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On the Existence of Solutions to BSDES under Sublinear Grown Condition

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Abstract: In this paper, we study one-dimensional BSDE's whose coefficient gis sublinear growth in z. We obtain a general existence result and a comparison theorem when g is linear growth in y and sublinear growth in z. Some known results are extended and generalized.
Keywords: one-dimensional BSDE, sublinear growth

1. Introduction

In this paper, we consider the following one-dimensional backward stochastic differential equation(BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dB_s ,t \in [0,T], \ (1.1)$$

Where ξ is a random variable called the terminal condition, the random function $g(w,t,y,z):\Omega\times[0,T]\times R\times R^d \to R$ is progressively measurable for each (y,z), called the generator of the BSDE(1.1), and B is a d-dimensional Brownian motion. The solution $(y.,z.)$ is a pair of adapted processes. The triple (ξ,T,g) is called the parameters of the BSDE(1.1).

It is by now well known that under standard assumptions where g is of linear growth and Lipschitz continuous with respect to (y,z), for any random variable $\xi \in L^2(\Omega, F_T, P; R^n)$,the BSDE(1) has a unique square integrable, adapted solution(see Pardoux and Peng [1]). Since then, there are many works attempting to relax the Lipschitz condition for getting the existence and uniqueness of solution, for instance Lepeltier and Martin [2], Bahlali [3], Kobylanski [4], Lepeltier and Martin [5], Briand and Hu [6], Briand et al. [7], and Fan and Liu [8] etc. In particular, in the case where $(g(t,0,0))_{t\in[0,T]}$ is a bounded process, and g is continuousand of linear growth in (y,z), Lepeltier and Martin [2] proved that there is atleast one solution to the BSDE(1.1). Furthermore, under the conditions that gis monotonic in y, has at most quadratic growth in z and ξ is bounded, Briand etal. [7] obtained some existence results on the solution to the BSDE(1.1).

Enlightened by these results, this paper generalizes the result in Lepeltier and Martin [2] and Kobylanski [4]. We prove that if g is linear growth in y and sublinear growth in z, and $(g(t,0,0))_{t\in[0,T]}$ is integrable, then for each integrable terminal condition ξ. the BSDE(1.1) has at least one solution. The content of this paper may be regarded as an extension and generalization of the corresponding result in Lepeltier and Martin [2], Kobylanski [4].

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminaries and assumptions. In Section 3, we obtain an existence theorem and a comparison theorem of solution for BSDE(1.1) when g is linear growth in y and sublinear growth in z.

2. Preliminaries

Let (Ω,F,P) be a complete probability space with a d-dimensional standard Brownian motion $\{B_t\}_{t\ge 0}$. The filtration $F = \{F_s, 0 \le s \le T\}$ is generated by $\{B_s\}_{0\le s\le T}$ and augmented by all P-null sets, i.e.,

$$F_s = \sigma\{B_r, r \le s\} \vee N_p\ , \ s \in [0,T],$$

where N_p is the set of all $P-null$ subsets and $T>0$ is a fixed real time horizon. For every positive integer n, we use $|\cdot|$ to denote the norm of Euclidean space R^n. For each real $p>0$, δ^p denotes the set of real-valued, adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\delta^p} := \left(E\left[\sup_{t\in[0,T]} |Y_t|^p\right]\right)^{1\wedge 1/p} < +\infty .$$

If $p\geq 1, \|\cdot\|_{\delta^p}$ is a norm on δ^p and if $p\in(0,1)$, $(X,X')\to\|X-X'\|_{\delta^p}$ defines adistance on δ^p. Under this metric, δ^p is complete. Moreover, let M^p denote the set of (equivalent classes of) (F_t)-progressively measurable, R^n-valued processes $\{Z_t, t\in[0,T]\}$ such that

$$\|Z\|_{M^p} = \left\{E\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right]\right\}^{1\wedge 1/p} < +\infty .$$

For $p\geq 1$, M^p is a Banach space endowed with this norm and for $p\in(0,1)$), M^p is a complete metric space with the resulting distance. We set $\delta = U_{p>1}\delta^p$ and letus recall that a continuous process $(Y_t)_{t\in[0,T]}$ belongs to the class (D) if the family{ Y_τ : τ is stopping time bounded by T} is uniformly integrable. For a process Y in the class (D), we put

$$\|Y\|_1 = \sup\{E[Y_\tau], \tau \text{ is } \textit{stopping time bounded by} T\}.$$

The space of (F_t)-progressively measurable continuous processes which belong to theclass (D) is complete under this norm.

Now, let terminal condition ξ is F_t -measurable and satisfies $E|\xi|<+\infty$, g be the(F_t)-progressively measurable generator of the BSDE(1). In this paper, by a solution to the BSDE(1) we mean a pair of (F_t)-adapted processes $(y.,z.)$ with values in $R\times R^d$ such that $dP-a.s., t\to y_t$ is continuous, $t\to z_t$ belongs to $L^2(0,L)$, $t\to g(t,y_t,z_t)$ belongs to $L^1(0,L)$ and $dP-a.s.,$ the BSDE(1) holds true for each $t\in[0,T]$.
The generator g of BSDE(1) is a random function g $g(w,t,y,z):\Omega\times[0,T]\times R\times R^d \to R$ which is progressively measurable for each (y, z) and satisfies the following assumptions:

(H1) $E\left[|\xi|+\int_0^T |g(s,0,0)|ds\right]<+\infty$;

(H2) There exist two constants $\mu>0, 0<\alpha<1$ such that $dP\times dt-a.s.$,
$\forall y_1,y_2,z_1,z_2, |g(w,t,y_1,z_1)-g(w,t,y_2,z_2)|\leq\mu|y_1-y_2|+\mu|z_1-z_2|^\alpha$.

The following result on BSDE(1) is referred to Fan and Liu [8].

Lemma 2.1. Under the assumptions (H1) and (H2), the BSDE(1.1) has a unique solution $(y.,z.)$ such that $y.$ is of class (D) and $z.\in M^\beta$ for some $\beta>\alpha$. Moreover, $(y.,z.)$ belongs to $\delta^\beta\times M^\beta$ for all $\beta\in(0,1)$.

3. The linear increasing case in y

First, we obtain a generalized comparison theorem of BSDE(1.1) which plays an important role in this paper.

Theorem3.1. Let g and g' be two generators of BSDEs and let $(y.,z.)$ and $(y'.,z'.)$ be respectively a solution for the BSDEs with parameters (ξ,T,g) and (ξ',T,g') such that both $y.$ and $y'.$ are of class (D), and both $z.$ and $z'.$belong to M^β for some $\beta>\alpha$. If $dP-a.s.,$ $\xi<\xi'$, g satisfies (H2) with $\alpha\in(0,1]$ and

$dP\times dt-a.s.$ $g(t,y_t',z_t')<g'(t,y_t',z_t')$ (resp. g' satisfies (H2) with $\alpha\in(0,1]$ and $dP\times dt-a.s.$, $g(t,y_t,z_t)<g'(t,y_t,z_t)$), then for each $t\in[0,T]$, we have

$$dP-a.s.,\ y_t\le y_t'.$$

Proof. We only prove the first case, the other case can be proved similarly. Let us fix $n\in N$ and denote τ_n the stopping time

$$\tau_n=\inf\left\{t\in[0,T]:\int_0^T|z_s|^2+|z_s'|^2\,ds\ge n\right\}\wedge T.$$

Tanaka's formula leads to the equation, setting $\hat{y}_t=y_t-y_t'$, $\hat{z}_t=z_t-z_t'$,

$$e^{\mu(t\wedge\tau_n)}\hat{y}^+_{t\wedge\tau_n}\le e^{\mu\tau_n}\hat{y}^+_{\tau_n}-\int_{t\wedge\tau_n}^{\tau_n}e^{\mu_s}I_{\hat{y}_s>0}\hat{z}_s\cdot dB_s+\int_{t\wedge\tau_n}^{\tau_n}e^{\mu_s}\{I_{\hat{y}_s>0}[g(s,y_s,z_s)-g'(s,y_s',z_s')]-\mu\hat{y}_s^+\}ds$$

Since $g(s,y_s,z_s)-g'(s,y_s',z_s')$ is non-positive, we have

$$g(s,y_s,z_s)-g'(s,y_s',z_s')=g(s,y_s,z_s)-g(s,y_s',z_s)+g(s,y_s',z_s)-g'(s,y_s',z_s').$$

and we deduce, using the assumptions (H2) of g, that

$$I_{\hat{y}_s>0}[g(s,y_s,z_s)-g'(s,y_s',z_s')]\le\mu\hat{y}_s^++\mu I_{\hat{y}_s>0}|z_s|^\alpha.$$

Thus, we get that for each $t\in[0,T]$,

$$e^{\mu(t\wedge\tau_n)}\hat{y}^+_{t\wedge\tau_n}\le e^{\mu\tau_n}\hat{y}^+_{\tau_n}-\int_{t\wedge\tau_n}^{\tau_n}e^{\mu_s}I_{\hat{y}_s>0}\hat{z}_s\cdot dB_s+\int_{t\wedge\tau_n}^{\tau_n}\mu e^{\mu_s}I_{\hat{y}_s>0}|z_s|^\alpha\,ds.\quad(3.1)$$

Note that $\hat{y}_.$ is of the class (D) and $\hat{z}_.$ belongs to M^β for some $\beta>\alpha$. By taking the conditional expectation with respect to F_t for two sides of inequality (3.1), sending into infinity and then using Jensens inequality, Doobs inequality and $Hölders$ inequality, we can deduce that $\hat{y}^+_.\in\delta$.

Furthermore, since $|x|^\alpha\le m|x|+1/m^\alpha$ for each $m\ge1$, by (3.1) we get that for each $m\ge1$,

$$\begin{aligned}e^{\mu(t\wedge\tau_n)}\hat{y}^+_{t\wedge\tau_n}&\le e^{\mu\tau_n}\hat{y}^+_{\tau_n}-\int_{t\wedge\tau_n}^{\tau_n}e^{\mu_s}I_{\hat{y}_s>0}\hat{z}_s\cdot dB_s+\int_{t\wedge\tau_n}^{\tau_n}\mu e^{\mu_s}I_{\hat{y}_s>0}(m|\hat{z}_s|+\frac{1}{m^\alpha})ds\\&\le e^{\mu\tau_n}\hat{y}^+_{\tau_n}+Te^{\mu T}\frac{\mu}{m^\alpha}-\int_{t\wedge\tau_n}^{\tau_n}e^{\mu_s}I_{\hat{y}_s>0}\hat{z}_s\cdot\left[-\frac{m\mu\hat{z}_s}{|\hat{z}_s|}I_{|\hat{z}_s|\ne0}ds+dB_s\right]\cdot\quad(3.2)\end{aligned}$$

Let P_m be the probability on (Ω,F) which is equivalent to P and defined by

$$\frac{dP_m}{dP}:=\exp\left\{m\mu\int_0^T\frac{\hat{z}_s}{|\hat{z}_s|}I_{|\hat{z}_s|\ne0}\cdot dB_s-\frac{1}{2}m^2\mu^2\int_0^T I_{|\hat{z}_s|\ne0}ds\right\}.$$

By taking the conditional expectation with respect to F_t under P_m for the two sides of the previous inequality, using Girsanovs theorem and then sending n to infinity, in view of $\hat{y}^+_.\in\delta$ and $\xi\le\xi'$, we can deduce that for each $t\in[0,T]$ and $m\ge1$, $e^{\mu t}\hat{y}_t^+\le Te^{\mu T}\mu/m$. Then letting $m\to\infty$ yields that $dP-a.s.$, $y_t\le y_t'$. The proof is complete.

Remark 3.1. Obviously, Theorem 3.1 generalize the Proposition 1 in Fan and Liu [9].

Let us now consider the generator $g(\omega,t,y,z):\Omega\times[0,T]\times R\times R^d\to R$ which isprogressively measurable for each (y, z) and satisfies the following assumptions:

(A1) For all (ω,t) the $g(\omega,t,\cdot,\cdot)$ is continuous;

(A2) There exist two constants $C\ge0$ and $0<\alpha<1$ such that $dP\times dt-a.s.$

$$\forall y,z,\ |g(\omega,t,y,z)| \le C(1+|y|+|z|^{\alpha}).$$

Before we prove our main results in this section we introduce a technical lemma.

Lemma 3.1. Assume that the generator g of the BSDE(1.1) satisfies (A1) and (A2). Then the sequence of functions

$$g_n(\omega,t,y,z) := \sup_{(u,v)\in R^{1+d}} \{g(\omega,t,u,v) - n(|y-u| + |z-v|^{\alpha})\}$$

is well defined for $n \ge C$ and it satisfies

(i) Linear growth in y and sublinear growth in z : $|g_n(\omega,t,y,z)| \le C(1+|y|+|z|^{\alpha})$;

(ii) Monotonicity in n: $\forall y,z, g_n(\omega,t,y,z)\downarrow$;

(iii) Lipschitz in y and $Hölders$ continuous in z: $\forall y,y',z,z'$,

$$|g_n(\omega,t,y,z) - g_n(\omega,t,y',z')| \le n(|y-y'| + |z-z'|^{\alpha});$$

(iv)Strong convergence: if $(y_n,z_n)\to(y,z), n\to\infty$, then

$$g_n(\omega,t,y_n,z_n)\to g(\omega,t,y,z), n\to\infty.$$

Proof. It is easy to see that, due to the assumption (A2) on g, g_n is well defined when $n \ge C$. And it's obvious that $g_n \ge g \ge -C(1+|y|+|z|^{\alpha})$. For $n \ge C$, we have from the assumption (A2)

$$g_n(\omega,t,y,z) \le \sup_{(u,v)\in R^{1+d}} \{C(1+|u|+|v|^{\alpha}) - C(|y-u|+|z-v|^{\alpha})\}.$$

For $\forall x,y, 0<\alpha\le 1$, using the fact that $|x+y|^{\alpha} \le |x|^{\alpha} + |y|^{\alpha}$, we have

$$g_n(\omega,t,y,z) \le \sup_{(u,v)\in R^{1+d}} \{C(1+|y|+|z|^{\alpha})\} = C(1+|y|+|z|^{\alpha}).$$

From above (i)holds.

(ii) is obvious.

Take $\varepsilon > 0$ and consider $(u_\varepsilon, v_\varepsilon)\in R^{1+d}$ such that

$$\begin{aligned} g_n(\omega,t,y,z) &< g(\omega,t,u_\varepsilon,v_\varepsilon) - n(|y-u_\varepsilon| + |z-v_\varepsilon|^{\alpha}) + \varepsilon \\ &= g(\omega,t,u_\varepsilon,v_\varepsilon) - n(|y'-u_\varepsilon| + |z'-v_\varepsilon|) + n(|y'-u_\varepsilon| + |z'-v_\varepsilon|) - n(|y-u_\varepsilon| + |z-v_\varepsilon|^{\alpha}) + \varepsilon \\ &\le g_n(\omega,t,y',z') + n(|y-y'| + |z-z'|^{\alpha}) + \varepsilon. \end{aligned}$$

Theorem, interchanging the place of (y,z)) and (y',z'), and because ε is arbitrary we deduce that (iii) holds.

Assume $(y_n,z_n)\to(y,z), n\to\infty$. For every n, we take $(u_n,v_n)\in R^{1+d}$ such that

$$g(\omega,t,y_n,z_n) \le g_n(\omega,t,y_n,z_n) \le g(\omega,t,u_n,v_n) - n(|y_n-u_n| + |z_n-v_n|^{\alpha}) + 1/n. \quad (3.3)$$

Since $g(\omega,t,y_n,z_n)$ is bounded, we get that (u_n) and (v_n) are bounded. Since gsatisfies (A2), $g(\omega,t,u_n,v_n)$is also bounded. Therefore, $\limsup_{n\to\infty} n|y_n-u_n| < \infty$, $\limsup_{n\to\infty} n|z_n-v_n| < \infty$, then we can get $u_n\to y, v_n\to z, n\to\infty$. Thus, from (3.3) we deduce that (iv) holds. The proof is complete.

We now give the following existence theorem for BSDE (1.1), which generalizes the corresponding result in Lepeltier and Martin [2] and Kobylanski [4].

Theorem 3.2. Assume that g satisfies the assumptions (A1) and (A2). Then, if $E|\xi| < \infty$, BSDE (1.1) has a solution $(y.,z.)$ such that $y.$ is of class (D) and $z. \in M^{\beta}$ for some $\beta > \alpha$. Moreover, there is a maximal solution $(\bar{y}.,\bar{z}.)$ of BSDE (1.1) insense that, for any other solution $(y.,z.)$ of Eq. (1.1), we have $y. \le \bar{y}.$.

Proof. Let g_n be defined as in Lemma 3.1, and also consider $h(\omega,t,y,z)=-C(f_t(\omega)+|y|+|z|^{\alpha})$, where C and $f_t(\omega)$ are taken from (A2). Then, it is easy to check that g_n and h are progressively measurable functions, satisfying (H1) and (H2). So, we get from Lemma 2.1 that, for $n\geq C$, the following BSDEs have a unique adapted solution $(y_\cdot^n, z_\cdot^n)$ and $(U_\cdot, V_\cdot)$ in $\delta^\beta\times M^\beta$, respectively:

$$y_t^n=\xi+\int_t^T g_n(s,y_s^n,z_s^n)ds-\int_t^T z_s^n\cdot dB_s\ ,t\in[0,T];$$

$$U_t=\xi+\int_t^T h(s,U_s,V_s)ds-\int_t^T V_s\cdot dB_s\ ,\ t\in[0,T].\ (3.4)$$

From the comparison theorem (Theorem 3.1) we obtain that

$$\forall n\geq m\geq C\ ,\ \forall n\geq m\geq C, y_\cdot^m\geq y_\cdot^n\geq U, a.s., t\in[0,T].\ (3.5)$$

Thus, since for each $n\geq 1$, $y_\cdot^n$ belongs to the class (D) and the space δ^β for each $\beta\in(0,1)$, there exists a process $y_\cdot$ which belongs also to the class (D) and the space δ^β for each $\beta\in(0,1)$ such that $\lim_{n\to\infty}\left\|y_t^n-y_t\right\|_1=0$ and

$$\forall\beta\in(0,1),\lim_{n\to\infty}E\left[\sup_{t\in[0,T]}\left|y_t^n-y_t\right|^\beta\right]=0.$$

Obviously, from $(U_\cdot, V_\cdot)$ in $\delta^\beta\times M^\beta$, there is a constant B depending on C,T,α ,and $E|\xi|$ such that $\|U\|<B$ and $\|V\|<B$. From (3.5), we have $\sup_{n\geq C}\left\|y^n\right\|\leq B$.Applying $It\hat{o}$'s formula to $(y_t^n)^2$, we have

$$\left|y_0^n\right|^2+\int_0^{\tau_k'}\left|z_s^n\right|^2 ds=\left|y_{\tau_k'}^n\right|^2+2\int_0^{\tau_k'}y_s^n g_n(s,y_s^n,z_s^n)ds-2\int_0^{\tau_k'}y_s^n z_s^n\cdot dB_s.\cdot$$

Therefore, from the (i) in Lemma 3.1, we have

$$\begin{aligned}\int_0^{T_k'}|z_s^n|^2\,ds&\leq|y_{T_k'}^n|^2+2C\int_0^{T_k'}|y_s^n|(1+|y_s^n|+|z_s^n|^\alpha)\,ds-2\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\\&\leq|y_{T_k'}^n|^2+2C\int_0^{T_k'}|y_s^n|(1+|y_s^n|+|z_s^n|+1)\,ds+2\left|\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\right|\\&\leq\frac{2TC}{\lambda^2}+(2CT+2C\lambda^2T+1)\sup_{s\in[0,T]}|y_s^n|^2+\frac{C}{\lambda^2}\int_0^{T_k'}|z_s^n|^2ds+2\left|\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\right|.\end{aligned}$$

Choosing $\lambda^2=2C$, we have

$$\int_0^{T_k'}|z_s^n|^2ds\leq\frac{4TC}{\lambda^2}+4CT+4C\lambda^2T+2)\sup_{s\in[0,T]}|y_s^n|^2+4\left|\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\right|.$$

Thus, since $y^n\in\sigma^\beta$ for each $\beta\in(0,1)$, we have

$$\begin{aligned}\mathrm{E}\left[\left(\int_0^{T_k'}|z_s^n|^2\,ds\right)^{\frac{\beta}{2}}\right]&\leq c_\beta\left(4CT+4C\lambda^2T+2\right)^{\frac{\beta}{2}}\mathrm{E}\left[\sup_{s\in[0,T]}|y_s^n|^\beta\right]\\&\quad+c_\beta\left(\frac{4TC}{\lambda^2}\right)^{\frac{\beta}{2}}+2^\beta c_\beta\mathrm{E}\left[\left|\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\right|^{\frac{\beta}{2}}\right],\end{aligned}$$

where c_β is a constant depending only on β. Furthermore, it follows from BDG's inequality that

$$\begin{aligned}2^\beta c_\beta\mathrm{E}\left[\left|\int_0^{T_k'}y_s^n z_s^n\cdot dB_s\right|^{\frac{\beta}{2}}\right]&\leq d_\beta\mathrm{E}\left[\left|\int_0^{T_k'}|y_s^n|^2|z_s^n|^2\,ds\right|^{\frac{\beta}{4}}\right]\\&\leq\frac{d_\beta^2}{2}\mathrm{E}\left[\sup_{s\in[0,T]}|y_s^n|^\beta\right]+\frac{1}{2}\mathrm{E}\left[\left(\int_0^{T_k'}|z_s^n|^2\,ds\right)^{\frac{\beta}{2}}\right]\end{aligned}$$

where d_β is a constant depending only on c_β and β. Thus, combining the above two inequality one knows

$$\mathrm{E}\left[\left(\int_0^{T_k}|z_s^n|^2\,ds\right)^{\frac{\beta}{2}}\right]\le[2c_\beta\left(4CT+4C\lambda^2T+2\right)^{\frac{\beta}{2}}+d_\beta^2]\mathrm{E}\left[\sup_{s\in[0,T]}|y_s^n|^\beta\right]+2c_\beta\left(\frac{4TC}{\lambda^2}\right)^{\frac{\beta}{2}}.$$

Letting $k\to\infty$ in above inequality, we have $\|z^n\|\le M$, where M depends only β, T, C. For each $n,m\ge C$, applying *Itô* 's formula to $|y^n-y^m|^2$ leads to the inequality

$$\int_0^{T_k}|z_s^n-z_s^m|^2\,ds=|y_{T_k}^n-y_{T_k}^m|^2+2\int_0^{T_k}\left(y_s^n-y_s^m\right)\left(g_n\left(s,y_s^n,z_s^m\right)-g_m\left(s,y_s^n,z_s^m\right)\right)ds$$
$$-2\int_0^{T}\left(y_s^n-y_s^m\right)\left(z_s^n-z_s^m\right)\cdot dB_s.$$

On the other hand, it follows form (i) in Lemma 3.1 and H¨ older inequality that

$$\int_0^{T}\left(y_s^n-y_s^m\right)\left(g_n\left(s,y_s^n,z_s^m\right)-g_m\left(s,y_s^m,z_s^m\right)\right)ds$$
$$\le 2\left(\int_0^T|y_s^n-y_s^m|^2\,ds\right)^{1/2}\left(\int_0^T\left(|g_n\left(s,y_s^n,z_s^m\right)|^2+|g_m\left(s,y_s^m,z_s^m\right)|^2\right)ds\right)^{1/2}$$
$$\le 4T\sup_{s\in[0,T]}|y_s^n-y_s^m|\left(\int_0^T\left(4+|y_s^m|^2+|y_s^n|^2+|z_s^n|^2+|z_s^m|^2\right)ds\right)^{1/2}.$$

Thus, for each $\beta\in(0,1)$, we have

$$\mathrm{E}\left[\left(\int_0^{T_k}|z_s^n-z_s^m|^2\,ds\right)^{\frac{\beta}{2}}\right]\le c_\beta\left(8T\left[(4T)^{\frac{\beta}{4}}+(2TB)^{\frac{1}{2}}+(2M)^{\frac{1}{2}}\right]\right)^{\frac{\beta}{2}}\mathrm{E}\sup_{s\in[0,T]}|y_s^n-y_s^m|^{\frac{\beta}{2}}$$
$$+2^\beta c_\beta\mathrm{E}\left[|\int_0^{T_k}\left(y_s^n-y_s^m\right)\left(z_s^n-z_s^m\right)dB_s|^{\frac{\beta}{2}}\right]+c_\beta\mathrm{E}\sup_{s\in[0,T]}|y_s^n-y_s^m|^\beta.$$

Furthermore, it follows from BDG's inequality that

$$2^\beta c_\beta\mathrm{E}\left[|\int_0^{T_k}\left(y_s^n-y_s^m\right)\left(z_s^n-z_s^m\right)dB_s|^{\frac{\beta}{2}}\right]\le d_\beta\mathrm{E}\left[|\int_0^{T_k}|y_s^n-y_s^m|^2|z_s^n-z_s^m|^2\,ds|^{\frac{\beta}{4}}\right]$$
$$\le\frac{d_\beta^2}{2}\mathrm{E}\left[\sup_{s\in[0,T]}|y_s^n-y_s^m|^\beta\right]+\frac{1}{2}\mathrm{E}\left[\left(\int_0^{T_k}|z_s^n-z_s^m|^2\,ds\right)^{\frac{\beta}{2}}\right],$$

Thus, combining the above inequality and letting $k\to\infty$ one knows that there exist two constants C_1,C_2 depending only on C,T,α,β, and $\mathrm{E}|\varepsilon|$ such that for all $n,m\ge C$

$$|z^n-z^m\|\le C_1\|y^n-y^m\|+C_2\|y^n-y^m\|^{1/2},$$

which means that, in view of the fact that $Z_\cdot^n$ belongs to M^β for each $\beta\in(0,1)$ and $n\ge 1$, there exists a process $Z_\cdot$ which belongs to also M^β such that

$$\lim_{n\to\infty}\mathrm{E}\left[\left(\int_0^T|z_s^n-z_s|^2\,ds\right)^{\beta/2}\right]=0.$$

Therefore, we complete our proof.

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