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# Discussion on a Kind of Sequence Limit

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**Abstract:** In this paper, we give four theorems and proved them. According to these four theorems, we deduce the solver method for the limit of a class of sequence $\{x_n\}$ by recursive relation $x_n = f(x_{n-1})$.
**Keywords:** limit of a sequence, recursion formula, mathematical induction limit of sequence limit of sequence

## PROBLEM POSING

For a special class of infinite series $(n=1,2,\cdots)$, If sequence of $\{x_n\}$ satisfies the recursive formula of $x_n = f(x_{n-1})$, we can discuss the limit of $\{x_n\}$ $(n=1,2,\cdots)$ according to the property of $f'(x)$ [1]. So we give four theorems as follows.

## FOUR THEOREMS

**Theorem 1** If in $[a,b]$ equation $x = f(x)$ has a unique root $\xi$, and $|f'(x)| \le q < 1$, $x_0$ is any real number in $|\xi - x_0| \le \min\{(\xi-a),(b-\xi)\}|$, then sequence:

$$x_1 = f(x_0), x_2 = f(x_1), \cdots, x_n = f(x_{n-1}) \cdots$$

convergence to $\xi$.

**Proof** According to $x_1 - f(x_0) = \xi - f(\xi)$ and Lagrange Mean Value Theorem[2]

$$|\xi - x_1| = |\xi - x_0|\,|f'(c)| \le |\xi - x_0|\, q < |\xi - x_0| \quad (\xi < c < x_0),$$

so $x_1$ is closer to $\xi$ than $x_0$, moreover, as $|\xi - x_0| \le \min\{(\xi-a),(b-\xi)\}$, so $x_1 \in [a,b]$.

By mathematical induction[3], we can deduce $x_n \in [a,b]$ $(n=1,2,\cdots)$. According to

$$x_{n+1} - f(x_n) = x_n - f(x_{n-1})$$

and Lagrange Mean Value Theorem:

$$|x_{n+1} - x_n| = |x_n - x_{n-1}|\,|f'(c_n)|.$$

Since $x_n \in [a,b]$, $c_n$ is between $x_n$ and $x_{n-1}$, so $c_n \in [a,b]$, $|f'(c_n)| \le q < 1$. Therefore

$$|x_{n+1} - x_n| \le |x_n - x_{n-1}|\, q.$$

And then

$$|x_{n+1} - x_n| \le q^n |x_1 - x_0|, \; |x_{n+p} - x_n| \le |x_1 - x_0| \le \frac{q^n}{1-q} |x_1 - x_0|.$$

When $n \to \infty$, $q^n \to 0$, so $\lim\limits_{n\to\infty} x_n$ is extant. By the continuity of $f(x)$, $\lim\limits_{n\to\infty} x_n = f(\lim\limits_{n\to\infty} x_{n-1})$, and by the uniqueness of $\xi$ in $[a,b]$, $\lim\limits_{n\to\infty} x_n = \xi$.

**Theorem 2** If $x = f(x)$ has real root $\xi_i$ $(i = 1, 2, 3 \cdots m)$, $f(x)$ has derivative in every point. And $|f'(\xi_i)| > 1$, for any real number $x_0$, the follow sequence is diverging:

$$x_1 = f(x_0), x_2 = f(x_1), ..., x_n = f(x_{n-1}), \quad \ldots \text{(If } i \neq j \text{, then } x_i \neq x_j\text{)}$$

**Proof** If $x_n = f(x_{n-1})$, sequence { $x_n$ } must converge to some real number $\xi_N$ $(1 \leq N \leq m)$.

By $x_n - f(x_{n-1}) = \xi_N - f(\xi_N)$ and Lagrange Mean Value Theorem, we can derive $\left|\dfrac{\xi_N - x_n}{\xi_N - x_{n-1}}\right| = |f'(c)|$.

Since c is between $\xi_N$ and $x_{n-1}$, $x_n \to \xi_N (n \to \infty)$, so $\left|\dfrac{\xi_N - x_n}{\xi_N - x_{n-1}}\right| \to |f'(\xi_N)|$ $(n \to \infty)$.

For any real number $\varepsilon$ in $(0, +\infty)$, when $n$ is large enough

$$|\xi_N - x_n| > |\xi_N - x_{n-1}|(|f'(\xi_N)| - \varepsilon)$$

is tenable.

As positive real number $\varepsilon$ is arbitrarily small, $|f'(\xi_N)| \leq 1$. it is contradict with $|f'(\xi_N)| > 1$, so { $x_n$ } must be diverging.

If $|f'(\xi_N)| > 1$, then $|f'(\xi_N)| - \varepsilon > 1$. But by

$$|\xi_N - x_n| > |\xi_N - x_{n-1}|(|f'(\xi_N)| - \varepsilon),$$

we can know $|\xi_N - x_n| > |\xi_N - x_{n-1}|$, the result is contradict with $x_n \to \xi_N$.

**Theorem 3** If in $[a, b]$ equation $x = f(x)$ has a unique root $\xi$, $f'(x) < 0$, and $f(a) \in [a, b]$, $f(b) \in [a, b]$, $x_0 \in [a, b]$, then sequence：

$$x_1 = f(x_0), x_3 = f(x_2), \cdots, x_{2m+1} = f(x_{2m}), \cdots$$

and $\quad x_2 = f(x_1), x_4 = f(x_3), \cdots, x_{2m} = f(x_{2m-1}), \cdots \qquad$ are both convergent.

**Proof** We might as well let $x_0 = b$ （$x_0$ is any value in $[a, b]$, the proof is same as this.）. According to $x_1 - f(b) = \xi - f(\xi)$ and $f'(x) < 0$, $\xi < b$, we can derive $x_1 < \xi$. By $x_2 - f(x_1) = \xi - f(\xi)$ and $f'(x) < 0$, $x_1 < \xi$, we can know $x_2 > \xi$. $x_{2m} > \xi > x_{2m+1}$ $(m = 0, 1, 2, \cdots)$ can be proved by mathematical induction.

Set $x_1^* = f(a)$, by $x_2 - f(x_1) = x_1^* - f(a)$ and $f'(x) < 0$, $x_1 \geq a$, we can deduce $x_2 \leq x_1^*$, we have known $x_1^* \leq b$ ($x_1^* \in [a, b]$), so $x_2 \leq b$.

As $x_3 - f(x_2) = x_1 - f(b)$ and $f'(x) < 0$, $x_2 \leq b$, then $x_1 \leq x_3$.

By $x_2 - f(x_1) = x_4 - f(x_3)$ and $f'(x) < 0$, $x_1 \leq x_3$, we can know $x_4 \leq x_2$.

So $x_{2m-1} \leq x_{2m+1}$, $x_{2m} \leq x_{2m-2}$ $(m = 0, 1, 2, \cdots)$ might be proved by mathematical induction. In conclusion. then

$$x_{2m-1} \leq x_{2m+1} \leq \xi < x_{2m} \leq x_{2m-2} \quad (m = 0, 1, 2, \cdots) \text{[3]},$$

so we can see $\lim\limits_{m \to \infty} x_{2m+1}$ and $\lim\limits_{m \to \infty} x_{2m}$ are both existent[4].

**Theorem 4** If in $[a, b]$ equation $x = f(x)$ has a unique root $\xi$, and $0 < f'(x) < 1$, $x_0 \in [a, b]$, then sequence:

$$x_1 = f(x_0), x_2 = f(x_1), \cdots, x_n = f(x_{n-1}), \cdots \quad \text{converge to } \xi.$$

**Proof** We might as well let $x_0 = b$. According to $0 < f'(x) < 1$, we can derive $1 - f'(x) > 0$. So $x - f(x)$ is monotone increasing, so that $b - f(b) > 0$, $a - f(a) < 0$. By $f'(x) > 0$, we can know $a - f(b) < a - f(a) < 0$. So monotone continuous function $x - f(b)$ in the two endpoints of $[a,b]$ has opposite signs. So that $x_1 = f(b) \in [a,b]$. As $x_1 - f(b) = \xi - f(\xi)$ and $f'(x) > 0$, $b > \xi$, then $x_1 > \xi$. So $x_n > \xi$ $(n = 0,1,2,\cdots)$ might be proved by mathematical induction. By $x_1 - f(b) = x_2 - f(x_1)$, $f'(x) > 0$, $b > x_1$, we can know $x_1 > x_2$. By mathematical induction we can prove $x_n > x_{n+1}$ $(n = 0,1,2,\cdots)$, so that $\xi < x_{n+1} < x_n$ $(n = 0,1,2,\cdots)$. So $\lim_{n\to\infty} x_n$ is existent. By $x_n = f(x_{n-1})$ we can infer

$$\lim_{n\to\infty} x_n = f(\lim_{n\to\infty} x_{n-1}).$$

Since $\xi$ is unique in $[a,b]$, we can infer $\lim_{n\to\infty} x_n = \xi$.

## APPLICATION EXAMPLE

**Example 1** Sequence $\{x_n\}$ is as follows.

$$x_1 = \sqrt{2}, x_2 = \sqrt{2+\sqrt{2}}, \cdots, x_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}, \cdots$$

Solving $\lim_{n\to\infty} x_n$ [5].

Solving $x = \sqrt{x+2}$, the solution is 2. In [0,4], the equation has a unique solution 2, and $|(\sqrt{x+2})'| = \left|\frac{1}{2\sqrt{x+2}}\right| < \frac{1}{2} < 1$, let $x_0 = 0$, it meets qualifications of Theorem 1, so $\lim_{n\to\infty} x_n = 2$.

**Example 2** Solving $\lim_{n\to\infty}(a + a^2 + \ldots + a^n)$ $(|a| < 1)$.

Solving $x = ax + a$, the solution is $\frac{a}{1-a}$. In $\left[0, \frac{2a}{1-a}\right]$ it has a unique solution $\frac{a}{1-a}$ (we might as well set $a > 0$), and $|(ax+a)'| = |a| < 1$. Let $x_0 = 0$, we can know it meets qualifications of Theorem 1, so $\lim_{n\to\infty}(a + a^2 + \ldots + a^n) = \frac{a}{1-a}$. If $a \le 0$ $(|a| < 1)$, $\lim_{n\to\infty}(a + a^2 + \ldots + a^n) = \frac{a}{1-a}$.

When $|a| < 1$,

$$\lim_{n\to\infty}(a + a^2 + \ldots + a^n) = \frac{a}{1-a}.$$

When $|a| > 1$, we can infer $|(ax+a)'| = |a| > 1$. Using theorem 2 we can know sequence:

$x_1 = a, x_2 = a + a^2, \ldots, x_n = a + a^2 + \ldots + a^n, \ldots$ ($i \ne j$时, $x_i \ne x_j$) is divergent.

## CONCLUSION

In general, through the four theorems we can deduce the solver method for the limit of a class of sequence $\{x_n\}$ by recursive relation $x_n = f(x_{n-1})$.

First, solving $x = f(x)$ we got all the real number solutions $\xi_i \quad (i=1,2,\cdots m)$. Sequence { $x_n$ } converge to and can only converge to some $\xi_i$ . And according to the features of sequence { $x_n$ }, we set a range $[a,b]$ which include a $\xi_i$ , by theorem 1, if we can find a $x_0$ that satisfied $|\xi_i - x_0| \le \min\{(\xi_i - a),(b-\xi_i)\}$, let $x_n = f(x_{n-1})$ $(n=0,1,2,\cdots)$ be tenable, then we can conclude $\lim_{n\to\infty} x_n = \xi_i$. By the theorem 4, if we can find a $x_0$ in $[a,b]$, let $x_n = f(x_{n-1})$ $(n=0,1,2,\cdots)$ be tenable, then we can conclude $\lim_{n\to\infty} x_n = \xi_i$. If $|f'(\xi_i)|>1$ $(i=1,2,\cdots)$, we can know sequence is divergent by theorem 4. If theorem 1, theorem 2, theorem 4 can not solve the question, we can use theorem 3 solve it. And when $|f'(\xi_i)|=1$, the convergence of sequence { $x_n$ } is uncertain.

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