Scholars Journal of Physics, Mathematics and Statistics
Sch. J. Phys. Math. Stat. 2015; 2(3A):329-334
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Existence of positive solutions for boundary value problems for a class of nonlinear high-order differential equation

Huijun Zheng
School of Mathematics and Statistics, Northeast Petroleum University, Daqing163318, Heilongjiang P.R. China

***Corresponding Author:**
Huijun Zheng
Email: zhenghuijun02@163.com

Abstract: In this paper, we investigate a kind of boundary value problem of nonlinear high-order differential equation. Making use of the fixed point theorems on cone and by constructing the correction function, we obtain the existence of positive solutions for it.
Keywords: Nonlinear high-order differential Equation, boundary value problem, Positive solution, Fixed point theorems on cone.
Mathematics Subject Classification: O175.

INTRODUCTION

In recent years, the existence and multiplicity of solutions for boundary value problems for nonlinear high-order ordinary differential equations, especially for even number order equation were widely investigated, and gave a lot of satisfactory results on condition of the conjugate boundary conditions or simpler boundary conditions in papers [1-4].

In present paper, by constructing the correction function, we investigate a kind of boundary value problem of nonlinear high-order differential equation with different boundary conditions, and based on the theorem of Krasnosellskii we obtain the existence of positive solutions for it.

Preliminary Notes

In this paper, we concern on the existence of positive solutions for the following nonlinear higher-order boundary value problem

$$\begin{cases} (-1)^n y^{(2n)}(x) = f(x, y(x)), & 0 < x < 1 \\ y^{(i)}(0) = 0, & 0 \le i \le k-1 \\ y^{(j)}(1) = 0, & k \le j \le 2n-1 \end{cases} \tag{1}$$

where $n \ge 2, 1 \le k \le 2n-1$.

We assume that

(H_1) $f \in C([0,1] \times [0,+\infty], (-\infty,+\infty))$ is continuous and nonnegative .

(H_2) $\exists N(x) \in L^1(0,1), N(x) > 0$ and $0 < \int_0^1 g(s)N(s)ds < +\infty$ satisfying

$$f(x,y) + N(x) \ge 0$$

when $y \ge 0$ for any $x \in (0,1)$.

Let $I = [0,1], E = C[I, R]$, then E be a Banach space with $\| u \| = \max_{x \in I} | u(x) |$.

In addition, we introduce space $L^1[0,1]$ with norm $\| u \|_1 = \int_0^1 | u(x) | dx$.

Theorem

Let B be a Banach space and $K \subset B$ a cone in B, and Ω_1, Ω_2 be open bounded subsets of B with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $\Phi : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator satisfying the condition

(i) $\|\Phi y\| \le \|y\|, y \in K \cap \partial \mathbf{\Omega}_1, \|\Phi y\| \ge \|y\|, y \in K \cap \partial \mathbf{\Omega}_2$;or

(ii) $\|\Phi y\| \ge \|y\|, y \in K \cap \partial \mathbf{\Omega}_1, \|\Phi y\| \le \|y\|, y \in K \cap \partial \mathbf{\Omega}_2$.

then Φ must have at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

MAIN RESULTS

Theorem [5] Let $G(t,s)$ be Grenn function for boundary value problem (1), then

$$G(x,s) = \begin{cases} \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^s u^{n-1}(u+x-s)^{n-1} \mathrm{d}u, 0 \le s \le x \le 1 \\ \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^x u^{n-1}(u+s-x)^{n-1} \mathrm{d}u, 0 \le x \le s \le 1 \end{cases}$$

with

$$\alpha(x)\mathrm{g}(s) \le G(x,s) \le \beta(x)\mathrm{g}(s)\ , \left|\frac{\partial G(x,s)}{\partial x}\right| \le \frac{2^{n-1}}{[(n-1)!]^2} s^{n-1} = cs^{n-1}, \tag{2}$$

where $\alpha(x) = \dfrac{x^n}{2n-1}, \beta(x) = \dfrac{x^{n-1}}{n}, g(s) = \dfrac{1}{[(n-1)!]^2} s^n$.

Lemma

For Grenn function for boundary value problem (1), we have

$$G(x,s) \le \frac{1}{[(n-1)!]^2}, G(x,s) \le \frac{2n-1}{[(n-1)!]^2} \alpha(x) \tag{3}$$

for any $x, s \in [0,1]$.

Proof

We can easily obtain $G(x,s) \le \mathrm{g}(s) \le \dfrac{1}{[(n-1)!]^2}$,

$$G(x,s) = \begin{cases} \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^s u^{n-1}(u+x-s)^{n-1} \mathrm{d}u, 0 \le s \le x \le 1 \\ \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^x u^{n-1}(u+s-x)^{n-1} \mathrm{d}u, 0 \le x \le s \le 1 \end{cases}$$

$$\le \begin{cases} \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^x x^{n-1} \mathrm{d}y = \dfrac{1}{[(n-1)!]^2} x^n, 0 \le s \le x \le 1 \\ \dfrac{1}{[(n-1)!]^2} \displaystyle\int_0^x y^{n-1} \mathrm{d}y = \dfrac{1}{[(n-1)!]^2} \dfrac{x^n}{n}, 0 \le x \le s \le 1 \end{cases}$$

$$\le \frac{x^n}{[(n-1)!]^2} = \frac{2n-1}{[(n-1)!]^2} \alpha(x)$$

Lemma

If $y(x) \in C^n[0,1]$ satisfying the conditions as follows

$$\begin{cases} (-1)^n y^{(2n)}(x) = h(x), 0 < x < 1 \\ y^{(i)}(0) = 0, 0 \le i \le k-1 \\ y^{(j)}(1) = 0, k \le j \le 2n-1, \end{cases}$$

then we have $y(x) \le \| y(x) \| \alpha(x), 0 \le x \le 1$, where $h(x) \ge 0$.

Proof

By theorem 2.2 and lemma 2.1, we get

$$\| y(x) \| = \max_{0 \le x \le 1} \int_0^1 G(x,s) h(s) \mathrm{d}s \le \int_0^1 g(s) h(s) \mathrm{d}s .$$

and so we can obtain

$$y(x) = \int_0^1 G(x,s) h(s) \mathrm{d}s \ge \alpha(x) \int_0^1 g(s) h(s) \mathrm{d}s \ge \| y \| \alpha(x) .$$

Lemma

Assume that (H_1) and (H_2) hold with $\omega(x) \in C^n[0,1]$ satisfying

$$\begin{cases} (-1)^n y^{(2n)}(x) = N(x), 0 < x < 1 \\ y^{(i)}(0) = 0, \quad 0 \le i \le k-1 \\ y^{(j)}(1) = 0, k \le j \le 2n-1 \end{cases}$$

that there exists a constant C such that $\omega(x) \le C\alpha(x)$,where $N(x) > 0, 0 \le x \le 1$.

Proof

For any $x \in [0,1]$, From Lemma 2.1, we have

$$\omega(x) = \int_0^1 G(x,s) N(s) \mathrm{d}s \le \frac{(2n-1)}{[(n-1)!]^2} \| N \|_1 \alpha(x) = C\alpha(x) ,$$

where $C = \dfrac{2n-1}{[(n-1)!]^2} \| N \|_1$.

For any $x \in I$,we define the operator correction function as

$$F(x,y) = g(x,y) + N(x), g(x,y) = \begin{cases} f(x,y), y \ge 0, \\ f(x,0), y < 0. \end{cases}$$

In this paper, we concern on the modified boundary value problem

$$\begin{cases} (-1)^n y^{(2n)}(x) = F(x, y(x) - \omega(x)), 0 < x < 1 \\ y^{(i)}(0) = 0, \quad 0 \le i \le k-1 \\ y^{(j)}(1) = 0, \quad k \le j \le 2n-1 \end{cases} \tag{4}$$

where $\omega(x)$ defined by Lemma 2.3.

Lemma

If $u(x) = y(x) + \omega(x)$ is a solution for problem (4) with $y(x) \ge 0$ for any $x \in [0,1]$, then $y(x)$ must be the positive solution for problem (1).

Proof

For any $x \in [0,1]$,if $u(x) = y(x) + \omega(x)$ is a solution for problem (4),then by definition of $F(x,y)$,we have

$$\begin{cases} (-1)^n [y^{(2n)}(x) + \omega^{(2n)}(x)] = F(x, y(x)), 0 < x < 1 \\ (y+\omega)^{(i)}(0) = 0, 0 \le i \le k-1 \\ (y+\omega)^{(j)}(1) = 0, k \le j \le 2n-1 \end{cases} \tag{5}$$

that is

$$\begin{cases} (-1)^n y^{(2n)}(x) = f(x, y(x)), 0 < x < 1 \\ y^{(i)}(0) = 0, 0 \le i \le k-1 \\ y^{(j)}(1) = 0, k \le j \le 2n-1 \end{cases} \tag{6}$$

So $y(x)$ is the positive solution for problem (1).

Clearly, the problem (4) is equivalent to the integral equation

$$y(x) = \int_0^1 G(x,s)F(s, y(s) - \omega(s))\mathrm{d}s$$

We define the operator $\Phi{:}K \to K$ by

$$(\Phi y)(x) = \int_0^1 G(x,s)F(s, y(s) - \omega(s))\mathrm{d}s$$

We define the cone $K = \{y \in E : y(x) \ge \| y \| \alpha(x), x \in [\theta, 1-\theta]\}$, Let $\sigma = \dfrac{\theta^n}{2(2n-1)}$.

Lemma

Assume that (H_1) and (H_2) hold,then $\Phi(K) \subset K$,and $\Phi : K \to K$ is completely continuous.

Proof

For any $y \in K, x \in [0,1]$, from(2)and(3)we have

$$\| \Phi y \| \le \int_0^1 g(s)F(s, y(s) - \omega(s))\mathrm{d}s$$

thus

$$\Phi y(x) \ge \alpha(x)\int_0^1 g(s)F(s, y(s) - \omega(s))\mathrm{d}s \ge \| \Phi y(x) \| \alpha(x)$$

This leads to $\Phi y(x) \ge \alpha(x) \| \Phi y \|$. Thus we get $y \in K$, that is, $\Phi(K) \subset K$.

By the Ascoli-Arzel as theorem, we can easily see that $\Phi : K \to K$ is completely continuous.

Theorem

Assume that (H_1) and (H_2) hold, with the function f satisfying

（Ⅰ） $\forall (x, y) \in [0,1] \times [\sigma R, R]$, such that $f(x, y) \ge MR$;

（Ⅱ） $\forall (x, y) \in [0,1] \times [0, r]$, such that $f(x, y) \le mr$,

for any $C < r < 2C < R$,where $m \le (\int_0^1 (g(s) + N(s))ds)^{-1}, M \ge (2\sigma \int_\theta^{1-\theta} g(s)ds)^{-1}$ and $mr \ge 1$,then problem (1) has at least one positive solution $y \in K$ satisfying $r \le y \le R$.

Proof

Let $\Omega_1 = \{y \in K : \| y \| < R\}$, then for any $x \in \partial\Omega_1, x \in [0,1]$ we have

$$y(x) - \omega(x) \ge y(x) - C\alpha(x) \ge y(x)(1 - \frac{C}{R}) \ge y(x)(1 - \frac{C}{2R}) \ge \frac{1}{2} \| y \| \alpha(x) = \frac{1}{2}\alpha(x)R .$$

And for any $x \in \partial\Omega_1, x \in [\theta, 1-\theta]$,we have

$$\sigma R = \frac{\theta^n}{2(2n-1)} R \le \frac{R}{2}\alpha(x) \le x(x) - \omega(x) \le \| x \| = R .$$

Thus, we obtain

$$\begin{aligned} \|\Phi y(x)\| &\ge \int_\theta^{1-\theta} G(x,s)F(s, y(s) - \omega(s))\mathrm{d}s \ge \int_\theta^{1-\theta} G(x,s)[p(s) f(s, y(s) - \omega(s))]\mathrm{d}s \\ &\ge \alpha(x)MR\int_\theta^{1-\theta} g(s)\mathrm{d}s \ge 2\sigma(2\sigma\int_\theta^{1-\theta} g(s)\mathrm{d}s)^{-1} R \cdot \int_\theta^{1-\theta} g(s)\mathrm{d}s = R = \| y \|. \end{aligned}$$

Let $\Omega_2 = \{y \in K : \| y \| < r\}$,then for any $y \in \partial\Omega_2, x \in [0,1]$,we can see that

$$y(x)-\omega(x)\le y(x)\le r$$

and

$$y(x)-\omega(x)\ge y(x)-C\alpha(x)\ge y(x)(1-\frac{C}{r})\ge 0$$

Therefore $\|\Phi y(x)\|\le \int_0^1 g(s)F(s,u(s)-\omega(s))\mathrm{d}s$

$$\le \int_0^1 [g(s)mr+N(s)]\mathrm{d}s = \| y \| \le mr\int_0^1 [\mathrm{g(s)}+N(s)]\mathrm{d}s \le r = \| y(x) \|$$

Thus, problem (1) has at least one positive solution $y\in K$ satisfying $r\le y\le R$.

From Lemma 2.3, we can see for any $\theta\le x\le 1-\theta$ that

$$u(x)\ge \| u \| \alpha(x)\ge r\alpha(x)>C\alpha(x)\ge \omega(x)$$

that is, $y(x)=u(x)-\omega(x)$ is the positive solution for problem (1).

Theorem

Assume that (H_1) and (H_2) hold with

(Ⅲ) $2(\sigma^2\int_\theta^{1-\theta} g(s)ds)^{-1} < \lim\limits_{y\to+\infty}\min\limits_{x\in[0,1]}\frac{f(x,y)}{y} < +\infty$;

(Ⅳ) $0\le \lim\limits_{y\to 0}\max\limits_{x\in[0,1]}\frac{f(x,y)}{y} < (\int_0^1 [g(s)+\omega(s)]ds)^{-1}$,

then problem (1) has at least one positive solution $y\in K$ satisfying $r\le y\le R$.

Proof

Let $A=\lim\limits_{y\to+\infty}\min\limits_{x\in[0,1]}\frac{f(x,y)}{y}$. Owing to condition(Ⅲ) we can see by taking

$$\varepsilon = A-2(\sigma^2\int_\theta^{1-\theta} g(s)ds)^{-1}>0,$$

that there exists a sufficiently large number $R\ne r$, such that

$$f(x,y)\ge (A-\varepsilon)y\ge (\sigma^2\int_\theta^{1-\theta} g(s)ds)^{-1}\sigma R=(\sigma\int_\theta^{1-\theta} g(s)ds)^{-1}R$$

for any $y\ge \sigma R$.

Let $M=(\sigma\int_\theta^{1-\theta} g(s)ds)^{-1}$, then we have

$$M>(2\sigma\int_\theta^{1-\theta} g(s)ds)^{-1} \text{ and } f(x,y)\ge MR, \sigma R\le y\le R.$$

Let $B=\lim\limits_{y\to 0}\max\limits_{x\in[0,1]}\frac{f(x,y)}{y}$. Owing to condition (IV), we can see by taking $\varepsilon=(\int_0^1 [g(s)+\omega(s)]ds)^{-1}-B>0$, For any $0\le y\le \rho$, if $y\ne 0$, we get

$$f(x,y)\le (B+\varepsilon)y\le (\int_0^1 [g(s)+\omega(s)]ds)^{-1}\rho$$

By letting $m=(\int_0^1 [g(s)+\omega(s)]ds)^{-1}$, $r=\rho$, we get $f(x,y)\le mr, 0\le y\le r$.

Thus, we see from Theorem 2.3 that problem (1) has at least one positive solution $y\in K$ satisfying $r\le y\le R$.

Acknowledgement: Project supported by Heilongjiang province education department natural science research item, China.
(12541076).

REFERENCES

1. Zhong M, Zhang X; Positive solutions of singularly perturbed (k,n-k) conjugate boundary value problems. Acta Math. Sci. Ser. A Chin. Ed, 2011; 31: 263-272.

2. Lingbin Kong, Junyu Wang; The Green's Function for (k,n-k) Conjugate Boundary Value Problems and Its Applications[J].Journal of Mathematical Analysis and Applications, 2001; 255: 404-422.
3. Yao Q; Positive solution to a singular (k,n-k) conjugate boundary value problem[J]. Appl. Math. Bohem, 2011; 136: 69-79.
4. Yang X; Green's function and positive solutions for higher-order ODE[J].Appl.Math. Computation, 2003; 136: 379-393.
5. Huijun Zheng, Lingbin Kong; The Green function and positive solution for 2m order nonlinear boundary value problems. Journal of daqing petroleum institute, 2010; 34: 106-110.