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Some Particular Examples for the Natural Lift Curve in Minkowski 3-Space

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Abstract: In this study, we give some particular examples for the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors.

Keywords: Natural Lift, Geodesic Spray

INTRODUCTİON

Thorpe gave the concepts of the natural lift curve and geodesic spray in [11]. Thorpe provied the natural lift $\bar{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M in [11]. Çalışkan, Sivridağ and Hacısalihoğlu studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [10]. They gave some interesting results about the original curve were obtaied, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$ in [10]. Some properties of $\bar{M}$ -vector field Z defined on a hypersurface M of $\bar{M}$ were studied by Agashe in [1]. $\bar{M}$ -integral curve of Z and $\bar{M}$ -geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\bar{\alpha}$ of the curve α (in $\bar{M}$) is an $\bar{M}$ -integral curve of the geodesic spray Z iff α is an $\bar{M}$ -geodesic in [4]. Bilici, Çalışkan and Aydemir studied $\left(\alpha,\alpha^*\right)$ being the pair of evolute-involute curves,the natural lift curve of the spherical indicatries of tangent, principal normal, binormal vectors of the involute curve α^* . They gave some interesting results about the evolute curve α were obtained, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute α^* should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [3]. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [6]. The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [6]. Çalşkan and Ergün defined $\bar{M}$ -vector field Z, $\bar{M}$ -geodesic spray, $\bar{M}$ -integral curve of Z, $\bar{M}$ -geodesic in [5]. The anologue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [5]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [12].

Let Minkowski 3-space $\mathbb{R}^3_1$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g\left(X,X\right)=-x_1^2+x_2^2+x_3^2$$

where $X=\left(x_1,x_2,x_3\right)\in\mathbb{R}^3$. A vector $X=\left(x_1,x_2,x_3\right)\in\mathbb{R}^3$ is said to be timelike if $g\left(X,X\right)<0$, spacelike if $g\left(X,X\right)>0$ and lightlike (or null) if $g\left(X,X\right)=0$. Similarly, an arbitrary curve $\alpha=\alpha\left(t\right)$ in $\mathbb{R}^3_1$ where t is a pseudo-arclength parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}\left(t\right)$ are respectively timelike, spacelike or null (lightlike), for every $t\in I\subset\mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1>0$ ($resp. x_1<0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (*resp.* $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$, [8] .

The Lorentzian sphere and hyperbolic sphere of radius 1 in $\mathbb{R}^3_1$ are given by

$$S_1^2 = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \ : \ g(X,X) = 1\}$$

and

$$H_0^2 = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \ : \ g(X,X) = -1\}$$

respectively,[8]. The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ are orthogonal if and only if $g(X,X) = 0$,[8]..

Now let X and Y be two vectors in $\mathbb{R}^3_1$, then the Lorentzian cross product is given by

$$X \times Y = (x_3y_2 - x_2y_3,\ x_1y_3 - x_3y_1,\ x_1y_2 - x_2y_1), \text{ [2].}$$

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve α . Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α , respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ . Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields.For this vectors, we can write

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N,$$

where $\times$ is the Lorentzian cross product, [2], in space $\mathbb{R}^3_1$. Then, Frenet formulas are given by

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = -\tau N, \text{ [12].}$$

The Frenet instantaneous rotation vector for the timelike curve is given by $W = \tau T + \kappa B.$

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field In this situation,

$$T \times N = B, \quad N \times B = T, \quad B \times T = -N,$$

Then, Frenet formulas are given by

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = \tau N, \text{ [12].}$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = \tau T - \kappa B.$

Let α be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that T and N are spacelike vector fields and B is a timelike vector field.In this situation,

$$T \times N = -B, \quad N \times B = T, \quad B \times T = N,$$

Then, Frenet formulas are given by,

$$\dot{T} = \kappa N,\ \dot{N} = -\kappa T + \tau B,\ \dot{B} = \tau N, \text{ [12].}$$

The Frenet instantaneous rotation vector for the spacelike space curve with a timelike binormal is given by $W = -\tau T + \kappa B.$

Theorem 1 : Let α be a unit speed timelike space curve. Then we have

1) $\kappa = 0$ *if and only if* α *is a part of a timelike straight line;*
2) $\tau = 0$ *if and only if* α *is a planar timelike curve;*
3) $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*
4) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a timelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$;

5) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a timelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$;

6) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| = \kappa$ *if and only if* α *can be parameterized by*

$$\alpha(s) = \frac{1}{6}\left(\kappa^2 s^3 + 6s, 3\kappa s^2, \kappa\tau s^3\right)$$, [12] .

Theorem2: Let α be a unit speed spacelike space curve with a spacelike binormal. Then we have

1) $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*
2) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(, \kappa\cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\sinh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 + \tau^2$,[12].

Theorem3: Let α be a unit speed spacelike space curve with a timelike binormal. Then we have

1) $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a circle;*
2) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$;

3) $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$;

4) $\kappa =$ *constant* >0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| = \kappa$ *if and only if* α *can be parameterized by*

$$\alpha(s) = \frac{1}{6}\left(\kappa\tau s^3, -\kappa^2 s^3 + 6s, 3\kappa s^2\right) \text{ [12]} .$$

Definition4: Let M be a hypersurface in $\mathbb{R}^3_1$ and let $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \text{ (for all } t \in I)$$

where X is a smooth tangent vector field on M , [8]. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M .

Definition 5: For any parametrized curve $\alpha : I \to M$, $\bar{\alpha} : I \to TM$ given by

$$\bar{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t)\right) = \dot{\alpha}(t)|_{\alpha(t)}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\bar{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\dot{\alpha}(t)$$

where D is the Levi-Civita connection on $\mathbb{R}^3_1$, [6].

Definition 6: A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$

$$X(V) = +\varepsilon g(S(V), V)N ,$$

where $\varepsilon = g(N, N)$, [6].

Theorem 7: The natural lift $\bar{\alpha}$ of the curve α is an integral curve of geodesic spray X if and only if α is a geodesic on M , [6].

Some Particular Examples for the Natural Lift Curve in Minkowski 3-Space

Let D , $\bar{D}$ and $\bar{\bar{D}}$ be connections in $\mathbb{R}^3_1$, S^2_1 and H^2_0 respectively and ξ be a unit normal vector field of S^2_1 and H^2_0 . Then Gauss Equations are given by the followings

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)\xi,$$

$$D_X Y = \bar{\bar{D}}_X Y + \varepsilon g(S(X), Y)\xi,$$

where $\varepsilon = g(\xi, \xi)$ and S is the shape operator of S^2_1 and H^2_0 .

Let α be a unit speed timelike space curve.

Corollary 8: If the natural lift $\overline{\alpha_T}$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^2\right)$, then the curve α is a part of an orthogonal hyperbola, [13].

Corollary 9: If the natural lift $\overline{\alpha_N}$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve α can be classified as

i. $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*

ii. $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a timelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$;

iii. $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a timelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$,[13].

Corollary 10: If the natural lift $\overline{\alpha_B}$ of α_B is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$, then we have $\kappa = 0$, $\tau = 1$. Therefore there is no curve α which holds this condition, [13].

Let α be a unit speed spacelike space curve with a spacelike binormal.

Corollary 11: If the natural lift $\overline{\alpha_T}$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve α is a part of an orthogonal hyperbola, [13].

Corollary 12: If the natural lift $\overline{\alpha_N}$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(H_0^2\right)$ then the curve α can be classified as

i. $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*

ii. $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(, \kappa\cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\sinh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 + \tau^2$,[13].

Corollary 13: If the natural lift $\overline{\alpha_B}$ of α_B is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then we have $\kappa = 0$, $\tau = 1$. Therefore there is no curve α which holds this condition,[13].

Let α be a unit speed spacelike space curve with a timelike binormal.

Corollary 14: If the natural lift $\overline{\alpha_T}$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve α is a part of a circle,[13].

Corollary 15: If the natural lift $\overline{\alpha_N}$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve α can be classified as

i. $\tau = 0$ *and* $\kappa =$ *constant* > 0 *if and only if* α *is a part of a circle;*

ii. $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa \sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa \cosh\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$ *;*

iii. $\kappa =$ *constant* > 0 , $\tau =$ *constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa \cos\left(\sqrt{K}s\right), \kappa \sin\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$,[13].

Corollary 16: If the natural lift $\overline{\alpha_B}$ of α_B is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$, then we have $\kappa = 0$, $\tau = 1$. Therefore there is no curve α which holds this condition, [13].

Example 17: *Let* $\alpha(s) = \left(\frac{2\sqrt{3}}{3}s, \frac{1}{3}\cos\left(\sqrt{3}s\right), \frac{1}{3}\sin\left(\sqrt{3}s\right)\right)$ *be a unit speed (timelike curve) timelike circular helix with;*

$$T(s) = \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right),$$

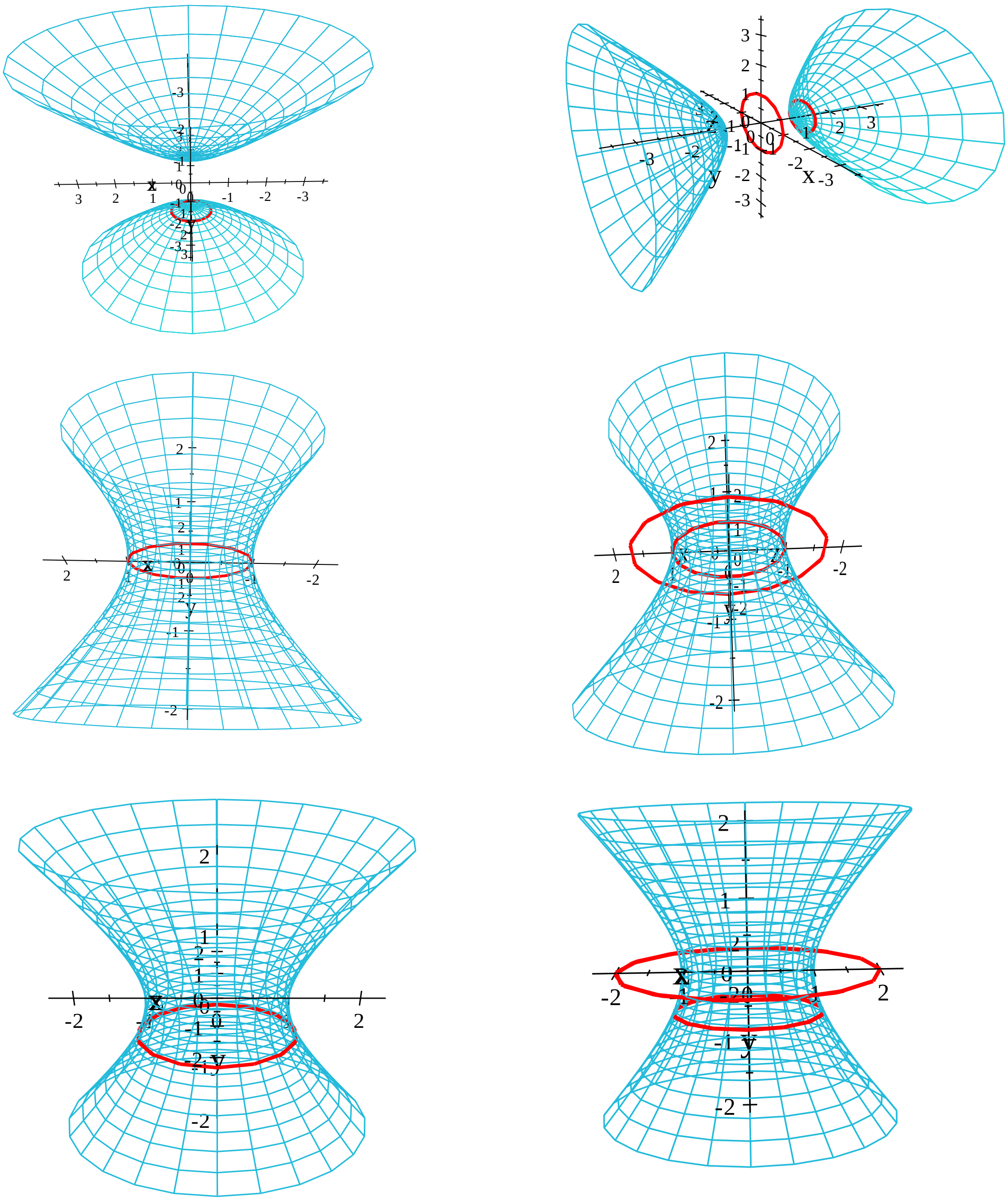
$$N(s) = \left(0, -\cos\left(\sqrt{3}s\right), -\sin\left(\sqrt{3}s\right)\right),$$

$$B(s) = \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), -\frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right) \text{ and}$$

$$\alpha_T(s) = \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right),$$

$$\alpha_N(s) = \left(0, -\cos\left(\sqrt{3}s\right), -\sin\left(\sqrt{3}s\right)\right),$$

$$\alpha_B(s) = \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), -\frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right)$$



Example 18: *Let* $\alpha(s)=\left(\frac{2}{3}\sinh\left(\sqrt{3}s\right),\frac{\sqrt{3}}{3}s,\frac{2}{3}\cosh\left(\sqrt{3}s\right)\right)$ *be a unit speed (timelike curve) timelike hyperbolic helix with;*

$$T(s)=\left(\frac{2\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{\sqrt{3}}{3},\frac{2\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right),$$

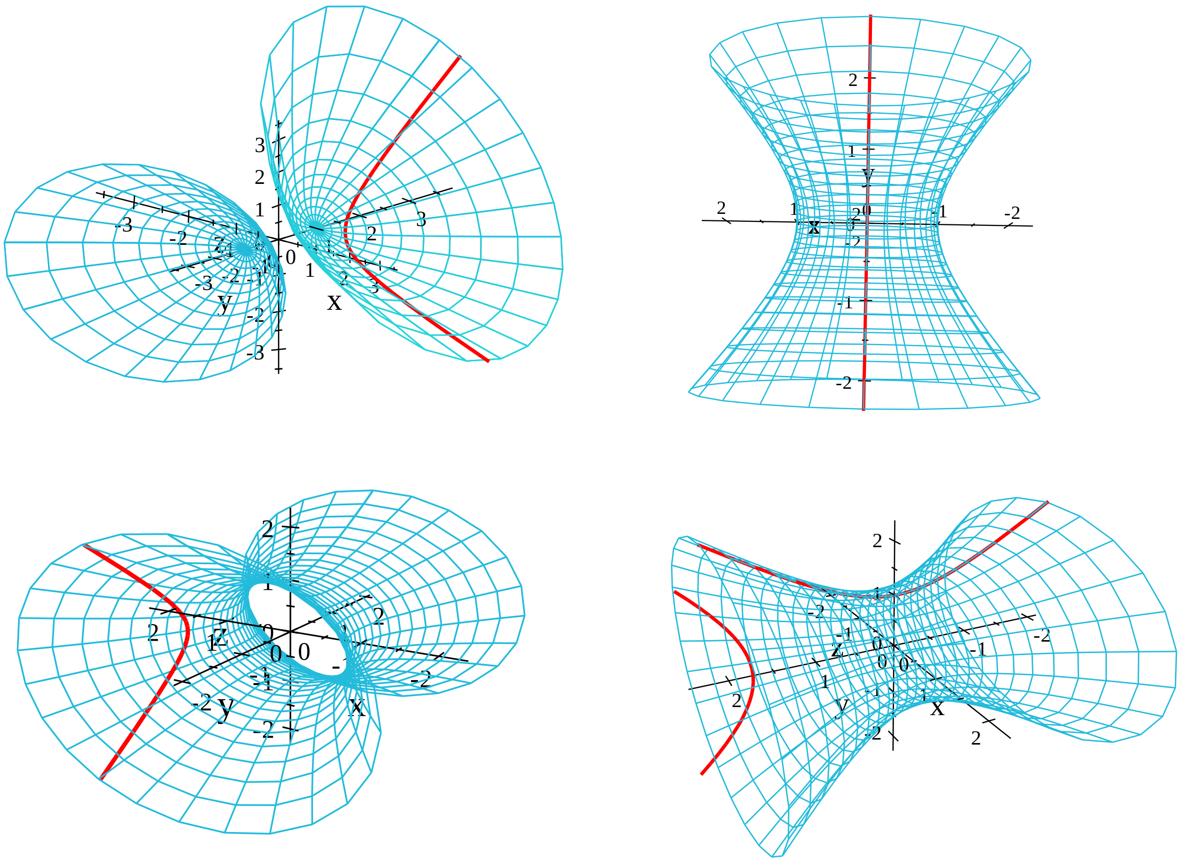
$$N(s)=\left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right),$$

$$B(s)=\left(-\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right) \text{ and}$$

$$\alpha_T(s)=\left(\frac{2\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{\sqrt{3}}{3},\frac{2\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right),$$

$$\alpha_N(s)=\left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right),$$

$$\alpha_B(s)=\left(-\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right),$$



Example 19: *Let* $\alpha(s)=\left(\frac{2}{5}\cosh\left(\sqrt{5}s\right),\frac{\sqrt{5}}{5}s,\frac{2}{5}\sinh\left(\sqrt{5}s\right)\right)$ *be a unit speed (spacelike curve with spacelike binormal) spacelike hyperbolic helix with;*

$$T(s)=\left(\frac{2\sqrt{5}}{5}\sinh\left(\sqrt{5}s\right),\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\cosh\left(\sqrt{5}s\right)\right),$$

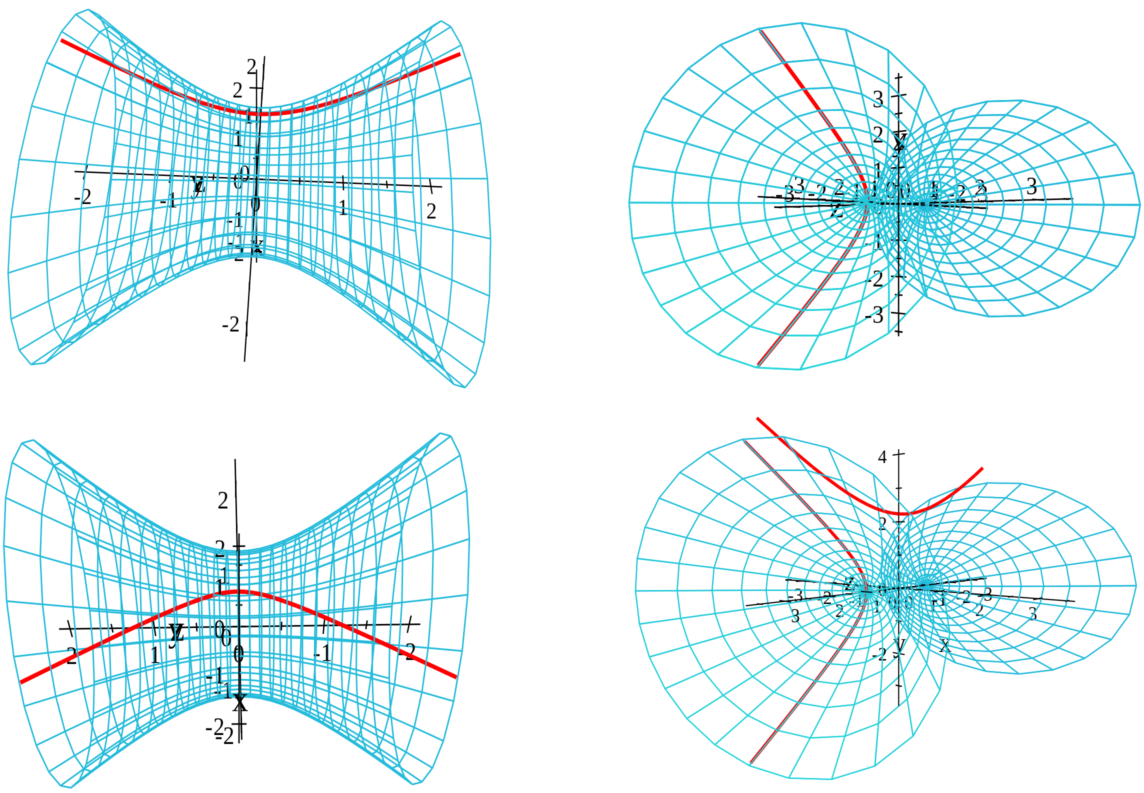
$$N(s)=\left(\cosh\left(\sqrt{5}s\right),0,\sinh\left(\sqrt{5}s\right)\right),$$

$$B(s)=\left(-\frac{\sqrt{5}}{5}\sinh\left(\sqrt{5}s\right),\frac{2\sqrt{5}}{5},-\frac{\sqrt{5}}{5}\cosh\left(\sqrt{5}s\right)\right) \text{ and}$$

$$\alpha_T(s)=\left(\frac{2\sqrt{5}}{5}\sinh\left(\sqrt{5}s\right),\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\cosh\left(\sqrt{5}s\right)\right),$$

$$\alpha_N(s)=\left(\cosh\left(\sqrt{5}s\right),0,\sinh\left(\sqrt{5}s\right)\right),$$

$$\alpha_B(s)=\left(-\frac{\sqrt{5}}{5}\sinh\left(\sqrt{5}s\right),\frac{2\sqrt{5}}{5},-\frac{\sqrt{5}}{5}\cosh\left(\sqrt{5}s\right)\right),$$



Example 20: *Let* $\alpha(s)=\left(\sinh(s),\sqrt{2}s,\cosh(s)\right)$ *be a unit speed (spacelike curve with timelike binormal)spacelike hyperbolic helix with;*

$$T(s)=\left(\cosh(s),\sqrt{2},\sinh(s)\right),$$

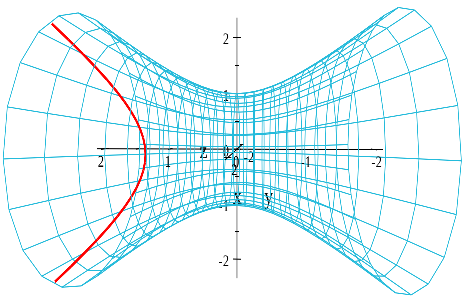
$$N(s)=\left(\sinh(s),0,\cosh(s)\right),$$

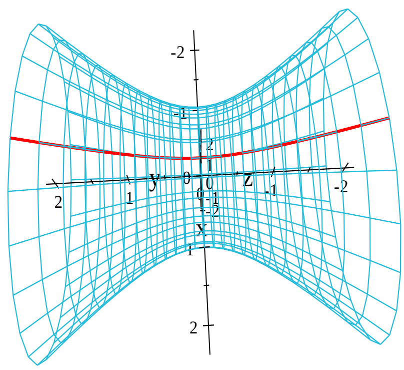
$$B(s)=\left(\sqrt{2}\cosh(s),1,\sqrt{2}\sinh(s)\right) \text{ and}$$

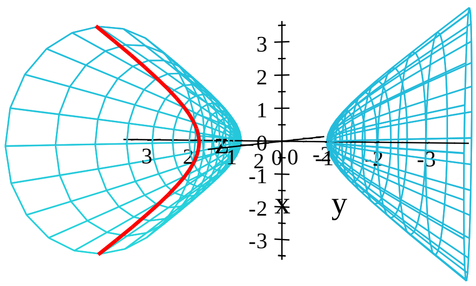
$$\alpha_T(s)=\left(\cosh(s),\sqrt{2},\sinh(s)\right),$$

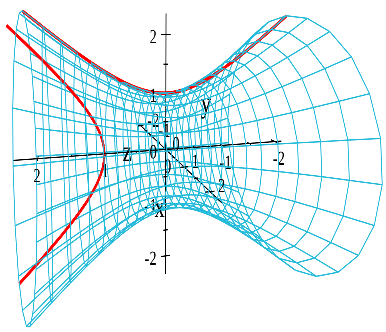
$$\alpha_N(s)=\left(\sinh(s),0,\cosh(s)\right),$$

$$\alpha_B(s)=\left(\sqrt{2}\cosh(s),1,\sqrt{2}\sinh(s)\right),$$









Example 21: *Let* $\alpha(s)=\left(\frac{\sqrt{3}}{3}s,\frac{2}{3}\cos\left(\sqrt{3}s\right),\frac{2}{3}\sin\left(\sqrt{3}s\right)\right)$ *be a unit speed (spacelike curve with timelike binormal) spacelike circular helix with;*

$$T(s)=\left(\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right),$$

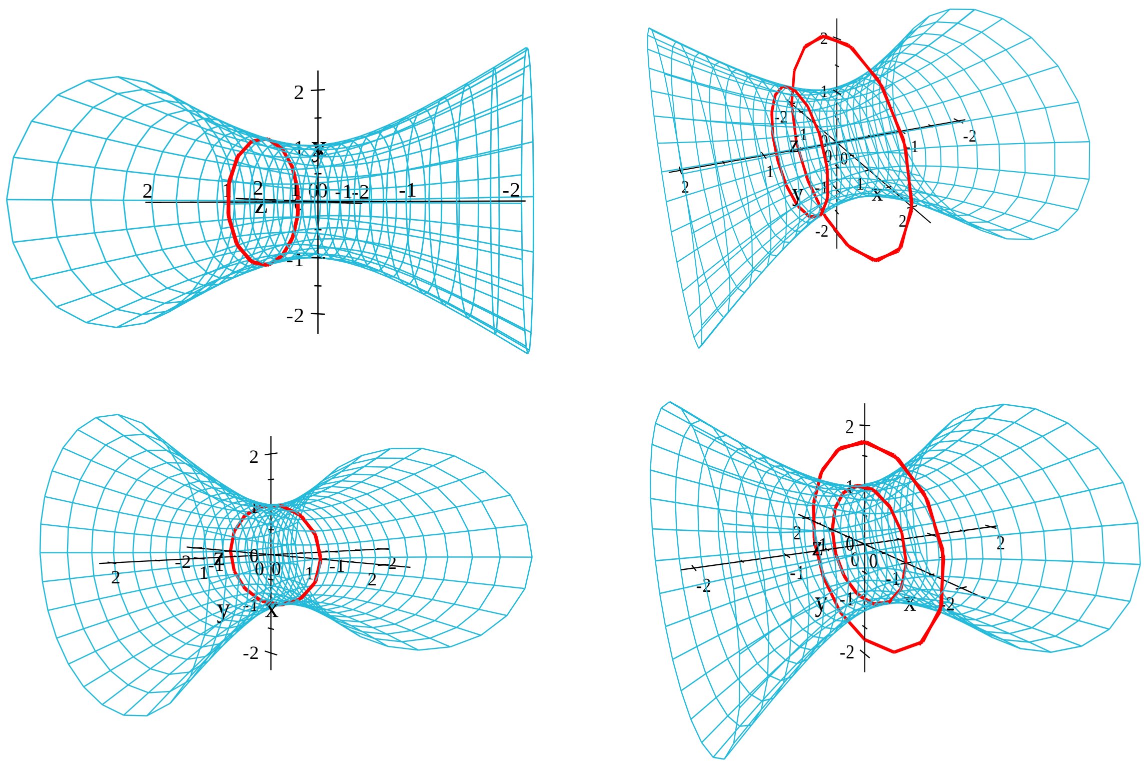
$$N(s)=\left(0,-\cos\left(\sqrt{3}s\right),-\sin\left(\sqrt{3}s\right)\right),$$

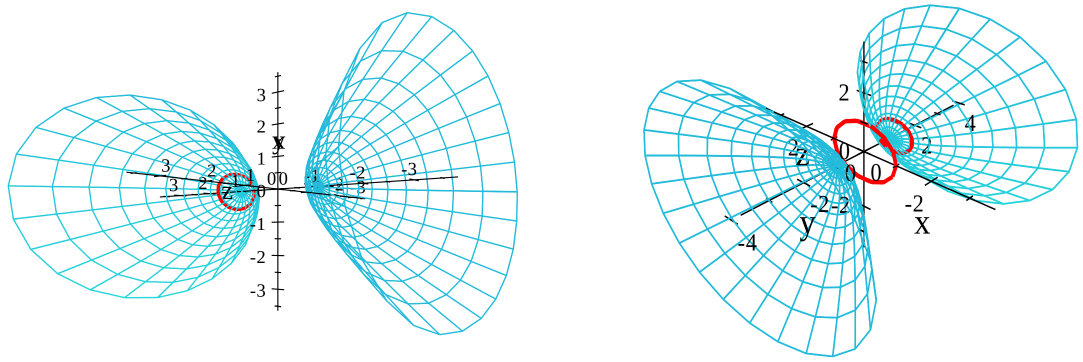
$$B(s)=\left(\frac{2\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right),\frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right) \text{ and}$$

$$\alpha_T(s)=\left(\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right),$$

$$\alpha_N(s)=\left(0,-\cos\left(\sqrt{3}s\right),-\sin\left(\sqrt{3}s\right)\right),$$

$$\alpha_B(s)=\left(\frac{2\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right),\frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right).$$





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