Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2015; 2(3A):315-319
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Generalized Likelihood Ratio Test for Normal Population Variance

LI Wenhe
College of Mathematics and Statistics, Northeast Petroleum University, Daqing 163318, China

***Corresponding Author:**
LI Wenhe
Email: xiongdi163@163.com

Abstract: Generalized likelihood ratio test (GLRT) is a very important method of hypothesis testing in mathematical statistics, which is widely applied. In this paper, GLRT is used to deduce the rejection region of hypothesis testing for the single normal population variance, with both known and unknown mean value.
Keywords: Normal distribution, Mean value, Hypothesis test, generalized likelihood ratio test

INTRODUCTION

Given the probability density function of the population as $f(x,\theta)$, where $\theta\in\Theta$ .For the testing issue: $H_0:\theta\in\Theta_0 \leftrightarrow H_1:\theta\in\Theta_1$, $\lambda(x)=\sup\limits_{\theta\in\Theta}L(\underset{\sim}{x},\theta)\Big/\sup\limits_{\theta\in\Theta_\circ}L(\underset{\sim}{x},\theta)$ is defined as the generalized likelihood ratio (GLR) of the sample ($x_1,x_2,\cdots x_n$).

The definition indicates that $\lambda(x)\geq 1$. Assuming that $\hat{\theta}$ and $\hat{\theta}_0$ represent the maximum likelihood estimation of θ at Θ and $\Theta_\circ$ respectively, we have:

$$\lambda(x)=\sup L(\underset{\sim}{x},\hat{\theta})\Big/\sup L(\underset{\sim}{x},\hat{\theta}_0)$$

If the original hypothesis H_0 is true, i.e. the truth value of θ is surely in Θ_0, then $\hat{\theta}$ is also in Θ_0 or very close to Θ_0, leading to $\sup\limits_{\theta\in\Theta_\circ}L(\underset{\sim}{x},\theta)=L(\underset{\sim}{x},\hat{\theta})\approx\sup\limits_{\theta\in\Theta}L(\underset{\sim}{x},\theta)$, and therefore $\lambda(x)\approx 1$. When $\lambda(x)$ is significantly larger than 1, there is $\sup\limits_{\theta\in\Theta_\circ}f(\underset{\sim}{x},\theta)<f(\underset{\sim}{x},\hat{\theta})$, namely, $\hat{\theta}$ is far away from $\Theta_\circ$. The truth value of $\hat{\theta}$ is quite close to that of θ, so it is highly possible that the truth value of θ is not in $\Theta_\circ$, i.e. the hypothesis H_0 is very possible invalid. As a result, the rejection region shall be $W_0=\{\underset{\sim}{x}\mid\lambda(\underset{\sim}{x})>\lambda_0\}$, in which, λ_0 satisfies:

$$\sup_{\theta\in\Theta_0}P(\underset{\sim}{X}\in W_0\mid\theta)=\alpha\ \ (0<\alpha<1)$$

In this study, GLRT was used to deduce, in detail, the rejection region of the one-sided hypothesis testing for the single normal population variance in different cases.

For the case with known mean value

Theorem 1

Suppose $X\sim N(\mu,\sigma^2)$ With $\mu=\mu_0$ known, the GLRT rejection region for the testing issue $H_0:\sigma^2=\sigma_0^2\leftrightarrow H_0:\sigma^2\neq\sigma_0^2$ is:

$$W_0=\{(x_1,x_2,\cdots,x_n)\mid\lambda(x)>\lambda_0\}=\{(x_{1,}x_2\cdots x_n)\mid m>c_1\ or\ m<c_2\} \quad (1)$$

Where c_1 and c_2 satisfy:

$$\int_0^{c_2}\chi_n^2(y)dy+\int_{c_1}^{+\infty}\chi_n^2(y)dy=\alpha$$

Where, $\chi_n^2(y)$ is a density function of the χ^2 distribution with *n* degrees of freedom.

Proof:

The likelihood function is:

$$L(\underset{\sim}{x};\sigma^2)=(\frac{1}{\sqrt{2\pi\sigma}})^n \exp\{-\sum_{i=1}^{n}(x_i-\mu_0)\Big/2\sigma^2\}$$

When $\sigma^2\in\Theta$, the maximum likelihood estimation of σ^2 is:

$$\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$$

$$\sup_{\sigma^2\in\Theta} L(\underset{\sim}{x},\sigma^2)=\left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right]^{-\frac{n}{2}}e^{-\frac{n}{2}}$$

When $\sigma^2\in\Theta_0$, $\sigma^2=\sigma_0^2$, we have

$$\sup_{\sigma^2\in\Theta_0} L(\underset{\sim}{x};\sigma^2)=(\frac{1}{\sqrt{2\pi}\sigma_0})^n \exp\{-\sum_{i=1}^{n}(x_i-\mu_0)^2\Big/2\sigma_0^2\}.$$

So we have

$$\lambda(x)=\sup_{\sigma^2\in\Theta} L(\underset{\sim}{x};\sigma^2)\Big/\sup_{\sigma^2\in\Theta_0} L(\underset{\sim}{x};\sigma^2)=\left(\sum_{i=1}^{n}(x_i-\mu_0)^2\Big/n\sigma_0^2\right)^{\frac{n}{2}}\exp\{\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma_0^2}-\frac{n}{2}\}$$

Let

$$m=\sum_{i=1}^{n}(x_i-\mu_0)^2\Big/\sigma_0^2$$

Then

$$\lambda(x)=(\frac{n}{m})^{n/2}e^{\frac{m}{2}-\frac{n}{2}}$$

When $m>n$, $\lambda(x)$ is increasing, while decreasing when $m<n$.

If H_0 is true, then:

$$m\sim\chi^2(n-1)$$

The rejection region therefore is:

$$W_0=\{(x_1,x_2,\cdots,x_n)\mid\lambda(x)>\lambda_0\}=\{(x_{1,}x_2\cdots x_n)\mid m>c_1\ or\ m<c_2\}$$

By the following formula

$$P\{(x_1,x_2,\cdots x_n)\in W\mid\sigma=\sigma_0\}=\alpha$$

We can get

$$\int_0^{c_2}\chi_n^2(y)dy+\int_{c_1}^{+\infty}\chi_n^2(y)dy=\alpha$$

Where $\chi_n^2(y)$ is a density function of the χ^2 distribution with *n* degrees of freedom

Theorem 2

Suppose $X\sim N(\mu,\sigma^2)$ With $\mu=\mu_0$ known, the GLRT rejection region for the testing issue $H_0:\sigma^2\le\sigma_0{}^2\leftrightarrow H_1:\sigma^2>\sigma_0{}^2$ is:

$$W_0=\left\{(x_1,x_2,\cdots,x_n)\middle|m>\chi_\alpha^2(n)\right\} \qquad (2)$$

Proof: The likelihood function is:

$$L(x;\sigma^2)=(\frac{1}{\sqrt{2\pi}\sigma})^n \exp\{-\sum_{i=1}^{n}(x_i-\mu_0)^2 \Big/ 2\sigma^2\}$$

When $\sigma^2 \in \Theta$, the maximum likelihood estimation of σ^2 is:

$$\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2=S_n^2$$

$$\sup_{\sigma^2\in\Theta} L(\underset{\sim}{x};\sigma^2)=\left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

When $\sigma^2 \in \Theta_0$, we have

$$\sup_{\sigma^2\in\Theta_0} L(\underset{\sim}{x};\sigma^2)=\begin{cases}\left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}} & \sigma_0{}^2\ge S_n^2\\ L(\underset{\sim}{x};\sigma_0{}^2) & \sigma_0{}^2< S_n^2\end{cases}$$

When $\sigma_0{}^2 > S_n^2$, we have

$$\lambda(x)\equiv 1$$

When $\sigma_0{}^2 < S_n^2$, we have

$$\lambda(x)=(\sum_{i=1}^{n}(x_i-\mu_0)^2 \Big/ n\sigma_0{}^2)^{\frac{n}{2}} \exp\{\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma_0{}^2}-\frac{n}{2}\}.$$

Because of $\frac{nS_n^2}{\sigma_0{}^2}>1$, $\lambda(x)$ is an increasing function about:

$$m=\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0{}^2}$$

Therefore, the rejection region is:

$$W_0=\left\{(x_1,x_2,\cdots,x_n)\middle|\lambda(x)>\lambda_0\right\}=\left\{(x_1,x_2,\cdots,x_n)\middle|m>C\right\}$$

and

$$P\left\{\text{reject } \mathrm{H}_0 \mid H_0\right\}=\mathrm{P}\left\{m>C\mid\sigma^2\le\sigma_0{}^2\right\}=\alpha$$

If H_0 is true, there is:

$$m=\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma_0{}^2}\sim\chi^2(n)$$

Hence

$$C=\chi_\alpha^2(n)$$

So we have

$$W_0=\left\{(x_1,x_2,\cdots,x_n)\middle|m>\chi_\alpha^2(n)\right\}$$

For the case with known variance
Theorem 3

Suppose $X\sim N(\mu,\sigma^2)$ With μ unknown, the GLRT rejection region for the testing issue $H_0:\sigma^2=\sigma_0^2 \leftrightarrow H_0:\sigma^2\ne\sigma_0^2$ is:

$$W_0=\{(x_1,x_2,\cdots,x_n)\,|\,\lambda(x)>\lambda_0\}=\{(x_1,x_2\cdots x_n)\,|\,m>c_1\ or\ m<c_2\} \quad (3)$$

Where c_1 and c_2 satisfy:

$$\int_0^{c_2}\chi^2_{n-1}(y)dy+\int_{c_1}^{+\infty}\chi^2_{n-1}(y)dy=\alpha$$

Where $\chi^2_{n-1}(y)$ is a density function of the χ^2 distribution with $n-1$ degrees of freedom.

Proof

The likelihood function is:

$$L(\underset{\sim}{x};\mu,\sigma^2)=(\frac{1}{\sqrt{2\pi\sigma}})^n\exp\{-\sum_{i=1}^{n}(x_i-\mu)\Big/2\sigma^2\}$$

When $(\mu,\sigma^2)\in\Theta$, $\mu=\bar{x}$, $\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$\sup_{(\mu,\sigma^2)\in\Theta}L(\underset{\sim}{x};\mu,\sigma^2)=\left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-\frac{n}{2}}e^{-\frac{n}{2}}$$

When $(\mu,\sigma^2)\in\Theta_0$, $\mu=\bar{x}$, $\sigma_0^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$\sup_{(\mu,\sigma^2)\in\Theta_0}L(\underset{\sim}{x};\mu,\sigma^2)=(\frac{1}{\sqrt{2\pi\sigma_0}})^n\exp\{-\sum_{i=1}^{n}(x_i-\bar{x})^2\Big/2\sigma_0^2\}.$$

Hence

$$\lambda(x)=\sup_{(\mu,\sigma^2)\in\Theta}L(\underset{\sim}{x};\mu,\sigma^2)\Big/\sup_{(\mu,\sigma^2)\in\Theta_0}L(\underset{\sim}{x};\mu,\sigma^2)$$

$$=\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\Big/n\sigma_0^2\right)^{\frac{n}{2}}\exp\{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2\sigma_0^2}-\frac{n}{2}\}$$

Let $m=\sum_{i=1}^{n}(x_i-\bar{x})^2\Big/\sigma_0^2$, then $\lambda(x)=(\frac{m}{n})^{\frac{n}{2}}e^{\frac{m}{2}-\frac{n}{2}}$. when $m>n$, $\lambda(x)$ is increasing, while decreasing when $m<n$. If H_0 is true, there is: $m\sim\chi^2(n-1)$.

As a result, the rejection region can be defined as:

$$W_0=\{(x_1,x_2,\cdots,x_n)\,|\,\lambda(x)>\lambda_0\}=\{(x_1,x_2\cdots x_n)\,|\,m>c_1\ or\ m<c_2\}$$

where c_1 and c_2 satisfy:

$$P\{\text{reject}\,H_0\,|\,H_0\}=P\{m>c_1\ or\ m<c_2\,|\,\sigma^2=\sigma_0^2\}=\alpha$$

Therefore,

$$\int_0^{c_2}\chi^2_{n-1}(y)dy+\int_{c_1}^{+\infty}\chi^2_{n-1}(y)dy=\alpha$$

where $\chi^2_{n-1}(y)$ is the density function of the χ^2 distribution with $n-1$ degrees of freedom.

Theorem 4

Suppose $X\sim N(\mu,\sigma^2)$ With μ unknown, the GLRT rejection region for the testing issue $H_0:\sigma^2\le{\sigma_0}^2\leftrightarrow H_1:\sigma^2>{\sigma_0}^2$ is:

$$W_0=\left\{(x_1,x_2,\cdots,x_n)\Big|m>\chi^2_\alpha(n-1)\right\} \quad (4)$$

Proof: The likelihood function is:

$$L(\underset{\sim}{x};\sigma^2)=(\frac{1}{\sqrt{2\pi\sigma}})^n\exp\{-\sum_{i=1}^{n}(x_i-\bar{x})^2\Big/2\sigma^2\}$$

When $\sigma^2 \in \Theta$, the maximum likelihood estimation of σ^2 is:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = S^2$$

$$\sup_{\sigma^2\in\Theta} L(\underset{\sim}{x};\sigma^2) = \left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

When $\sigma^2 \in \Theta_0$, we have

$$\sup_{\sigma^2\in\Theta_0} L(\underset{\sim}{x};\sigma^2) = \begin{cases} \left[1\Big/2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}}, & \sigma_0^2 \geq S^2 \\ L(\underset{\sim}{x};\sigma_0^2), & \sigma_0^2 < S^2 \end{cases}$$

When $\sigma_0^2 > S^2$, we have

$$\lambda(x) \equiv 1$$

When $\sigma_0^2 < S^2$, we have

$$\lambda(x) = \left(\sum_{i=1}^{n}(x_i-\bar{x})^2 \Big/ n\sigma_0^2\right)^{\frac{n}{2}} \exp\left\{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2\sigma_0^2} - \frac{n}{2}\right\}.$$

Because of $\frac{nS^2}{\sigma_0^2} > 1, \lambda(x)$ is an increasing function about:

$$m = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma_0^2}$$

Therefore, the rejection region is:

$$W_0 = \left\{(x_1, x_2, \cdots, x_n) \middle| \lambda(x) > \lambda_0\right\} = \left\{(x_1, x_2, \cdots, x_n) \middle| m > C\right\}$$

and

$$P\left\{\text{reject } H_0 \mid H_0\right\} = P\left\{m > C \mid \sigma^2 \leq \sigma_0^2\right\} = \alpha$$

Hence

$$C = \chi_\alpha^2(n-1)$$

So we have

$$W_0 = \left\{(x_1, x_2, \cdots, x_n) \middle| m > \chi_\alpha^2(n-1)\right\}$$

CONCLUSIONS

In this paper, by using the generalized likelihood ratio test, four conclusions are obtained:

- The rejection region of the two-sided hypothesis testing for normal population variance with the known mean value(1);
- The rejection region of the one-sided hypothesis testing(2);
- The rejection region of the two-sided hypothesis testing for normal population variance with the unknown mean value(3);
- The rejection region of the one-sided hypothesis testing (4).

ACKNOWLEDGEMENTS

I would like to thank the referees and the editor for their valuable suggestions.

REFERENCES

1. CHEN Xiru. An introduction to mathematical statistics. Science press, 1981.
2. CHEN Jiading. The notes of mathematical statistics. Higher education press, 1993.