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One-Sided Generalized Likelihood Ratio Test for Normal Population Mean Value

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Abstract: Generalized likelihood ratio test (GLRT) is a very important method of hypothesis testing in mathematical statistics, which is widely applied. In this paper, GLRT is used to deduce the rejection region of hypothesis testing of mean value for the single normal population with both known and unknown variance.
Keywords: Normal distribution, Mean value, One-sided hypothesis test, Generalized likelihood ratio test.

INTRODUCTION

Given the probability density function of the population as $f(x,\theta)$, where $\theta \in \Theta$.For the testing issue: $H_0:\theta\in\Theta_0 \leftrightarrow H_1:\theta\in\Theta_1$, $\lambda(x)=\sup\limits_{\theta\in\Theta} L(\underset{\sim}{x},\theta) \Big/ \sup\limits_{\theta\in\Theta_\circ} L(\underset{\sim}{x},\theta)$ is defined as the generalized likelihood ratio (GLR) of the sample ($x_1, x_2, \cdots x_n$).

The definition indicates that $\lambda(x)\geq 1$. Assuming that $\hat{\theta}$ and $\hat{\theta}_0$ represent the maximum likelihood estimation of θ at Θ and $\Theta_\circ$ respectively, we have:

$$\lambda(x)=\sup L(\underset{\sim}{x},\hat{\theta}) \Big/ \sup L(\underset{\sim}{x},\hat{\theta}_0)$$

If the original hypothesis H_0 is true, i.e. the truth value of θ is surely in Θ_0, then $\hat{\theta}$ is also in Θ_0 or very close to Θ_0, leading to $\sup\limits_{\theta\in\Theta_\circ} L(\underset{\sim}{x},\theta)=L(\underset{\sim}{x},\hat{\theta})\approx\sup\limits_{\theta\in\Theta} L(\underset{\sim}{x},\theta)$, and therefore $\lambda(x)\approx 1$. When $\lambda(x)$ is significantly larger than 1, there is $\sup\limits_{\theta\in\Theta_\circ} f(\underset{\sim}{x},\theta) < f(\underset{\sim}{x},\hat{\theta})$, namely, $\hat{\theta}$ is far away from $\Theta_\circ$. The truth value of $\hat{\theta}$ is quite close to that of θ, so it is highly possible that the truth value of θ is not in $\Theta_\circ$, i.e. the hypothesis H_0 is very possible invalid. As a result, the rejection region shall be $W_0=\{\underset{\sim}{x} \mid \lambda(\underset{\sim}{x})>\lambda_0\}$, in which, λ_0 satisfies:

$$\sup_{\theta\in\Theta_0} P(\underset{\sim}{X}\in W_0 \mid \theta)=\alpha \quad (0<\alpha<1)$$

In this study, GLRT was used to deduce, in detail, the rejection region of the one-sided hypothesis testing for the single normal population mean value in different cases.

For the case with known variance

Theorem 1

Suppose $X\sim N(\mu,\sigma^2)$ with $\sigma^2=\sigma_0^2$ known, $\mu\in\Theta=(-\infty,+\infty)$,
the GLRT rejection region for the testing issue $H_0:\mu\leq\mu_0 \leftrightarrow H_1:\mu>\mu_0$ is:

$$W_0=\left\{\underset{\sim}{x} \mid \bar{X}-\mu_0>Z_\alpha\cdot\sigma\Big/\sqrt{n}\right\} \tag{1}$$

Proof

The likelihood function is:

$$L(\underset{\sim}{x};\mu)=(\frac{1}{\sqrt{2\pi}\sigma_0})^n\exp\{-\sum_{i=1}^{n}(x_i-\mu)^2\Big/2{\sigma_0}^2\}$$
$$=(\frac{1}{\sqrt{2\pi}\sigma_0})^n\exp\{-[\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2]\Big/2{\sigma_0}^2\}$$

When $\mu\in\Theta$, the maximum likelihood estimation of normal distribution is $\hat{\mu}=\overline{X}$.

$$\sup_{\mu\in\Theta}L(\underset{\sim}{x};\mu)=L(\underset{\sim}{x};\hat{\mu})=(\frac{1}{\sqrt{2\pi}\sigma_0})^n\exp\{-\sum_{i=1}^{n}(x_i-\bar{x})^2\Big/2{\sigma_0}^2\}$$
$$\sup_{\mu\in\Theta_0}L(\underset{\sim}{x};\mu)=\begin{cases}L(\overline{X}), & \overline{X}\le\mu_0\\ L(\mu_0), & \overline{X}>\mu_0\end{cases}$$

When $\overline{X}\le\mu_0$,we have $\lambda(x)\equiv1$.So we only need to conside the case $\overline{X}>\mu_0$:

$$\lambda(x)=\sup_{\mu\in\Theta}L(\underset{\sim}{x};\mu)\Big/\sup_{\mu\in\Theta_0}L(x;\mu)=\exp\{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2\}$$

Therefore, the rejection region is:

$$W_0=\left\{\underset{\sim}{x}\,|\,\lambda(x)>\lambda_0\right\}=\left\{\underset{\sim}{x}\,|\,\overline{X}-\mu_0>C\right\}$$

where C satisfy:

$$P\{\text{reject}\,H_0\,|\,H_0\}=P\{\bar{x}-\mu_0>C\,|\,\mu=\mu_0\}=\alpha$$

That is

$$P\left\{\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}>\frac{C}{\sigma/\sqrt{n}}\,|\,\mu=\mu_0\right\}=\alpha$$

Hence

$$\frac{C}{\sigma/\sqrt{n}}=Z_\alpha,\ C=Z_\alpha\cdot\sigma/\sqrt{}$$

So we have

$$W_0=\left\{\underset{\sim}{x}\,|\,\overline{X}-\mu_0>Z_\alpha\cdot\sigma/\sqrt{n}\right\}$$

In the same way we can prove the following Theorem 2.

Theorem 2

Suppose $X\sim N(\mu,\sigma^2)$ With $\sigma^2=\sigma_0^2$ known, $\mu\in\Theta=(-\infty,+\infty)$, the GLRT rejection region for the testing issue $H_0=\mu\ge\mu_0\leftrightarrow H_1:\mu<\mu_0$ is:

$$W_0=\left\{(X_1,X_2,\cdots,X_n)\,|\,\overline{X}-\mu_0<-Z_\alpha\cdot\sigma/\sqrt{n}\right\}\qquad(2)$$

For the case with unknown variance

Theorem 3

Suppose $X\sim N(\mu,\sigma^2)$ with σ^2 unknown, $\mu\in\Theta=(-\infty,+\infty)$, the GLRT rejection region for the testing issue $H_0:\mu\le\mu_0\leftrightarrow H_1:\mu>\mu_0$ is:

$$W_0=\left\{(X_1,X_2,\cdots,X_n)\,|\,t>t_\alpha(n-1)\right\}\qquad(3)$$

Proof:

The likelihood function is:

$$L(\underset{\sim}{x};\mu,\sigma^2)=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\exp\{-\sum_{i=1}^{n}(X_i-\mu)^2\Big/2\sigma^2\}$$

When $(\mu,\sigma^2)\in\Theta$, the maximum likelihood estimation of normal distribution is $\hat{\mu}=\overline{X}$, $\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\mu})^2$

Hence

$$\sup_{(\mu,\sigma^2)\in\Theta} L(\underset{\sim}{x};\mu,\sigma^2) = \left[2\pi \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

$$\sup_{(\mu,\sigma^2)\in\Theta_0} L(\underset{\sim}{x};\mu,\sigma^2) = \begin{cases} \sup\limits_{(\mu,\sigma^2)\in\Theta} L(\underset{\sim}{x};\mu,\sigma^2), & \bar{X} \le \mu_0 \\ \left[2\pi \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2\right]^{-\frac{n}{2}} e^{-\frac{n}{2}}, & \bar{X} > \mu_0 \end{cases}$$

Because of $\bar{X} \le \mu_0$,we have

$$\lambda(x) \equiv 1$$

So we only need to conside the case $\bar{X} > \mu_0$.

$$\lambda(x) = \sup_{(\mu,\sigma^2)\in\Theta} L(\underset{\sim}{x};\mu,\sigma^2) \Big/ \sup_{(\mu,\sigma^2)\in\Theta_0} L(\underset{\sim}{x};\mu,\sigma^2) = (1+\frac{t^2}{n-1})^{\frac{n}{2}}$$

Where

$$t = t(x) = \frac{\sqrt{n(n-1)}(\bar{X}-\mu_0)}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}} = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} > 0$$

and

$$t \sim t(n-1)$$

So we get

$$W_0 = \left\{\left(X_1, X_2, \cdots, X_n\right) \mid \lambda(x) > \lambda_0\right\} = \left\{\left(X_1, X_2, \cdots, X_n\right) \mid t > C\right\}$$

where C satisfy:

$$P(\underline{X} \in W_0 \mid (\mu_0, \sigma)) = \alpha$$

If the conditions H_0 is satisfied, we have $t \sim t(n-1)$.

Hence

$$C = t_\alpha(n-1)$$

Thus The rejection region therefore of the generalized likelihood ratio test is:

$$W_0 = \left\{\left(X_1, X_2, \cdots, X_n\right) \mid t > t_\alpha(n-1)\right\}$$

In the same way we can prove the following Theorem 4.

Theorem 4: Suppose $X \sim N(\mu, \sigma^2)$ with σ^2 unknown, the GLRT rejection region for the testing issue $H_0 : \mu \ge \mu_0 \leftrightarrow H_1 : \mu < \mu_0$ is:

$$W_0 = \left\{\left(X_1, X_2, \cdots X_n\right) \mid t < -t_\alpha(n-1)\right\} \quad (4)$$

CONCLUSIONS

In this paper, by using the generalized likelihood ratio test, two conclusions are obtained:

- The rejection region of hypothesis testing for normal population mean value with the known variance (1)(2);
- The rejection region of hypothesis testing for normal population mean value with the unknown variance (3)(4).

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