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Strongly Gorenstein Syzygy Modules and Strongly Gorenstein Flat Dimensions

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Abstract: For any ring R and any positive integer n, We investigate the relation between the Strongly Gorenstein-n-syzygy with the n-syzygy of a module as well as the relation between the Gorenstein-n-syzygy and the n-syzygy of a module. We obtain a criterion for computing strongly Gorenstein flat dimension of modules.
Keywords: strongly Gorenstein flat modules, strongly Gorenstein n- syzygy modules, n-syzygy modules, strongly Gorenstein flat dimension.
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INTRODUCTION

In classical homological algebra, the notion of finitely generated projective modules is an important and fundamental research object. As a generalization of this notion, Auslander and Bridger introduced in [1] the notion of finitely generated modules of Gorenstein dimension zero over a left and right Noetherian ring. Over a general ring, Enochs and Jenda introduced in [8] the notion of Gorenstein projective modules (not necessarily finitely generated) . It is well known that these two notions coincide for finitely generated modules over a left and right Noetherian ring. In particular, Gorenstein projective modules share many nice properties of projective modules (e.g. [1, 3, 4, 5, 8, 11]).

In [6], a particular case of Gorenstein projective modules which is called strongly Gorenstein flat modules was introduced. Dual to the definition of strongly Gorenstein flat modules, in [12], a particular case of Gorenstein injective modules which is called Gorenstein FP-injective modules was introduced. We notice that strongly Gorenstein flat and Gorenstein FP-injective modules are also called Ding projective and Ding injective modules in [9], respectively.

The notion of a syzygy module was defined via the projective resolution of modules as follows. For a positive integer n, a module $A\in$ Mod R is called an n-syzygy module (of M)there exists an exact sequence $0\to A\to P_{n-1}\to\Lambda\to P_1\to P_0\to M\to 0$ in Mod R with all P_i projective. Recall from [10], a module A is called Gorenstein n-syzygy module (of M) if there exists an exact sequence $0\to A\to Q_{n-1}\to\Lambda\to Q_1\to Q_0\to M\to 0$ in Mod R with all Q_i Gorenstein projective. Analogously, we call A strongly Gorenstein n-syzygy module (of M) if there exists an exact sequence $0\to A\to X_{n-1}\to\Lambda\to X_1\to X_0\to M\to 0$ in Mod R with all X_i strongly Gorenstein flat.

In Section 2, we establish the relation between the strongly Gorenstein syzygy with the syzygy of a module as well as the relation between the Gorenstein-n-syzygy and the n-syzygy of a module. Then we get a criterion for compute strongly Gorenstein flat dimensions. Let M be a left R-module and $n\geq 0$. We prove that the strongly Gorenstein flat dimension of M is at most n if and only if for every non-negative integer t such that $0\leq t\leq n$, there exists an exact sequence $0\to X_{n-1}\to\Lambda\to X_1\to X_0\to M\to 0$ in Mod R such that X_t is strongly Gorenstein flat and X_i is projective for $i\neq t$.

Unless stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary left R-modules . We use Mod R to denote the class of right R-modules, For any right R-module M, we denote by pd (M), id (M) and fd (M) the projective dimension, injective dimension and flat dimension of M respectively. Let

M and N be right R-modules. Hom(M , N) (respectively Ext$^i(M,N)$ means Hom$_R(M,N)$ (respectively Ext$_R^i(M,N)$ for an integer $i \geq 1$ throughout this paper. Given a class X of R-modules, a sequence is Hom(-, X)-exact if it is exact after applying the functor Hom(-, X)for all $X \in$ X . The sequence is Hom(X ,-)-exact if it is exact after applying the functor Hom(X, -) for all $X \in$ X . For unexplained concepts and notations, we refer the reader to [1, 2, 13].

THE MAIN RESULTS

Recall from [6], a left R-module M is called strongly Gorenstein flat if there exists a Hom(-, F)-exact exact sequence: $\Lambda \to P_1 \to P_0 \to P^0 \to P^1 \to \Lambda$ of projective left R-modules with M =ker($P^0 \to P^1$) where F stands for the class of flat R-modules.

We use SGfd$_R(M)$ to denote the strongly Gorenstein flat dimension of a module M in Mod R, which is defined as the smallest non-negative integer n such that there exists an exact sequence $0 \to G_n \to \Lambda \to G_1 \to G_0 \to M \to 0$ with each G_i strongly Gorenstein flat. If no such n exists, set SGfd$_R(M)$ $=\infty$.

We use SGF(R) to denote the class of strongly Gorenstein flat R-modules. By [14, Lemma 2.3], SGF(R) is p rojectively resolving over any ring.

Lemma 2.1 ([14, Lemma 2.3]) SGF(R) is projectively resolving, and closed under arbitrary direct sums and direct summands.

Lemma 2.2 Let $0 \to M_3 \to M_2 \to M_1 \to 0$ be an exact sequence in Mod R with $M_3 \neq 0$. If M_1 is strongly Gorenstein flat, then SGfd$_R(M_3)$=SGfd$_R(M_2)$.

Proof. By [14, Proposition 2.7 and Proposition 2.8], it is easy to get the assertion.
The following result plays a crucial role in this paper.

Proposition 2.3 Let $0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0$ be an exact sequence in Mod R with G_0 and G_1 strongly Gorenstein flat. Then we have the following exact sequences:

$$0 \to A \to P \to G \to M \to 0 \quad \text{and} \quad 0 \to A \to H \to Q \to M \to 0$$

in Mod R with P, Q projective and G, H strongly Gorenstein flat.

Proof . Because G_1 is strongly Gorenstein flat, there exists an exact sequence $0 \to G_1 \to P \to G_2 \to 0$ in Mod R with P projective and G_2 strongly Gorenstein flat. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & \mathrm{Im}(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathrm{Im}(f) & \to & G_0 & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B & \to & G & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & G_2 & = & G_2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

In the sequence $0 \to G_0 \to G \to G_2 \to 0$, both G_0 and G_2 are strongly Gorenstein flat, then so is G by Lemma 2.1. Assembling the sequences $0 \to A \to P \to B \to 0$ and $0 \to B \to G \to M \to 0$, we obtain the exact sequence $0 \to A \to P \to G \to M \to 0$ of left R- modules with P projective and G strongly Gorenstein flat, as desired.

Next we shall prove the second desired sequence. Since G_0 is strongly Gorenstein flat, there exists an exact sequence $0 \to G_3 \to Q \to G_0 \to 0$ of left R modules with Q projective and G_3 strongly Gorenstein flat, Dually, one easily gets the desired result by taking pull-back diagram.

For a positive integer n, a module $A \in$ Mod R is called an n-syzygy module (of M)there exists an exact sequence $0 \to A \to P_{n-1} \to \Lambda \to P_1 \to P_0 \to M \to 0$ in Mod R with all P_i projective. Recall from [10], a module A is called Gorenstein n-syzygy module (of M) if there exists an exact sequence $0 \to A \to Q_{n-1} \to \Lambda \to Q_1 \to Q_0 \to M \to 0$ in Mod R with all Q_i Gorenstein projective.

Definition 2.4 For a positive integer n, a module $A \in$ Mod R is a strongly Gorenstein n-syzygy module (of M) if there exists an exact sequence $0 \to A \to X_{n-1} \to \Lambda \to X_1 \to X_0 \to M \to 0$ in Mod R with all X_i strongly Gorenstein flat.

The following theorem is the main result in this section.

Theorem 2.5 Let n be a positive integer. If $0 \to A \to G_{n-1} \to \Lambda \to G_1 \to G_0 \to M \to 0$ is an exact sequence of R-modules with all G_i strongly Gorenstein flat, then we have the following

(1) There exist exact sequences $0 \to A \to P_{n-1} \to \Lambda \to P_1 \to P_0 \to N \to 0$ and $0 \to M \to N \to G \to 0$ of R-modules with all P_i projective and G strongly Gorenstein flat. In particular, a left R-module is an n-syzygy if and only if it is a strongly Gorenstein n-syzygy.

(2) There exist exact sequences $0 \to A \to Q_{n-1} \to \Lambda \to Q_1 \to Q_0 \to H \to 0$ and \$0 $0 \to H \to B \to A \to 0$ of R-modules with all Q_i projective and H strongly Gorenstein flat.

Proof. (1) We assume that $n > 1$, and then proceed by induction on n. If $n = 1$, then 0 $0 \to A \to G_0 \to M \to 0$ is exact with G_0 strongly Gorenstein flat. Thus we have the exactness of $0 \to G_0 \to P_0 \to G \to 0$, where P_0 is projective and G is strongly Gorenstein flat.
The desired result follows from the pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & G_0 & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & P_0 & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now let $n > 2$ and let $0 \to A \to G_{n-1} \to \Lambda \to G_1 \to G_0 \to M \to 0$ be an exact sequence of R-modules with all G_i strongly Gorenstein flat. Set $W = \mathrm{Coker}(G_{n-1} \to G_{n-2})$, and we have the exact sequence $0 \to A \to G_{n-1} \to G_{n-2} \to W \to 0$. By Proposition 2.3, we get an exact sequence \$0 $0 \to A \to P_{n-1} \to T_{n-2} \to W \to 0$ with P_{n-1} projective and T_{n-2} strongly Gorensteinflat. Put A_1 $=\mathrm{Im}(P_{n-1} \to T_{n-2})$.Then we obtain the exactness of $0 \to A_1 \to T_{n-2} \to G_{n-3} \to \Lambda \to G_0 \to M \to 0$

Therefore, the assertion follows by the induction hypothesis.

(2) We suppose that $n > 1$, and then proceed by induction on n. If $n = 1$, then there exists an exact sequence $0 \to A \to G_0 \to M \to 0$ with G_0 strongly Gorenstein flat. Since G_0 is strongly Gorenstein flat, one gets the exactness of $0 \to H \to Q_0 \to G_0 \to 0$, where G_0 is projective and H is strongly Gorenstein flat . Now we consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L & \to & Q_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now suppose $n > 2$ and let $0 \to A \to G_{n-1} \to \Lambda \to G_1 \to G_0 \to M \to 0$ be exact with all G_i strongly Gorenstein flat. Put $X = \mathrm{Ker}\,(G_1 \to G_0)$, and we obtain the exactness of $0 \to X \to G_1 \to G_0 \to M \to 0$.The remainder proof is similar to (1), one easily checks the result holds by Proposition 2.3 and by induction.

Corollary 2.6 Let $M \in \mathrm{Mod}\ R$ and n a non-negative integer. If $\mathrm{SGfd}_R(M) = n$, then there exists an exact sequence $0 \to M \to N \to G \to 0$ in $\mathrm{Mod}\,R$ with $\mathrm{pd}_R(N) = n$ and G strongly Gorenstein flat.

Proof. Let $M \in \mathrm{Mod}\ R$ with $\mathrm{SGfd}_R(M) = n$. Then one uses Proposition 2.5(1) with $A = 0$

to get an exact sequence $0\to M\to N\to G\to 0$ in Mod R with $\mathrm{pd}_R(N)\le n$ and G strongly Gorenstein flat. By Lemma 2.2, $\mathrm{SGfd}_R(N)=n$\$, and thus $\mathrm{pd}_R(N)=n$.

We denote $\mathrm{S}^2\mathrm{GF}(R)$={ $A\in$ Mod R there exists a Hom(-, F) exact exact sequence $\Lambda\to G_1\to G_0\to G^0\to G^1\to\Lambda$ in Mod R with all G_i and G^i in $\mathrm{SGF}(R)$ and $A\cong\mathrm{Im}(G_0\to G^0)$.

The following result means that an iteration of the procedure used to define the strongly Gorenstein flat modules yields exactly the strongly Gorenstein flat modules.

Theorem 2.7 $\mathrm{S}^2\mathrm{GF}(R)=\mathrm{SGF}(R)$

Proof. It is clear that $\mathrm{SGF}(R)\subseteq \mathrm{S}^2\mathrm{GF}(R)$. In the following, we prove the converse inclusion.

Let $\Lambda\to G_1\to G_0\to G^0\to G^1\to\Lambda$ be a Hom(-, F) exact exact sequence in Mod R with all G_i and G^i in $\mathrm{SGF}(R)$ and $A\cong\mathrm{Im}(G_0\to G^0)$. Then $\mathrm{Ext}^i(A,$ F)=0 for any $i\ge 1$.

Put $A_i=\mathrm{Im}(G^i\to G^{i+1})$ for any $i\ge 0$. By Theorem 2.5, there exist exact sequences $0\to A\to P^0\to N^0\to 0$ and $0\to A^0\to N^0\to G\to 0$ in Mod R with P^0 is projective and G strongly Gorenstein flat such that the former one is Hom(-, F) exact. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A^0 & \to & N^0 & \to & G & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & G^1 & \to & G' & \to & G & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & A^1 & = & A^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By Lemma 2.1 and the exactness of the middle row in the above diagram, G' is strongly Gorenstein flat. Because the first column in the above diagram is Hom(-, F) exact exact, $\mathrm{Ext}^i(A^1,$ F)=0. It yields that the middle column in the above diagram is also Hom(-, F) exact exact, and so we get a Hom(-, F) exact exact sequence $0\to N^0\to G'\to G^2\to G^3\to\Lambda$ in Mod R. Then by the above argument, we have Hom(-, F) exact exact sequence $0\to N^0\to P^1\to N^1\to 0$ and $0\to N^1\to G''\to G^3\to G^4\to\Lambda$ in Mod R. We proceed in this manner to get a Hom(-, F) exact exact sequence $0\to A\to P^0\to P^1\to\Lambda$ 0 in Mod R with all P^i projective . Thus A is strongly Gorenstein flat by [14, Lemma 2.1], and therefore, $\mathrm{S}^2\mathrm{GF}(R)\subseteq\mathrm{SGF}(R)$. The proof is finished.

By [14, Theorem 2.8], we have that $\mathrm{SGfd}_R(M)\le n$ if and only if there exists an exact sequence which is defined as the smallest non-negative integer n such that there exists an exact sequence $0\to G_n\to P_{n-1}\to\Lambda\to P_1\to P_0\to M\to 0$ in Mod R with each P_i projecitve and G_n strongly Gorenstein flat. The following theorem generalizes this result.

Theorem 2.8 For any right R-module M and any integer $n\ge 0$, the following are equivalent:

(1)$\mathrm{SGfd}_R(M)\le n$;

(2) for every non-negative integer t with $0 \le t \le n$, there exists an exact sequence $0 \to X_{n-1} \to \Lambda \to X_1 \to X_0 \to M \to 0$ in Mod R such that X_t is strongly Gorenstein flat and X_i is projective for $i \ne t$.

Proof. (2) $\Rightarrow$ (1) It is obvious.

(1) $\Rightarrow$ (2) We proceed by induction on n . If SGfd $_R(M) \le 1$, then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in Mod R with G_0 and G_1 strongly Gorenstein flat. Note that G_0 is strongly Gorenstein flat, one gets a short exact sequence $0 \to G_2 \to P_0 \to G_0 \to 0$, where P_0 is projective and G_2 is strongly Gorenstein flat. We consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_2 & = & G_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G_1' & \to & P_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & G_1 & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since both G_1 and G_2 are strongly Gorenstein flat, G_1' is also strongly Gorenstein flat by Lemma 2.1. Thus we get the exact sequence $0 \to G_1' \to P_0 \to M \to 0$ with P_0 projective and G_1' strongly Gorenstein flat. Because G_1 is strongly Gorenstein flat, there exists an exact sequence $0 \to G_1 \to P_1 \to G_3 \to 0$ with P_1 projective and G_3 strongly Gorenstein flat. Consider the pushout of $G_1 \to G_0$ and $G_1 \to P_1$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G_1 & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & P_1 & \to & G_0' & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_3 & = & G_3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In the sequence $0 \to G_0 \to G_0' \to G_3 \to 0$, both G_0' and G_3 are strongly Gorenstein flat, hence so is G_0' by Lemma 2.1. Therefore, one has the exact sequence $0 \to P_1 \to G_0' \to M \to 0$ with P_1 projective and G_1' strongly Gorenstein flat.

Next we suppose $n \ge 2$. Then there is an exact sequence ,

$$0 \to G_n \to G_{n-1} \to \Lambda \to G_1 \to G_0 \to M \to 0 \qquad (*)$$

in Mod R where all G_i are strongly Gorenstein flat for $1 \le i \le n$. Put $L =$ Coker($G_3 \to G_2$), and we have the exact sequence $0 \to L \to G_1 \to G_0 \to M \to 0$. By Lemma 2.4, one gets the exactness of

$0 \to L \to G_1^{'} \to P_0 \to M \to 0$ with P_0 projective and $G_1^{'}$ strongly Gorenstein flat . Hence we obtain the following exact sequence of R -modules $0 \to G_n \to \Lambda \to G_2 \to G_1^{'} \to P_0 \to M \to 0$.

Set $N = \mathrm{Im}(G_1^{'} \to P_0)$, it is clear that $\mathrm{SGfd}_R(N) \le n-1$. By the induction hypothesis, there exists an exact sequence $0 \to X_{n-1} \to \Lambda \to X_t \to \Lambda \to X_1 \to P_0 \to M \to 0$ such that P_0 is projective and X_t is strongly Gorenstein flat and X_i is projective for $i \ne t$ and $1 \le t \le n$.

Now it remains to show (2) for the case $t = 0$. In the sequence $(*)$, put $N = \mathrm{Coker}(G_2 \to G_1)$. One gets the exactness of $0 \to G_n \to G_{n-1} \to \Lambda \to G_2 \to G_1 \to N \to 0$.

By the induction hypothesis, we have an exact sequence of R -modules $0 \to P_n \to P_{n-1} \to \Lambda \to P_2 \to G_1^{'} \to N \to 0$ with $G_1^{'}$ strongly Gorenstein flat and all P_i is projective for $2 \le i \le n$. Set $K = \mathrm{Coker}(P_3 \to P_2)$. The exactness of $0 \to K \to G_1^{'} \to G_0 \to M \to 0$ gives rise to an exact sequence $0 \to K \to P_1 \to G_0^{'} \to M \to 0$ with P_1 projective and $G_0^{'}$ strongly Gorenstein flat by Lemma 2.4. Therefore, we have the exact sequence $0 \to P_n \to \Lambda \to P_1 \to G_0^{'} \to M \to 0$ where $G_0^{'}$ is strongly Gorenstein flat and all P_i are projective for $1 \le i \le n$. This completes the proof.

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