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The Stability of the Triangular Points of the Restricted three Body Problem when both the Primaries are Triaxial Rigid Bodies

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Abstract: The location and the stability of the triangular points of the planar restricted three body problem have been discussed when both the primaries are triaxial rigid bodies considering the case of stationary rotational motion of the bigger primary and of the smaller primary are $\left(\theta_2=\varphi_2=\pi/2\ ,\psi_2=0\right)$ and $\left(\theta_1=\psi_1=\pi/2,\varphi_1=0\right)$ respectively.
Keywords: restricted three body problem; triangular points; triaxial rigid bodies.

INTRODUCTION

The problem of stability conditions of triangular libration points was assumed by Gascheau [2] and then by Routh[5].

In recent times many perturbing forces i.e., oblateness and radiation forces of the primaries, Coriolis and centrifugal forces etc., have been included in the study of the restricted three body problem. Bhatnagar and Gupta[1] show the existence of 36 stationary motions each corresponding to the constant values of the non-cyclic generalized coordinates and thus depending on the Eulerian angles of both the bodies. Khanna and Bhatnagar [3] have studied the problem when the smaller primary is a triaxial rigid body. Also Sharma et.al.[5] have studied the problem when both the primaries are triaxial rigid bodies in the case of stationary rotational motion $\left(\theta_i,\ \psi_i \text{ and } \varphi_i\right)$ are small quantities.

In this paper we consider the restricted three body problem when both the primaries are triaxial rigid bodies with the stationary rotational motion $\left(\theta_1=\psi_1=\pi/2,\varphi_1=0\right)$ of the bigger primary, and $\left(\theta_2=\varphi_2=\pi/2\ ,\psi_2=0\right)$ of the smaller primary.

EQUATIONS OF MOTION

We shall adopt the notation and terminology of Szebehly [7]. As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of masses of the primaries is also taken one. The unit of time is chosen so as to make the gravitational constant unity. Besides this the principle axes of the primaries are oriented to the synodic axes by Euler's angels $\left(\theta_i\ ,\ \psi_i\ ,\ \varphi_i\ ,\left(i=1,2\right)\right)$. Since the axes are supposed to rotate with the same angular velocity as that of the rigid bodies and the bodies are moving around their center of mass without rotation, the Euler's angles remain constant throughout the motion. Using dimensionless variables,

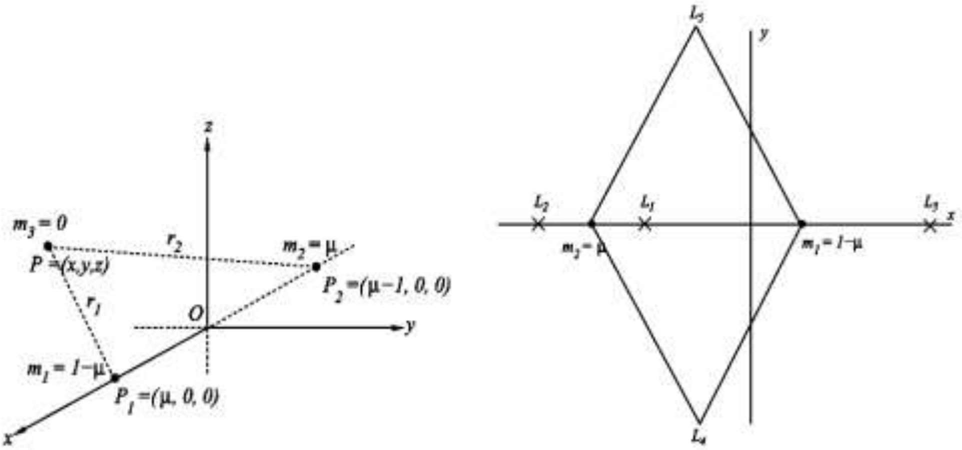

Fig-1: Left: the circular restricted three body problem in the synodical reference system with a dimensional units. Right: the five equilibrium points associated with the problem.

the equations of motion of the infinitesimal mass m_3 in a synodic coordinate system (x, y) are

$$\ddot{x} - 2n\dot{y} = \frac{\partial\Omega}{\partial x}$$

and

$$\ddot{y} + 2n\dot{x} = \frac{\partial\Omega}{\partial y} \tag{1}$$

where

$$\Omega = \frac{n^2}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)}{2m_1 r_1^3}\left[I_1 + I_2 + I_3 - 3I\right]$$
$$+ \frac{\mu}{2m_2 r_2^3}\left[I'_1 + I'_2 + I'_3 - 3I'\right] \tag{2}$$

$$r_1^2 = (x-\mu)^2 + y^2$$

and

$$r_2^2 = (x+1-\mu)^2 + y^2 \tag{3}$$

Here μ is the ratio of mass of the smaller primary to the total mass of primaries and $0 \le \mu \le 1/2$, i.e., $\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$ with $m_1 \ge m_2$ being the masses of the primaries.

I_1, I_2, I_3 are the principal moments of inertia of the triaxial rigid body of mass m_1 at its center of mass, with a, b, c as its axes. *I* is the moment of inertia about a line joining the center of the rigid body of mass m_1 and the infinitesimal body of mass m_3 and is given by

$$I = I_1 l_1'^2 + I_2 m_1'^2 + I_3 n_1'^2,$$

where l_1', m_1' *and* n_1' are the directional cosines of the line respect to its principal axes.

I'_1, I'_2, I'_3 are the principal moments of inertia of the triaxial rigid body of mass m_2 at its center of mass, with a', b', c' as its axes. I' is the moment of inertia about a line joining the center of the rigid body of mass m_2 and the infinitesimal body of mass m_3 and is given by

$$I' = I_1' \mathrm{l}_2'^2 + I_2' m_2'^2 + I_3' n_2'^2$$

where l'_2, m'_2 *and* n'_2 are the directional cosines of the line respect to its principal axes.

We denote the unit vectors along the principle axes at $p_1 \left(or\ \ p_2\right)$ by $\mathbf{i},\mathbf{j},\mathbf{k}$ and the unit vectors parallel to the synodic axes by $\mathbf{I},\mathbf{J},\mathbf{K}$ with the help of Euler's angles $\left(\theta_i\ ,\ \psi_i\ ,\ \varphi_i\ ,\left(i\ =1,2\right)\right)$. They are connected by Synge and Griffith[7] (1959),

$$\mathbf{I} = a_{1i}\,\mathbf{i} + b_{1i}\,\mathbf{j} + c_{1i}\,\mathbf{k}$$

$$\mathbf{J} = a_{2i}\,\mathbf{i} + b_{2i}\,\mathbf{j} + c_{2i}\,\mathbf{k}$$

and

$$\mathbf{K} = a_{3i}\,\mathbf{i} + b_{3i}\,\mathbf{j} + c_{3i}\,\mathbf{k},$$

$$\left(i\ =1,2\right),$$

where

$$a_{1i} = -Sin\ \varphi_i\ \ Sin\,\psi_i\ + Cos\ \theta_i\ \ Cos\,\varphi_i\ \ Cos\,\psi_i\ ,$$

$$a_{2i} = Cos\ \varphi_i\ \ Sin\,\psi_i\ + Cos\ \theta_i\ \ Sin\,\varphi_i\ \ Cos\,\psi_i\ ,$$

$$a_{3i} = -Sin\ \theta_i\ \ Cos\ \psi_i\ ,$$

$$b_{1i} = -Sin\ \varphi_i\ \ Cos\ \psi_i\ - Cos\ \theta_i\ \ Cos\,\varphi_i\ \ Sin\,\psi_i\ ,$$

$$b_{2i} = Cos\ \varphi_i\ \ Cos\ \psi_i\ - Cos\ \theta_i\ \ Sin\,\varphi_i\ \ Sin\,\psi_i\ ,$$

$$b_{3i} = Sin\ \theta_i\ \ Sin\,\psi_i\ ,$$

$$c_{1i} = Sin\ \theta_i\ \ Cos\ \varphi_i\ ,$$

$$c_{2i} = Sin\ \theta_i\ \ Sin\ \varphi_i\ ,$$

and

$$c_{3i} = Cos\ \theta_i\ \ ,\ \left(i\ =1,2\right).$$

The axes $O\left(xyz\right)$ have been defined by Szebehely [7]. Now, Ω in equation (2) can be written as

$$\Omega = \left(1-\mu\right)\left(\frac{1}{r_1}+\frac{n^2 r_1^2}{2}\right)+\mu\left(\frac{1}{r_2}+\frac{n^2 r_2^2}{2}\right)$$
$$+\frac{\left(1-\mu\right)}{2r_1^3}\left[2\left(A_1+A_2+A_3\right)-3\frac{1}{r_1^2}\left\{\begin{array}{l}\left(A_2+A_3\right)\left(a_{11}\left(x-\mu\right)+a_{21}y\right)^2\\+\left(A_1+A_3\right)\left(b_{11}\left(x-\mu\right)+b_{21}y\right)^2+\\\left(A_2+A_1\right)\left(c_{11}\left(x-\mu\right)+c_{21}y\right)^2\end{array}\right\}\right] \qquad (4)$$
$$+\frac{\mu}{2r_2^3}\left[2\left(A'_1+A'_2+A'_3\right)-3\frac{1}{r_2^2}\left\{\begin{array}{l}\left(A'_2+A'_3\right)\left(a_{12}\left(x+1-\mu\right)+a_{22}y\right)^2\\+\left(A'_1+A'_3\right)\left(b_{12}\left(x+1-\mu\right)+b_{22}y\right)^2\\+\left(A'_2+A'_1\right)\left(c_{12}\left(x+1-\mu\right)+c_{22}y\right)^2\end{array}\right\}\right],$$

where

$$A_1 = \frac{a^2}{5R^2}, A_2 = \frac{b^2}{5R^2}, A_3 = \frac{c^2}{5R^2}$$

$$A'_1 = \frac{a'^2}{5R^2}, A'_2 = \frac{b'^2}{5R^2}, A'_3 = \frac{c'^2}{5R^2} \tag{5}$$

and R is the distance between the primaries. The mean motion, n is given by

$$\begin{aligned} n^2 &= 1 + \frac{3}{2}\left[2\left(A_1 + A_2 + A_3\right) - 3a_{11}^2\left(A_2 + A_3\right) - 3b_{11}^2\left(A_1 + A_3\right) - 3c_{11}^2\left(A_2 + A_1\right)\right] \\ &+ \frac{3}{2}\left[2\left(A'_1 + A'_2 + A'_3\right) - 3a_{12}^2\left(A'_2 + A'_3\right) - 3b_{12}^2\left(A'_1 + A'_3\right) - 3c_{12}^2\left(A'_2 + A'_1\right)\right]. \end{aligned} \tag{6}$$

Equation (1) permit an integral analogous to Jacobi integral

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

The liberation points are the singularities of the manifold

$$f\left(x, y, \dot{x}, \dot{y}\right) = \dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

Therefore, these points are the solutions of the equations

$$\Omega_x = 0\ ,\ \Omega_y = 0\,.$$

We have Ω_x *and* Ω_y are established by Sharma [4]. Let $\left(\theta_1 = \psi_1 = \pi/2, \varphi_1 = 0\right)$ of the bigger primary, in this case $a_{21} = b_{31} = c_{11} = 1$ and the other elements are equal to zero; $\left(\theta_2 = \varphi_2 = \pi/2\ , \psi_2 = 0\right)$ of the smaller primary $a_{32} = b_{12} = -1, c_{22} = 1$ and the other elements are equal to zero,

$$\begin{aligned} \Omega_x &= \left(1-\mu\right)\left(\frac{-1}{r_1^2} + n^2 r_1\right)\frac{\left(x-\mu\right)}{r_1} + \mu\left(\frac{-1}{r_2^2} + n^2 r_2\right)\frac{\left(x+1-\mu\right)}{r_2} \\ &- \frac{3\left(1-\mu\right)\left(x-\mu\right)}{2r_1^5}\left[\left(4A_1 + 4A_2 + 2A_3\right) - \frac{5}{r_1^2}\left(\left(A_1 + A_2\right)\left(x-\mu\right)^2 + \left(A_2 + A_3\right)y^2\right)\right] \\ &- \frac{3\left(\mu\right)\left(x+1-\mu\right)}{2r_2^5}\left[\left(4A'_1 + 4A'_3 + 2A'_2\right) - \frac{5}{r_2^2}\left(\left(A'_1 + A'_3\right)\left(x+1-\mu\right)^2 + \left(A'_1 + A'_2\right)y^2\right)\right] = 0, \\ \Omega_y &= \left(1-\mu\right)\left(\frac{-1}{r_1^2} + n^2 r_1\right)\frac{y}{r_1} + \mu\left(\frac{-1}{r_2^2} + n^2 r_2\right)\frac{y}{r_2} \\ &- \frac{3\left(1-\mu\right)y}{2r_1^5}\left[\left(2A_1 + 4A_2 + 4A_3\right) - \frac{5}{r_1^2}\left(\left(A_1 + A_2\right)\left(x-\mu\right)^2 + \left(A_2 + A_3\right)y^2\right)\right] \\ &- \frac{3\left(\mu\right)y}{2r_2^5}\left[\left(4A'_1 + 4A'_2 + 2A'_3\right) - \frac{5}{r_2^2}\left(\left(A'_1 + A'_3\right)\left(x+1-\mu\right)^2 + \left(A'_1 + A'_2\right)y^2\right)\right] = 0, \end{aligned} \tag{7}$$

where

$$n^2 = 1 + \frac{3}{2}\left[\left(2A_3 - A_2 - A_1\right)\right] + \frac{3}{2}\left[\left(2A'_2 - A'_1 - A'_3\right)\right].$$

TRIANGULAR LIBERATION POINTS

The triangular liberation points are the solutions of the equation (7).

If the values of A_i, A'_i $\left(i = 1,2,3\right)$ are equal to zero we simply get $r_1 = r_2 = 1$. When A_i, A'_i $\left(i = 1,2,3\right)$ aren't equal to zero we suppose that

$r_1 = 1 + \alpha$ *and* $r_2 = 1 + \beta$ where $\alpha, \beta = 1$. (8)

Putting the values of r_1 *and* r_2 from equation (8) in equation (3), we get

$$x = \mu - \frac{1}{2} + \beta - \alpha$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}(\beta+\alpha)\right] \tag{9}$$

Putting the values of r_1 *and* r_2 from equation (8) and *x*, *y* from equation (7), rejecting higher order terms, we get

$$\alpha = \frac{1}{8}(9A_1 - 2A_2 - 7A_3 + 4A_1') - \frac{(1-9\mu)}{16(1-\mu)}A_2' - \frac{(7+\mu)}{16(1-\mu)}A_3'$$

and

$$\beta = \frac{(1+\mu)}{4\mu}A_2 - \frac{(1-3\mu)}{4\mu}A_1 - A_3\frac{(15+7\mu)}{16\mu}(A_3' - A_2') \tag{10}$$

STABILITY ANALYSIS

Assuming ξ *and* η denote small displacement of the infinitesimal particle from the equilibrium points.

$$X = X_o + \xi,\ Y = Y_o + \eta \tag{11}$$

Now

$$\Omega_x = \Omega_x(x,y) = \Omega_x(x_0+\xi, y_0+\eta)$$

Expanding by taylor's expansion and considering only first orders, we have

$$\begin{aligned} \Omega_x &= \Omega_x^0(x,y) + \xi\Omega_{xx}^0 + \eta\Omega_{xy}^0 \\ \Omega_y &= \Omega_y^0(x,y) + \xi\Omega_{yx}^0 + \eta\Omega_{yy}^0 \end{aligned} \tag{12}$$

Where Ω_x^0 is the value of Ω_x at the point (x_0, y_0) and similarly the other values $\Omega_{xx}^0, \Omega_{xy}^0, \Omega_y^0, \Omega_{yx}^0$ and Ω_{yy}^0 are the respective values at the points (x_0, y_0).

At the equilibrium points we have

$$\Omega_x^0 = \Omega_y^0 = 0$$

Hence the equation of motion of the infinitesimal particle is

$$\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} &= \xi\Omega_{xx}^0 + \eta\Omega_{xy}^0, \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} &= \xi\Omega_{yx}^0 + \eta\Omega_{yy}^0. \end{aligned} \tag{13}$$

In order to solve equation (16) substitute

$$\xi = Ae^{\lambda t}, \quad \eta = Be^{\lambda t} \tag{14}$$

Where *A*,*B* and λ are parameters. This gives that

$$\begin{aligned} A\left(\lambda^2 - \Omega_{xx}^0\right)e^{\lambda t} + B\left(-2n\lambda - \Omega_{xy}^0\right)e^{\lambda t} &= 0, \\ A\left(2n\lambda - \Omega_{xy}^0\right)e^{\lambda t} + B\left(\lambda^2 - \Omega_{yy}^0\right)e^{\lambda t} &= 0. \end{aligned} \tag{15}$$

The set of equation (15) has nontrivial solution if

$$\lambda^4 + \left(4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0\right)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2 = 0 \tag{16}$$

Where Ω_{xx}^o, Ω_{yy}^o *and* Ω_{xy}^o is defined at L_4 when the primaries are triaxial rigid bodies as

$$\begin{aligned} \Omega_{xx}^o = n^2 + (1-\mu)\left(\frac{-1}{4} + \frac{9}{4}\alpha - 3\beta\right) + \mu\left(\frac{-1}{4} + \frac{9}{4}\beta - 3\alpha\right) + \frac{3(1-\mu)}{32}(41A_1 - 4A_2 - 37A_3) \\ + \frac{3\mu}{32}(41A_3' - 4A_1' - 37A_2') \end{aligned} \tag{17}$$

$$\Omega_{yy}^{o}=n^{2}+(1-\mu)\left(\frac{5}{4}-\frac{21\alpha}{4}+3\beta\right)+\mu\left(\frac{5}{4}+3\alpha-\frac{21\beta}{4}\right)+\frac{3}{32}(1-\mu)(-27A_{1}+268A_{2}+143A_{3})$$
$$+\frac{3\mu}{32}(268AA_{1}+143AA_{2}-27AA_{3}), \quad (18)$$

and

$$\Omega_{xy}^{o}=\frac{\sqrt{3}}{4}(1-\mu)(-3+7\alpha+4\beta)-\frac{\sqrt{3}}{4}\mu(-3+4\alpha+7\beta)-\frac{15\sqrt{3}}{16}(1-\mu)(3A_{1}-8A_{2}-7A_{3})$$
$$-\frac{15\sqrt{3}}{16}\mu(8A_{1}'+7A_{2}'-3A_{3}') \quad (19)$$

We can rewrite equation (16) as

$$\Lambda^{2}+P\Lambda+Q=0 \quad (20)$$

Where

$$P=\left(4n^{2}-\Omega_{xx}^{0}-\Omega_{yy}^{0}\right),$$

$$Q=\Omega_{xx}^{0}\Omega_{yy}^{0}-\left(\Omega_{xy}^{0}\right)^{2},$$

and

$$\Lambda=\lambda^{2}$$

The stability of the triangular points requires that $\Lambda=\lambda^{2}$ must be negative to obtain pure imaginary roots, i.e. the discrinnant of equation (17) is $P^{2}-4Q<0$ that is the condition for stability implies that:

$$-(1-27\mu+27\mu^{2})+\frac{3}{8}(48-265\mu+225\mu^{2})A_{1}+\frac{3}{4}(293-649\mu+360\mu^{2})A_{2}$$
$$+\frac{3}{8}(386-537\mu+135\mu^{2})A_{3}+\frac{3}{2}(2-52\mu+213\mu^{2})A_{1}'$$
$$+\frac{3}{16}(-207-265\mu+1788\mu^{2})A_{2}'-\frac{3}{16}(479-983\mu+276\mu^{2})A_{3}'<0 \quad (21)$$

If A_{i}, A_{i}' (i = 1, 2, 3) are equal to zero, then the stability condition is $\mu_{0}<0.0385208965$ Szebehly [7]. And if A_{i}, A_{i}' (i = 1, 2, 3) are not equal to zero, we suppose that $\mu_{crit.}=\mu_{0}+p_{1}A_{1}+p_{2}A_{2}+p_{3}A_{3}+p_{4}A_{1}'+p_{5}A_{2}'+p_{6}A_{3}'$ where $p_{1},p_{2},p_{3},p_{4},p_{5}$ and p_{6} are to be determined therefore we have

$$p_{1}=\frac{48-265\mu_{0}+225\mu_{0}^{2}}{72(-1+2\mu_{0})}, \quad p_{2}=\frac{293-649\mu_{0}+360\mu_{0}^{2}}{36(-1+2\mu_{0})}$$

$$p_{3}=\frac{386-537\mu_{0}+135\mu_{0}^{2}}{72(-1+2\mu_{0})}, \quad p_{4}=\frac{2-52\mu_{0}+213\mu_{0}^{2}}{18(-1+2\mu_{0})}$$

$$p_{5}=\frac{-207-265\mu_{0}+1788\mu_{0}^{2}}{144(-1+2\mu_{0})}, \quad p_{6}=\frac{-479+983\mu_{0}-276\mu_{0}^{2}}{144(-1+2\mu_{0})}$$

And the stability condition for the triangular points is

$$\mu_{crit.}<0.0385208965+0.5737263333A_{1}+8.0819275078A_{2}+5.5003483611A_{3}$$
$$+0.0188389082A_{1}'-1.61433589314A_{2}'-3.3222244623A_{3}' \quad (22)$$

CONCLUSION

We construct the location and the stability condition of the triangular points of the restricted three body problem with triaxial primaries considering a stationary rotational motion $\left(\theta_1 = \psi_1 = \pi/2, \varphi_1 = 0\right)$ of the bigger primary and $\left(\theta_2 = \varphi_2 = \pi/2, \psi_2 = 0\right)$ of the smaller primary and we conclude that the stability condition is depend on that orientations. Also we found the stability condition in our particular case.

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