Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2015; 2(3A):289-299 **ISSN 2393-8064 (Online)**

Positive Solution for Nonlinear Higher Order $(k,n-k)$ Conjugate Boundary Value Problem

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Abstract: The nonlinear $(k,n-k)$ conjugate boundary problem

$$(-1)^{n-k}y^{(n)}(x)=f(x,y(x)),\ y^{(i)}(0)=y^{(j)}(1)=0,\ 0\le i\le k-1, k\le j\le n-1,$$

is studied in this paper, where $0<x<1, n\ge 1\ 0<k<n$. The upper and lower boundary of the Green's function and derivative estimate are given, the existence result of positive solution for the problem is proved by using Krasnoselskii fixed point theorem in cones.
Keywords: the conjugate boundary problem, positive solution, the fixed point theorem

1.INTRODUCTION

In recent years, positive solutions for nonlinear higher order $(k,n-k)$ conjugate boundary value problem

$$(-1)^{n-k}y^{(n)}(x)=f(x,y(x)), 0<x<1, n\ge 1, 0<k<n \qquad (1.1)$$

$$y^{(i)}(0)=y^{(j)}(1)=0, 0\le i\le k-1, 0\le j\le n-k-1 \qquad (1.2)$$

has been studied extensively. Many authors established existence results of positive solutions where the nonlinear term is nonnegative continuous and meet superlinear or sublinear $[1-16]$.When the nonlinear term is negative value or changes number, the boundary value problem (1.1), (1.2) come more from the chemical reaction problem, some authors consider the conjugate boundary value problem (1.1), (1.2) ,and get some results $[19-22]$.

This paper deals with the following nonlinear $(k,n-k)$ conjugate boundary value problem

$$(-1)^{n-k}y^{(n)}(x)=f(x,y(x)), 0<x<1, n\ge 1, 0<k<n \qquad (1.3)$$

$$y^{(i)}(0)=y^{(j)}(1)=0, 0\le i\le k-1, k\le j\le n-1 \qquad (1.4)$$

The condition of boundary problem (1.3), (1.4) in $x=1$ is different from boundary value problem (1.1), (1.2) .Our purpose is to construct the Green's function $G(x,s)$ of conjugate boundary value problem (1.3), (1.4) ,and establish the existence result of positive solution to the problem by using Krasnoselskii fixed point theorem in cones.

$G(x,s)$ is called Green's function of nonlinear higher order $(k,n-k)$ conjugate boundary value problem (1.3), (1.4) ,if it satisfies

$$(-1)^{n-k}G^{(n)}(x,s)=\delta(x-s), 0<x<1, 0<s<1,$$

$$G^{(i)}(0,s)=G^{(j)}(1,s)=0, 0\le i\le k-1, k\le j\le n-1$$

where $G^{(m)}(x,s)=\dfrac{\partial^m G(x,s)}{\partial x^m}$,and $\delta(x)$ is the Dirac δ function.

The function $y(x)$ is called the positive solution of the boundary value problem (1.3), (1.4), if it satisfies $y(x)\in C^{n-1}(0,1]\cap C^{k-1}[0,1)$ and (1.3), (1.4) hold.

Our assumptions are as follows:

$(H_1)h(x)$ is nonnegative continuous for $x \in (0,1)$,and $0 < \int_0^1 \frac{g(s)}{s} h(s)\,\mathrm{d}\,s < +\infty$;

$(H_2)\, f(y)$ is nonnegative continuous for $[0,+\infty)$,

where $g(x) = \dfrac{x^{n-k}}{(k-1)!(n-k-1)!}$

The main results for this paper are as follows.

Theorem 1 Assume that $G(x,s)$ is the Green's function of the conjugate boundary value problem (1.3), (1.4), then

$$G(x,s) = \begin{cases} \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^s u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}\,u, s \le x \\ \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^x u^{k-1}(u-x+s)^{n-k-1}\,\mathrm{d}\,u, x \le s \end{cases}$$

and

$$\alpha(x)g(s) \le G(x,s) \le \beta(x)g(s)$$
$$\left|\frac{\partial G(x,s)}{\partial x}\right| \le \frac{(n-1)x^{k-1}}{s^2} g(s)$$

where

$$\alpha(x) = \frac{x^k}{n-1}, \beta(x) = \frac{x^{k-1}}{\min\{k, n-k\}}.$$

Theorem 2 Assume that $(H_1),(H_2)$ hold, then the nonlinear conjugate boundary value problem (1.3) and (1.4) has at least one positive solution $y(x) \in C^{n-1}(0,1] \cap C^{k-1}[0,1)$,
if the following conditions is satisfies

$$\text{(i) } f_0 = 0,\ f_\infty = +\infty \text{或} \text{ (ii) } f_0 = +\infty,\ f_\infty = 0$$

where $f_0 = \lim\limits_{y\to 0} \dfrac{f(y)}{y}$, $f_\infty = \lim\limits_{y\to\infty} \dfrac{f(y)}{y}$.

To establish the results of Theorem 2,We require the following Krasnoselskii Cone fixed point theorem.

Theorem 3 Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ and let $\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(1) $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$;or

(2) $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$,

then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2.GREEN'S FUNCTION STRUCTURE

In this section, we structure the Green's function of nonlinear higher order $(k, n-k)$ conjugate boundary value problem (1.3), (1.4), and prove the conclusion of theorem 1 is right.Let

$$G(x,s) = \begin{cases} \dfrac{(-1)^{n-k}(x-s)^{n-1}}{(n-1)!} + \displaystyle\sum_{m=0}^{n-1} \frac{a_m}{m!} x^m, & s \le x \\ \displaystyle\sum_{m=0}^{n-1} \frac{a_m}{m!} x^m, & x \le s \end{cases} \tag{2.1}$$

where $a_m = a_m(s), 0 \le m \le n-1$ are undetermined coefficients, so for each fixed s, $G(x,s)$ is continuously differentiable function about x under $n-2$ order. And

$$G^{(n-1)}(x,s) = \begin{cases} (-1)^{n-k} + a_{n-1}, & s < x \\ a_{n-1}, & x < s \end{cases}$$

therefore $(-1)^{n-k} G^{(n)}(x,s) = \delta(x-s)$ for all x and s.

Next, determined the coefficient $a_m, 0 \le m \le n-1$, make $G(x,s)$ satisfied condition of $G^{(i)}(0,s) = G^{(j)}(1,s) = 0$, $0 \le i \le k-1, k \le j \le n-1$. From

$$G^{(i)}(0,s) = 0, 0 \le i \le k-1,$$

we get $a_m = 0, 0 \le m \le k-1$. Then by $G^{(j)}(1,s) = 0, k \le j \le n-1$, the following equations can be obtained

$$\begin{cases} (n-k-1)!a_k + \dfrac{(n-k-1)!}{1!}a_{k+1} + \cdots + \dfrac{(n-k-1)!}{(n-k-2)!}a_{n-2} + a_{n-1} = (-1)^{n-k-1}(1-s)^{n-k-1} \\ (n-k-2)!a_{k+1} + \dfrac{(n-k-2)!}{1!}a_{k+2} + \cdots + \dfrac{(n-k-2)!}{(n-k-3)!}a_{n-2} + a_{n-1} = (-1)^{n-k-1}(1-s)^{n-k-2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 2!a_{n-3} + \dfrac{2!}{1!}a_{n-2} + a_{n-1} = (-1)^{n-k-1}(1-s)^2 \\ a_{n-2} + a_{n-1} = (-1)^{n-k-1}(1-s) \\ a_{n-1} = (-1)^{n-k-1} \end{cases}$$

The following use Cramer's method to solve $a_m, k \le m \le n-1$, since

$$D = \begin{vmatrix} (n-k-1)! & \dfrac{(n-k-1)!}{1!} & \dfrac{(n-k-1)!}{2!} & \cdots & \dfrac{(n-k-1)!}{(n-k-2)!} & 1 \\ 0 & (n-k-2)! & \dfrac{(n-k-2)!}{1!} & \cdots & \dfrac{(n-k-2)!}{(n-k-3)!} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 2! & \dfrac{2!}{1!} & 1 \\ 0 & 0 & 0 & 0 & 1! & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \prod_{i=1}^{n-k-1} i!$$

$D_{k-1+j} = (-1)^{n-k-1} \times$

$$\begin{vmatrix} (n-k-1)! & \frac{(n-k-1)!}{1!} & \frac{(n-k-1)!}{1!} & \cdots & (1-s)^{n-k-1} & \cdots & \frac{(n-k-1)!}{(n-k-2)!} & 1 \\ 0 & (n-k-2)! & \frac{(n-k-2)!}{1!} & \cdots & (1-s)^{n-k-2} & \cdots & \frac{(n-k-2)!}{(n-k-3)!} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & (n-k-j+1)! & \cdots & (1-s)^{n-k-j+1} & \cdots & \frac{(n-k-j+1)!}{(n-k-j)!} & 1 \\ & & & & (1-s)^{n-k-j} & \cdots & \frac{(n-k-j)!}{(n-k-j-1)!} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1-s & \cdots & 1! & 1 \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 1 \end{vmatrix}$$

$$= (-1)^{n-k-1} \prod_{i=n-k-j+1}^{n-k-1} i! \times$$

$$\begin{vmatrix} (1-s)^{n-k-j} & \frac{(n-k-j)!}{1!} & \frac{(n-k-j)!}{2!} & \cdots & \frac{(n-k-j)!}{(n-k-j-2)!} & \frac{(n-k-j)!}{(n-k-j-1)!} & 1 \\ (1-s)^{n-k-j-1} & (n-k-j-1)! & \frac{(n-k-j-1)!}{1!} & \cdots & \frac{(n-k-j-1)!}{(n-k-j-3)!} & \frac{(n-k-j-1)!}{(n-k-j-2)!} & 1 \\ (1-s)^{n-k-j-2} & 0 & (n-k-j-2)! & \cdots & \frac{(n-k-j-2)!}{(n-k-j-4)!} & \frac{(n-k-j-2)!}{(n-k-j-3)!} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^2 & 0 & 0 & \cdots & 2! & \frac{2!}{1!} & 1 \\ (1-s) & 0 & 0 & \cdots & 0 & 1! & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

(The i row minus the $i+1$ row, in order to do so, $i=1,2,\cdots,n-k-1$,then press column extracted common factor, and then the last column to expand)

$$= (-1)^{n-k-1}(-s)(n-k-j-1) \prod_{i=n-k-j+1}^{n-k-1} i! \times$$

$$\begin{vmatrix} (1-s)^{n-k-j-1} & \frac{(n-k-j-1)!}{1!} & \frac{(n-k-j-1)!}{2!} & \cdots & \frac{(n-k-j-1)!}{(n-k-j-3)!} & \frac{(n-k-j-1)!}{(n-k-j-2)!} & 1 \\ (1-s)^{n-k-j-2} & (n-k-j-2)! & \frac{(n-k-j-2)!}{1!} & \cdots & \frac{(n-k-j-2)!}{(n-k-j-4)!} & \frac{(n-k-j-2)!}{(n-k-j-3)!} & 1 \\ (1-s)^{n-k-j-3} & 0 & (n-k-j-3)! & \cdots & \frac{(n-k-j-3)!}{(n-k-j-5)!} & \frac{(n-k-j-3)!}{(n-k-j-4)!} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^2 & 0 & 0 & \cdots & 2! & \frac{2!}{1!} & 1 \\ 1-s & 0 & 0 & \cdots & 0 & 1! & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

(According to the above method repeatedly to do so, to the determinant of reduced-order)

$$= (-1)^{n-k-1}(-s)^{n-k-j-1} \prod_{i=1, i\neq n-k-j}^{n-k-1} i! \begin{vmatrix} 1-s & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)^{n-k-1}(-s)^{n-k-j} \prod_{i=1, i\neq n-k-j}^{n-k-1} i!$$

By using Cramer's ruler, have

$$a_{k-1+j} = \frac{D_{k-1+j}}{D} = \frac{(-1)^{n-k-1}(-s)^{n-k-j}}{(n-k-j)!}, \; j = 1, 2, \cdots, n-k$$

make $m = k-1+j$, then we get

$$a_m = (-1)^{n-k-1}\frac{(-s)^{n-m-1}}{(n-m-1)!}, m = k, k+1, \cdots, n-1$$

Put a_m into (2.1) we can get

$$G(x,s) = \begin{cases} \frac{(-1)^{n-k}(x-s)^{n-1}}{(n-1)!} + \sum_{m=k}^{n-1} \frac{(-1)^{n-k-1}(-s)^{n-m-1}x^m}{m!(n-m-1)!}, & s \le x \\ \sum_{m=k}^{n-1} \frac{(-1)^{n-k-1}(-s)^{n-m-1}x^m}{m!(n-m-1)!}, & x \le s \end{cases}$$

$$= \begin{cases} \frac{(-1)^{n-k}(x-s)^{n-1}}{(n-1)!} + \sum_{i=0}^{n-k-1} \frac{(-1)^{n-k-1}(-s)^{i}x^{n-1-i}}{i!(n-1-i)!}, & s \le x \\ \sum_{i=0}^{n-k-1} \frac{(-1)^{n-k-1}(-s)^{i}x^{n-1-i}}{i!(n-1-i)!}, & x \le s \end{cases} \qquad (2.2)$$

Following we transformed $G(x,s)$ into an integral formula, since

$$\sum_{i=0}^{n-k-1} \frac{(-1)^{n-k-1}(-s)^{i}x^{n-1-i}}{i!(n-1-i)!}$$

$$= \frac{(-1)^{n-k-1}}{(k-1)!(n-k-1)!} \sum_{i=0}^{n-k-1} \frac{(n-k-1)!(-s)^{i}x^{n-1-i}(k-1)!(n-k-1-i)!}{i!(n-k-1-i)!(n-1-i)!}$$

$$= \frac{(-1)^{n-k-1}}{(k-1)!(n-k-1)!} \sum_{i=0}^{n-k-1} \frac{(n-k-1)!(-s)^{i}x^{n-1-i}}{i!(n-k-1-i)!} \int_0^1 w^{k-1}(1-w)^{n-k-1-i}\,\mathrm{d}w$$

$$=\frac{(-1)^{n-k-1}}{(k-1)!(n-k-1)!}\int_0^1 x^k w^{k-1}\sum_{i=0}^{n-k-1}\frac{(n-k-1)!(-s)^i[x(1-w)]^{n-k-1-i}}{i!(n-k-1-i)!}\,\mathrm{d}w$$

$$=\frac{(-1)^{n-k-1}}{(k-1)!(n-k-1)!}\int_0^1 x^k w^{k-1}[x(1-w)-s]^{n-k-1}\,\mathrm{d}w \text{ (let } u=s-x(1-w)\text{)} \qquad (2.3)$$

$$=\frac{1}{(k-1)!(n-k-1)!}\int_{s-x}^{s} u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u$$

$$=\frac{1}{(k-1)!(n-k-1)!}[\int_0^s u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u-\int_0^{s-x} u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u]$$

(let $u=(s-x)v$ in the second term in equation)

$$=\frac{1}{(k-1)!(n-k-1)!}[\int_0^s u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u-(-1)^{n-k}(x-s)^{n-1}\int_0^1 v^{n-k-1}(1-v)^{k-1}\,\mathrm{d}v]$$

$$=\frac{1}{(k-1)!(n-k-1)!}\int_0^s u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u-\frac{(-1)^{n-k}(x-s)^{n-1}}{(n-1)!} \qquad (2.4)$$

where, the following result

$$\int_0^1 v^{n-k-1}(1-v)^{k-1}\,\mathrm{d}v=\frac{(k-1)!(n-k-1)!}{(n-1)!}$$

is used. In addition, we know from (2.3)

$$\sum_{i=0}^{n-k-1}\frac{(-1)^{n-k-1}(-s)^i x^{n-1-i}}{i!(n-1-i)!}$$

$$=\frac{1}{(k-1)!(n-k-1)!}\int_0^1 x^k w^{k-1}[(s-x(1-w]^{n-k-1}\,\mathrm{d}w \text{(let } t=x(1-w)\text{)}$$

$$=\frac{1}{(k-1)!(n-k-1)!}\int_0^x (x-t)^{k-1}(s-t)^{n-k-1}\,\mathrm{d}t \text{ (let } u=x-t\text{)}$$

$$=\int_0^x u^{k-1}(u+s-x)^{n-k-1}\,\mathrm{d}u \qquad (2.5)$$

put (2.4), (2.5) into (2.1), we have

$$G(x,s)=\begin{cases}\dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^s u^{n-k-1}(u+x-s)^{k-1}\,\mathrm{d}u, & s\le x\\[2ex] \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^x u^{k-1}(u+s-x)^{n-k-1}\,\mathrm{d}u, & x\le s\end{cases} \qquad (2.6)$$

by using the binomial theorem in (2.6), we obtain

$$G(x,s)=\begin{cases}\dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^s u^{n-k-1}\sum_{i=0}^{k-1}\frac{(x-s)^i u^{k-1-i}}{i!(n-1-i)!}\,\mathrm{d}u, s\le x\\[2ex] \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\int_0^x u^{k-1}\sum_{i=0}^{n-k-1}\frac{(s-x)^i u^{n-k-1-i}}{i!(n-k-1-i)!}\,\mathrm{d}u, x\le s\end{cases}$$

$$=\begin{cases}\dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\sum_{i=0}^{k-1}\frac{(x-s)^i s^{n-1-i}(k-1)!}{i!(k-1-i)!(n-1-i)}, s\le x\\[2ex] \dfrac{1}{(k-1)!(n-k-1)!}\displaystyle\sum_{i=0}^{n-k-1}\frac{(s-x)^i x^{n-1-i}(n-k-1)!}{i!(n-k-1-i)!(n-1-i)}, x\le s\end{cases}$$

$$\geq \begin{cases} \dfrac{1}{(k-1)!(n-k-1)!(n-1)} x^{k-1}s^{n-k}, s \leq x \\ \dfrac{1}{(k-1)!(n-k-1)!(n-1)} x^{k}s^{n-k-1}, x \leq s \end{cases}$$

$$\geq \frac{x^{k-1}s^{n-k}}{(k-1)!(n-k-1)!(n-1)} = \alpha(x)g(s)$$

$$G(x,s) \leq \begin{cases} \dfrac{s^{n-k}x^{k-1}}{(k-1)!(n-k-1)!(n-k)}, s \leq x \\ \dfrac{s^{n-k-1}x^{k}}{(k-1)!(n-k-1)!k}, x \leq s \end{cases}$$

$$\leq \begin{cases} \dfrac{s^{n-k}x^{k-1}}{(k-1)!(n-k-1)!\min\{k,n-k\}}, s \leq x \\ \dfrac{s^{n-k-1}x^{k}}{(k-1)!(n-k-1)!\min\{k,n-k\}}, x \leq s \end{cases}$$

$$= \begin{cases} \dfrac{x^{k-1}g(s)}{\min\{k,n-k\}}, s \leq x \\ \dfrac{x^{k}g(s)}{s\min\{k,n-k\}}, x \leq s \end{cases}$$

$$\leq \frac{x^{k-1}}{\min\{k,n-k\}} g(s) = \beta(x)g(s)$$

and hence

$$\alpha(s)g(s) \leq G(x,s) \leq \beta(x)g(s)$$

Other, when $s \leq x$,we have

$$\frac{\partial G(x,s)}{\partial x} = \frac{1}{(k-1)!(n-k-1)!} \sum_{i=1}^{k-1} \frac{(k-1)!i(x-s)^{i-1}s^{n-1-i}}{i!(k-1-i)!(n-1-i)}$$

$$\left|\frac{\partial G(x,s)}{\partial x}\right| \leq \frac{k-1}{(k-1)!(n-k-1)!(n-k)} \sum_{j=0}^{k-2} \frac{(k-2)!}{j!(k-2-j)!}(x-s)^j s^{k-2-j}x^{n-k}$$

$$= \frac{k-1}{(k-1)!(n-k-1)!(n-k)} s^{n-k}x^{k-2}$$

$$\leq \frac{n}{(k-1)!(n-k-1)!} s^{n-k}x^{k-2}$$

when $x \leq s$,we have

$$\frac{\partial G(x,s)}{\partial x} = \frac{1}{(k-1)!(n-k-1)!}[\sum_{i=1}^{n-k-1} \frac{-i(n-k-1)!(s-x)^{i-1}x^{n-1-i}}{i!(n-k-1-i)!(n-1-i)} + \sum_{i=0}^{n-k-1} \frac{(n-k-1)!(s-x)^{i}x^{n-2-i}}{i!(n-k-1-i)!}] \text{ thus}$$

$$\left|\frac{\partial G(x,s)}{\partial x}\right| \leq \frac{1}{(k-1)!(n-k-1)!}[\sum_{j=0}^{n-k-2} \frac{(n-k-1)!(s-x)^j x^{n-k-2-j}x^k}{j!(n-k-2-j)!k} + s^{n-k-1}x^{k-1}]$$

$$= \frac{1}{(k-1)!(n-k-1)!}[\frac{n-k-1}{k} s^{n-k-2}x^k + s^{n-k-1}x^{k-1}]$$

$$\le \frac{n}{(k-1)!(n-k-1)!} s^{n-k-1} x^{k-1}$$

therefore

$$\left|\frac{\partial G(x,s)}{\partial x}\right| \le \begin{cases} \frac{n}{(k-1)!(n-k-1)!} s^{n-k} x^{k-2}, s \le x \\ \frac{n}{(k-1)!(n-k-1)!} s^{n-k-1} x^{k-1}, x \le s \end{cases}$$

$$\le \frac{n}{(k-1)!(n-k-1)!} s^{n-k-1} x^{k-1} = \frac{n x^{k-1} g(s)}{s}$$

3.THE EXISTENCE OF POSITIVE SOLUTION

This section establish the positive solution of the conjugate boundary value problem $(1.3),(1.4)$ and prove the conclusion of theorem 2.

Let $C[0,1]$ be the Banach space which consisting of all continuous functions in $[0,1]$, define a cone in $C[0,1]$ as following

$$K = \{y \in C[0,1]; y(x) \ge \alpha(x)\|y\|/\|\beta\|\}, \|\beta\| = \max_{x\in[0,1]} |\beta(x)| \qquad (3.1)$$

Define a mapping $\Phi : C^+[0,1] \to C^+[0,1]$,

$$(\Phi y)(x) = \int_0^1 G(x,s)h(s) f(y(s))\,\mathrm{d}s$$

Lemma 1 $\Phi : K \to K$ is a completely continuous mapping.

Proof Let $y \in K$, we have from theorem 1

$$(\Phi y)(x) \ge \alpha(x)\int_0^1 g(s)h(s) f(y(s))\,\mathrm{d}s$$

$$\ge \frac{\alpha(x)}{\|\beta\|} \max_{x\in[0,1]} \beta(x) \int_0^1 g(s)h(s) f(y(s))\,\mathrm{d}s$$

$$\ge \alpha(x)\|\Phi y\|/\|\beta\|$$

That is to say $\Phi y \in K$

Let $K_N = \{y \in K; \|y\| \le N\}$, $M = \max_{0\le y\le N} f(y)$,for $y \in K_N$, we have by the theorem 1,

$$\|\Phi y\| \le M\|\beta\|\frac{\alpha(x)}{\|\beta\|}\int_0^1 g(s)h(s)\,\mathrm{d}s$$

This shows $\Phi(K_N)$ is bounded. According condition (H_1) and theorem 1, we obtain

$$|(\Phi y)'(x)| \le \int_0^1 \left|\frac{\partial G(x,s)}{\partial x}\right| h(s) f(y(s))\,\mathrm{d}s$$

$$\le Mnx^{k-1}\int_0^1 \frac{g(s)}{s} h(s)\,\mathrm{d}s$$

where $q = Mn\int_0^1 \frac{g(s)}{s} h(s)\,\mathrm{d}s$, therefore

$$\int_0^1 |(\Phi y)'(x)|\,\mathrm{d}s \le q\int_0^1 x^{k-1}\,\mathrm{d}x = \frac{q}{k}$$

This shows $\Phi(K_N)$ is a compact set in K .From continuity of $f(y)$, easily know Φ is a continuous mapping, from Ascoli−Arzela theorem, Φ is a completely continuous operator.

Now, we prove the theorem 2. First we prove the case (i) .

Since $f_0 = 0$, there exists a $r > 0$ such that $f(y) \le \varepsilon y$ for $0 < y < r$, where $\varepsilon > 0$ satisfies $\varepsilon\|\beta\|\int_0^1 g(s)h(s)\,\mathrm{d}s \le 1$

Let $K_r = \{y \in C[0,1]; \|y\| < r\}$, for $y \in K \cap \partial K_r$

$$\|\Phi y\| \le \varepsilon r\|\beta\|\int_0^1 g(s)h(s)\,\mathrm{d}s \le r = \|y\|$$

this shows $\|\Phi y\| \le \|y\|$, $y \in K \cap \partial K_r$

Further, Since $f_\infty = +\infty$, there exist $R > r > 0$ such that $f(y) \ge My$ for $y \ge RA(\sigma)$, where

$$A(\sigma) = \frac{1}{\|\beta\|}\min_{x\in[\sigma,1-\sigma]}\alpha(x),\ \sigma \in (0,\frac{1}{2})$$

and $M > 0$ satisfies $MA(\sigma)\min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)\,\mathrm{d}s \ge 1$

Let $K_R = \{y \in C[0,1]; \|y\| < R\}$, for $y \in K \cap \partial K_R$, we have from (3.1)

$$\min_{x\in[\sigma,1-\sigma]} y(x) \ge \min_{x\in[\sigma,1-\sigma]}\alpha(x)\|y\|\big/\|\beta\| = A(\sigma)\|y\| = RA(\sigma)$$

thus

$$\|\Phi y\| \ge \min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)f(y(s))\,\mathrm{d}s$$

$$\ge MRA(\sigma)\min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)\,\mathrm{d}s \ge R = \|y\|$$

this shows $\|\Phi y\| > \|y\|$ for $y \in K \cap \partial K_R$, and from theorem 3 that Φ has a fixed point in $y \in K \cap (\overline{K}_R \setminus K_r)$. First prove situation

Next we prove the case (ii). Since $f_0 = \infty$, we can choose $r > 0$ such that $f(y) \ge My$ for $0 < y \le r$, where M satisfies $MA(\sigma)\min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)\,\mathrm{d}s \ge 1$.

For $y \in K \cap \partial K_r$,we have

$$\|\Phi y\| \ge \min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)f(y(s))\,\mathrm{d}s$$

$$\ge M\min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)y(s)\,\mathrm{d}s$$

$$\ge MrA(\sigma)\min_{x\in[\sigma,1-\sigma]}\alpha(x)\int_\sigma^{1-\sigma} g(s)h(s)\,\mathrm{d}s \ge r = \|y\|$$

and hence $\|\Phi y\| \ge \|y\|$ for $y \in K \cap \partial K_r$

Further, since $f_\infty = 0$, there exist $R_0 > r > 0$ such that $f(y) \le \varepsilon y$ for $y \ge R_0$, where $\varepsilon > 0$ satisfies $\varepsilon\|\beta\|\int_0^1 g(s)h(s)\,\mathrm{d}s \le 1$. Let

$$R = \frac{\|\beta\|\max_{0\le y\le R_0} f(y)\int_0^1 g(s)h(s)\,\mathrm{d}s}{1-\varepsilon\|\beta\|\int_0^1 g(s)h(s)\,\mathrm{d}s} + R_0$$

then we have for $y \in K \cap K_R$

$$\|\Phi y\| \le \|\beta\| \int_0^1 g(s)h(s) f(y(s))\,\mathrm{d}s$$

$$\le \|\beta\| [\int_{0\le y(s)\le R_0} g(s)h(s) f(y(s))\,\mathrm{d}s + \int_{R_0\le y(s)\le R} g(s)h(s) f(y(s))\,\mathrm{d}s]$$

$$\le \|\beta\| (\max_{0\le y\le R_0} f(y) + \varepsilon R)\int_0^1 g(s)h(s)\,\mathrm{d}s \le R = \|y\|$$

That is to say $\|\Phi y\| \le \|y\|$ for $y \in K \cap \partial K_R$. From theorem 3 that Φ has a fixed point $y \in K \cap (\overline{K}_R \setminus K_r)$ and $y(x)$ is the positive solution of integral equation

$$y(x) = \int_0^1 G(x,s)h(s) f(y(s))\,\mathrm{d}s \qquad (3.2)$$

Because $y(x)$ satisfies (3.2), we obtain

$$y^{(i)}(x) = \int_0^1 \frac{\partial^i G(x,s)}{\partial x^i} h(s) f(y(s))\,\mathrm{d}s$$

furthermore

$$y^{(i)}(0) = \int_0^1 G^{(i)}(0,s)h(s) f(y(s))\,\mathrm{d}s = 0\,, 0 \le i \le k-1$$

$$y^{(j)}(1) = \int_0^1 G^{(j)}(1,s)h(s) f(y(s))\,\mathrm{d}s = 0\,, k \le j \le n-1$$

From theorem 1 we get

$$(-1)^{n-k} y^{(n)}(x) = \int_0^1 (-1)^{n-k} \frac{\partial^n G(x,s)}{\partial x^n} h(s) f(y(s))\,\mathrm{d}s$$

$$= \int_0^1 \delta(x-s)h(s) f(y(s))\,\mathrm{d}s = h(x) f(y(x))$$

this shows $y(x)$ is the positive solution of nonlinear higher order $(k, n-k)$ conjugate boundary value problem (1.3), (1.4)

Acknowledgement: Project supported by Heilongjiang province education department natural science research item, China. (12541076).

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