  $x^2 = 15y^2 - 11^t$  . Scholars Journal of the Engineering and Technology, 2014; 2(2A):156-158.