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A note on certain analytic functions

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Abstract: A theorem involving analytic functions is considered and then its certain consequences are given.
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INTRODUCTION

Let H_n denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, (n, p \in N = \{1, 2, \cdots\}), \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$.

A function $f \in H_n$ is said to be in the class $S^*(\lambda)$ of starlike functions of order λ in U if it satisfies the following inequality:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda, 0 \le \lambda < 1, p \in N, z \in U. \tag{2}$$

Further, a function $f \in H_n$ is said to be in the class $N(\lambda)$ in U (see[8][9]),if it satisfies the following inequality:

$$Re\left(f'(z)\left(\frac{z}{f(z)}\right)^{(}1+\lambda)\right) > 0, 0 < \lambda < 1, z \in U. \tag{3}$$

Recently Obradovic and Owa introduced and studied the following class of analytic functions defined as follows (see[1- 3]).

DEFINITION

A function $f \in H_n$ is said to be a member of the class $B(\alpha, \mu, \lambda)$ if and only if

$$\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \lambda, (p \in N, z \in U) \tag{4}$$

for some $\mu \ge 0, 0 \le \lambda < 1, \alpha \in C$.
Note that condition (4) implies that

$$Re\left((1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu}\right) > \alpha, z \in U. \tag{5}$$

The class $B(1, \mu, \lambda)$ is the class which has been introduced and studied by Obradovic [1] (see also [4][5][6]).

To prove our main result, we need the following Lemma:
LEMMA

(see [7]). Let the function $w(z)$ be(nonconstant) analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then

$$z_0 w'(z_0)=kw(z_0), \tag{6}$$

where $k\geq 1$ is a real number.

MAIN RESULTS AND THEIR CONSEQUENCES

Theorem

Let $f\in H_n, w\in C$, $\{0\}, \mu\geq 0, 0\leq \alpha<1,$ and also let the function H be defined by

$$H(z)=\left(\frac{(1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}}{(1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}-1}\right)\left(1+\frac{zf''(z)}{f'(z)}-(1+2\mu)\frac{zf'(z)}{f(z)}+2\mu\right). \tag{7}$$

If $H(z)$ satisfies one of the following conditions:

$$Re\{H(z)\}=\begin{cases} <|w|^{-2}\ Re\{w\} & if\ Re\{w\}>0, (8)\\ \neq 0 & if\ Re\{w\}=0, (9)\\ >|w|^{-2}\ Re\{w\} & if\ Re\{w\}<0.\end{cases} \tag{10}$$

or

$$Im\{H(z)\}=\begin{cases} <|w|^{-2}\ Im\{\overline{w}\} & if\ Im\{\overline{w}\}>0, (11)\\ \neq 0 & if\ Im\{\overline{w}\}=0, (12)\\ >|w|^{-2}\ Im\{\overline{w}\} & if\ Im\{\overline{w}\}<0.\end{cases} \tag{13}$$

then

$$\left|\left((1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}-1\right)^{w}\right|<1-\lambda, \tag{14}$$

where the value of complex power in (10) is taken to be as its principal value.

Proof.

We define the function Ω by

$$\left((1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}-1\right)^{w}=(1-\lambda)\Omega(z), \tag{15}$$

where $w\in C$, $\{0\}, \mu\geq 0, 0\leq\lambda<1, z\in U, f\in H_n$.

We see clearly that the function Ω is regular in U and $\Omega(0)=0$ .Making use of the logarithmic differentiation of both sides of (11) with respect to the known complex variable z, and if we make use of equality (11) once again, then we find that

$$\begin{aligned} &wz\left((1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}-1\right)^{-1}\\ &\left((1-\alpha)(\frac{z}{f(z)})^{\mu}+\alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}-1\right)'=\frac{z\Omega'(z)}{\Omega(z)},\end{aligned} \tag{16}$$

which yields

$$H(z)=\frac{\overline{w}}{|w|^2}\frac{z\Omega'(z)}{\Omega(z)}, w\in C,\ \{0\}, z\in U. \tag{17}$$

Assume that there exists a point $z_0\in U$ such that

$$\max_{|z|<|z_0|} |\Omega(z)| = |\Omega(z_0)| = 1, z \in U. \tag{18}$$

Applying Lemma 1.1, we can then write

$$z_0\Omega'(z_0) = c\Omega(z_0), c \geq 1. \tag{19}$$

Then (13) yields

$$Re\{H(z_0)\} = Re\{\frac{\overline{w}}{|w|^2}\frac{z_0\Omega'(z_0)}{\Omega(z_0)}\} = Re\{c\overline{w}|w|^{-2}\}, \tag{20}$$

so that

$$Re\{H(z_0)\} = \begin{cases} \geq |w|^{-2}\ Re\{w\} & if\ Re\{w\} > 0, (21) \\ = 0 & if\ Re\{w\} = 0, (22) \\ \leq |w|^{-2}\ Re\{w\} & if\ Re\{w\} < 0, \end{cases} \tag{23}$$

or

$$Im\{H(z_0)\} = \begin{cases} \geq |w|^{-2}\ Im\{\overline{w}\} & if\ Im\{\overline{w}\} > 0, (24) \\ = 0 & if\ Im\{\overline{w}\} = 0, (25) \\ \leq |w|^{-2}\ Im\{\overline{w}\} & if\ Im\{\overline{w}\} < 0. \end{cases} \tag{26}$$

But the inequalities in (17) and (18) contradict, respectively, the inequalities in (8) and (9). Hence, we conclude that $|\Omega(z)| < 1$ for all $z \in U$. Consequently, it follows from (11) that

$$|((1-\alpha)(\frac{z}{f(z)})^{\mu} + \alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu} - 1)^{w}| = (1-\lambda)|\Omega(z)| < 1-\lambda, \tag{27}$$

Therefore, the desired proof is completed.

This theorem has many interesting and important consequences in analytic function theory and geometric function theory. We give some of these with their corresponding geometric properties.

First, if we choose the value of the parameter w as a real number with $w = \delta \in R$, $\{0\}$ in the theorem, then we obtain the following corollary.

Corollary

Let $f \in H_n, \delta \in R$, $\{0\}, \mu \geq 0, 0 \leq \lambda < 1,$ and let the function H be defined by (7). Also, if $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} = \begin{cases} < \frac{1}{\delta} & if\ \delta > 0, (28) \\ & \qquad orIm\{H(z)\} \neq 0, \\ > -\frac{1}{\delta} & if\ \delta < 0, \end{cases} \tag{29}$$

then

$$Re\{(1-\alpha)(\frac{z}{f(z)})^{\mu} + \alpha\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\mu}\} > 1-(1-\lambda)^{1/\delta}. \tag{30}$$

Putting $w = 1$ in the theorem, we get the following corollary.

Corollary

Let $f \in H_n(, 0 \leq \lambda < 1, z \in U,$ and let the function H be defined by (7). Also, if $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 or Im\{H(z)\} \neq 0, \tag{31}$$

then $f \in B(\alpha, \mu, \lambda)$.

Setting $w=1, \alpha=1$ and $\mu=0$ in the theorem, we have the following corollary.

Corollary

Let $f \in H_n, 0 \leq \lambda < 1, z \in U,$ and let the function H be defined by

$$H(z) = (\frac{zf'(z)}{zf'(z) - f(z)})(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}). \tag{32}$$

If $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 or Im\{H(z)\} \neq 0, \tag{33}$$

then $Re\{zf'(z)/f(z)\} > \lambda$, that is, f is starlike of order λ in U.

Setting $w=1, \alpha=1$ and $\mu=1$ in the theorem, we have the following corollary.

Corollary

Let $f \in H_n, 0 \leq \lambda < 1, z \in U,$ and let the function H be defined by

$$H(z) = (\frac{z^2 f'(z)}{z^2 f'(z) - f^2(z)})(3 + \frac{zf''(z)}{f'(z)} + 3\frac{zf'(z)}{f(z)}). \tag{34}$$

If $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 or Im\{H(z)\} \neq 0, \tag{35}$$

then $Re\{f'(z)(\frac{z}{f(z)})^2\} > \lambda$, that is, f is in $N(\lambda)$.

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