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On The Ternary Quadratic Diophantine Equation $x^2+y^2-xy=103z^2$

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Abstract: The ternary quadratic Diophantine equation represented by $x^2+y^2-xy=103z^2$ is analyzed for its non-zero distinct integer solutions. A few interesting properties between the solutions and special figurate numbers are obtained.
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NOTATIONS USED:

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

$$Pr_n=n(n+1)$$

$$S_n=6n(n-1)+1$$

INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-24] for quadratic equation with three unknowns. This communication concerns with yet another interesting equation $x^2+y^2-xy=103z^2$ representing non- homogeneous quadractic equation with three unknowns for determining its infinitely many non-zero integral points. Also , a few interesting relations among the solution are presented.

METHOD OF ANALYSIS

The ternary quadratic Diophantine equation to be solved is

$$x^2+y^2-xy=103z^2 \tag{1}$$

The substitution of the linear transformations

$$x=u+v;\ y=u-v\quad (u\neq 0, v\neq 0) \tag{2}$$

In(1) leads to

$$u^2+3v^2=103z^2 \tag{3}$$

Take

$$z=z(a,b)=a^2+3b^2 \tag{4}$$

where a , b are non-zero distinct integers. Different patterns of solutions of (1) are illustrated below .

PATTERN: 1

Write 103 as

$$103=(10+i\sqrt{3})(10-i\sqrt{3}) \tag{5}$$

Substituting (4) , (5) in (3) and employing the method of factorization , we've

$$(u+i\sqrt{3}v)\,(u-i\sqrt{3}v)=(10+i\sqrt{3})\,(10-i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{3}v)=(10+i\sqrt{3})(a+i\sqrt{3}b)^2 \tag{6}$$

$$(u-i\sqrt{3}v)=(10-i\sqrt{3})(a-i\sqrt{3}b)^2 \tag{7}$$

Equating the real and imaginary parts in (6)

$$u=u(a,b)=10a^2-6ab-30b^2$$

$$v=v(a,b)=a^2+20ab-3b^2$$

Substituting the values of u and v in (2) , we've

$$x=x(a,b)=11a^2+14ab-33b^2 \tag{8}$$

$$y=y(a,b)=9a^2-26ab-27b^2 \tag{9}$$

Thus (8) , (9) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES

- ❖ $x(a,1)-22t_{3,a}\equiv -33 \pmod 3$
- ❖ $x(1,b)+66t_{3,b}\equiv 11 \pmod{47}$
- ❖ $x(2,b)+66t_{3,b}\equiv 44 \pmod{61}$
- ❖ $y(2a,1)-6S_a\equiv -28 \pmod{16}$
- ❖ $y(a,1)-t_{20,a}\equiv -9 \pmod{18}$
- ❖ $x(a,1)+y(a,1)-11Pr_a-9Pr_a\equiv -28 \pmod{32}$

Note: 1

Write 103 as

$$103=(-10+i\sqrt{3})\,(-10-i\sqrt{3}) \tag{10}$$

Substituting (4) , (10) in (3) and employing the method of factorization , we've

$$(u+i\sqrt{3}v)\,(u-i\sqrt{3}v)=(-10+i\sqrt{3})\,(-10-i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{3}v)=(-10+i\sqrt{3})(a+i\sqrt{3}b)^2 \tag{11}$$

$$(u-i\sqrt{3}v)=(-10-i\sqrt{3})(a-i\sqrt{3}b)^2 \tag{12}$$

Equating the real and imaginary parts in (11)

$$u=u(a,b)=-10a^2-6ab+30b^2$$

$$v=v(a,b)=a^2-20ab-3b^2$$

Substituting the values of u and v in (2), we've

$$x=x(a,b)=-9a^2-26ab+27b^2 \tag{13}$$

$$y=y(a,b)=-11a^2+14ab+33b^2 \tag{14}$$

Thus (13) , (14) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

PATTERN: 2

Write (1) as

$$x^2 + y^2 - xy = 103z^2 = 103z^2 * 1 \tag{14a}$$

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \tag{15}$$

Substituting (4) , (5), (15) in (14a) , it is written in the factorizable form as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (10 + i\sqrt{3})(10 - i\sqrt{3})(a + i\sqrt{3}b)^2 (a - i\sqrt{3}\, b)^2 * \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}v) = (10 + i\sqrt{3})(a + i\sqrt{3}b)^2 * \frac{(1+i\sqrt{3})}{2} \tag{16}$$

$$(u - i\sqrt{3}v) = (10 - i\sqrt{3})(a - i\sqrt{3}b)^2 * \frac{(1-i\sqrt{3})}{2} \tag{17}$$

Equating the real and imaginary parts in (16)

$$u = \frac{1}{2}(7a^2 - 66ab - 21b^2) \tag{18}$$

$$v = \frac{1}{2}(11a^2 + 14ab - 33b^2) \tag{19}$$

Replacing a by 2A , b by 2B in (18) , (19) and (4) ,we've

$$u = u(A,B) = 14A^2 - 132AB - 42B^2$$

$$v = v(A,B) = 22A^2 + 28AB - 66B^2$$

$$z = z(A,B) = 4A^2 + 12B^2 \tag{20}$$

Substituting the values of u and v in (2) , we've

$$x = x(A,B) = 36A^2 - 104AB - 108B^2 \tag{21}$$

$$y = y(A,B) = -8A^2 - 160AB + 24B^2 \tag{22}$$

Thus (21) , (22) and (20) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES

- $x(2,b) + 108Pr_b \equiv 144 \pmod{100}$
- $y(2,b) - t_{50,b} \equiv -32 \pmod{297}$
- $x(a,1) - z(a,1) - 64t_{3,a} \equiv -96 \pmod{136}$
- $z(a,a+1) - S_a - 20P^5_{a+1} + 9 \equiv 0 \pmod{0}$

Note : 2
Write 1 as

$$1 = \frac{(-1+i\sqrt{3})(-1-i\sqrt{3})}{4} \tag{23}$$

Substituting (4) , (10) ,(23) in (14a) , it is written in the factorizable form as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (-10 + i\sqrt{3})(-10 - i\sqrt{3})(a + i\sqrt{3}b)^2 (a - i\sqrt{3}\, b)^2 * \frac{(-1+i\sqrt{3})(-1-i\sqrt{3})}{4}$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}v) = (-10 + i\sqrt{3})(a + i\sqrt{3}b)^2 * \frac{(-1 + i\sqrt{3})}{2} \quad (24)$$

$$(u - i\sqrt{3}v) = (-10 - i\sqrt{3})(a - i\sqrt{3}b)^2 * \frac{(-1 - i\sqrt{3})}{2} \quad (25)$$

Equating the real and imaginary parts in (24)

$$u = \frac{1}{2}(7a^2 + 66ab - 21b^2) \quad (26)$$

$$v = \frac{1}{2}(-11a^2 + 14ab + 33b^2) \quad (27)$$

Replace a by 2A , b by 2B in (26) , (27) and (4)

$$u = u(A,B) = 14A^2 + 132AB - 42B^2$$

$$v = y(A,B) = -22A^2 + 28AB + 66B^2$$

$$z = z(A,B) = 4A^2 + 12B^2 \quad (28)$$

Substituting the values of u and v in (2) , we've

$$x = x(A,B) = -8A^2 + 160AB + 24B^2 \quad (29)$$

$$y = y(A,B) = 36A^2 + 104AB - 108B^2 \quad (30)$$

Thus (29) , (30) and (28) represent non-zero distinct integral solutions of (1) in two parameters.

PATTERN: 3

The ternary quadratic equation (3) can be written as

$$(u + 10z)(u - 10z) = 3(z + v)(z - v) \quad (31)$$

The above equation is written in the form of ratio as

$$\frac{(u + 10z)}{(z - v)} = \frac{3(z + v)}{(u - 10z)} = \frac{a}{b} \qquad (b \neq 0) \quad (32)$$

The equation (32) is equivalent to the following two equations

$$bu + av + z(10b - a) = 0 \quad (33)$$

$$au + z(-10a - 3b) - 3vb = 0 \quad (34)$$

Applying the method of cross multiplication , we get ,

$$\frac{u}{-10a^2 + 30b^2 - 6ab} = \frac{v}{-a^2 + 3b^2 + 20ab} = \frac{z}{-(a^2 + 3b^2)}$$

Therefore ,

$$u = u(a,b) = -10a^2 + 30b^2 - 6ab$$

$$v = y(a,b) = -a^2 + 3b^2 + 20ab$$

$$z = z(a,b) = -(a^2 + 3b^2) \quad (35)$$

Substituting in the values of u and v in (2) , we've

$$x = x(a,b) = -11a^2 + 14ab + 33b^2 \quad (36)$$

$$y = y(a,b) = -9a^2 - 26ab + 27b^2 \quad (37)$$

Thus (36) , (37) and (35) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES

- $x(a,1) + 11Pr_a \equiv 8 \pmod{25}$
- $y(a+1, a+1) + 8S_a - 40Pr_a \equiv 0 \pmod{104}$
- $x(a+1, a+1) - 4Pr_a \equiv 0 \pmod{4}$
- $y(a+1, a+1) + z(a+1, a+1) + 8S_a - 44Pr_a \equiv -4 \pmod{100}$

Note: 3

The ternary quadratic equation (3) can be written as

$$(u+10z)(u-10z) = 3(z+v)(z-v) \tag{38}$$

The above equation is written in the form of ratio as

$$\frac{(u+10z)}{3(z-v)} = \frac{(z+v)}{(u-10z)} = \frac{a}{b} \qquad (b \neq 0) \tag{39}$$

The equation (39) is equivalent to the following two equations

$$bu + 3av + z(10b - 3a) = 0 \tag{40}$$

$$-au + z(10a + b) - vb = 0 \tag{41}$$

Applying the method of cross multiplication , we get ,

$$\frac{u}{30a^2 - 10b^2 + 6ab} = \frac{v}{-3a^2 - b^2 - 20ab} = \frac{z}{(3a^2 + b^2)}$$

Therefore ,

$$u = u(a,b) = 30a^2 - 10b^2 + 6ab$$

$$v = v(a,b) = 3a^2 - b^2 - 20ab$$

$$z = z(a,b) = 3a^2 + b^2 \tag{42}$$

Substituting the values of u and v in (2) , we've

$$x = x(a,b) = 33a^2 - 14ab - 11b^2 \tag{43}$$

$$y = y(a,b) = 27a^2 + 26ab - 9b^2 \tag{44}$$

Thus (43) , (44) and (42) represent non-zero distinct integral solutions of (1) in two parameters.

Note: 4

The ternary quadratic equation (3) can be written as

$$(u+10z)(u-10z) = 3(z+v)(z-v) \tag{45}$$

The above equation is written in the form of ratio as

$$\frac{(u+10z)}{3(z+v)} = \frac{(z-v)}{(u-10z)} = \frac{a}{b} \ , \qquad (b \neq 0) \tag{46}$$

The equation (46) is equivalent to the following two equations

$$bu - 3av + z(10b - 3a) = 0 \tag{47}$$

$$au + z(-10a - b) + vb = 0 \tag{48}$$

Applying the method of cross multiplication , we get ,

$$\frac{u}{30a^2-10b^2+6ab}=\frac{v}{-3a^2+b^2+20ab}=\frac{z}{(3a^2+b^2)}$$

$$u=u(a,b)=30a^2-10b^2+6ab$$

$$v=v(a,b)=-3a^2+b^2+20ab$$

$$z=z(a,b)=3a^2+b^2 \qquad (49)$$

Substituting the values of u and v in (2) , we've

$$x=x(a,b)=27a^2+26ab-9b^2 \qquad (50)$$

$$y=y(a,b)=33a^2-14ab-11b^2 \qquad (51)$$

Thus (50) , (51) and (49) represent non-zero distinct integral solutions of (1) in two parameters.

PATTERN: 4

Write equation (3) as

$$u^2=103z^2-3v^2 \qquad (*)$$

Substitute

$$z=X-3T$$

$$v=X-103T \qquad (**)$$

in (*) . we get,

$$u^2=100X^2-30900T^2 \qquad (52)$$

$$u^2=D\alpha^2+\beta^2 \qquad (***)$$

$$T=2pq \qquad (53)$$

$$u=30900p^2-q^2 \qquad (54)$$

$$x=3090p^2+\frac{q^2}{10} \qquad (55)$$

Replace 'p' by 'P' and 'q' by '10Q' in (53) , (54) , (55)

$$T=20PQ \qquad (56)$$

$$u=30900P^2-100Q^2 \qquad (57)$$

$$X=3090P^2+10Q^2 \qquad (58)$$

From (**) , we get ,

$$z=3090P^2+10Q^2-60PQ \qquad (59)$$

$$v=3090P^2+10Q^2-2060PQ \qquad (60)$$

Substituting the values of u and v in (2) , we've

$$x=x(P,Q)=33990P^2-90Q^2-2060PQ \qquad (61)$$

$$y=y(P,Q)=27810P^2-110Q^2+2060PQ \qquad (62)$$

Thus (61) , (62) and (59) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES

- ❖ $x(1,q)+90Pr_q\equiv 500\ (\text{mod } 1970)$
- ❖ $y(1,q)+110Pr_q\equiv 1770\ (\text{mod } 2170)$

- $x(1,q)+y(1,q)+200Pr_q \equiv 0 \pmod{200}$
- $z(1,q)-10Pr_q \equiv 10 \pmod{70}$

Note : 5

Instead of (**) and we may consider

$$z = X+3T$$

$$v = X+103T$$

Then we get the values ,

$$x = x(P,Q) = 33990\,P^2 - 90Q^2 + 2060PQ$$

$$y = y(P,Q) = 27810\,P^2 - 110Q^2 - 2060PQ$$

$$z = z(P,Q) = 3090\,P^2 + 10Q^2 + 60PQ$$

The above equations represent non-zero distinct integral solutions of (1) in two parameters.

PATTERN: 5

Write equation (3) as

$$u^2 = 103z^2 - 3v^2 \qquad (*)$$

Substitute

$$\left.\begin{aligned} z &= X-3T \\ v &= X-103T \end{aligned}\right. \qquad (**)$$

in (*) . we get,

$$u^2 = 100X^2 - 30900T^2 \qquad (52)$$

$$(10X+u)(10X-u) = 30900T^2 \qquad (63)$$

Equation (63) solved by two ways ,

$$\left\{\begin{aligned} 10X+u &= 30900T \\ 10X-u &= T \end{aligned}\right\} \qquad (64)$$

Equation (64) solved and we get,

$$X = \frac{30901T}{20} \qquad (65a)$$

$$u = \frac{30899T}{2} \qquad (65b)$$

Replace T by 20k in equation (65a),(65b) , (**). we get,

$$\begin{aligned} X &= 30901k \\ u &= 308990k \\ v &= 28841k \\ z &= 30841k \end{aligned} \qquad (66)$$

Substituting the values of u and v in (2) , we've

$$x = 337831k \qquad (67)$$

$$y = 280149k \qquad (68)$$

Thus (67) , (68) and (66) represent non-zero distinct integral solutions of (1) .

PATTERN: 6

Write equation (3) as

$$u^2 = 103z^2 - 3v^2 \tag{*}$$

Substitute

$$\begin{aligned} z &= X - 3T \\ v &= X - 103T \end{aligned} \tag{**}$$

in (*) . we get,

$$u^2 = 100X^2 - 30900T^2 \tag{52}$$

$$(10X + u)(10X - u) = (309T)(100T) \tag{aa}$$

Equation (aa) solved by two ways ,

$$\left\{\begin{matrix} 10X + u = 309T \\ 10X - u = 100T \end{matrix}\right\} \tag{69}$$

Equation (69) solved and we get,

$$X = \frac{409T}{20} \tag{70a}$$

$$u = \frac{209T}{2} \tag{70b}$$

Replace T by 20k in equation (70a),(70b) , (**). we get,

$$\begin{aligned} X &= 409k \\ u &= 2090k \\ v &= -1651k \\ z &= 349k \end{aligned} \tag{71}$$

Substituting the values of u and v in (2) , we've

$$x = 439k \tag{72}$$

$$y = 3741k \tag{73}$$

Thus (72) , (73) and (71) represent non-zero distinct integral solutions of (1) .

CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$x^2 + y^2 - xy = 103z^2$$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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