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The minimum group congruence on a GV-semi group

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Abstract: A GV-semi group is a semi group in which some power of each element is regular and every regular element lies in a subgroup. The minimum group congruence on a GV-semi group is investigated by means of establishing a subset and a binary relation. A description of the minimum group congruence on a GV-semi group is given. Moreover, the particular Transparent description on the minimum group congruence is obtained in the case of a completely regular semi group.
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1.Introduction

Throughout this paper, we follow the notation and conventions of Howie [6]. Recall that a semi group is said to be eventually regular if each of its element has some power is regular. In particular, every regular element of an eventually regular semi group lies in a subgroup, then it is said to be a GV-semi group. From the definition of GV-semi groups, we conclude that the case of GV-semi groups is the special case of eventually regular semi groups. Edward, P. M [2] was successful in showing that many results for regular semi groups can be obtained for eventually regular semi groups. The strategy to study eventually regular semi groups is to generalize known results for regular semi groups to eventually regular semi groups. Group congruences on regular semi groups have been investigated by many algebraists. Latorre, D. R [5] explored group congruences on regular semi groups extensively and gave the representation of group congruences on regular semi groups. Hanumantha R.S [4] generalized the results in [5] for regular semi groups to eventually regular semi groups. Moreover, group congruences on E-inversive semi groups were studied in [7, 8].

In this paper, the author explores the minimum group congruences on a GV-semi group by means of establishing a subset and a binary relation. The minimum group congruence on a GV-semi group is characterized. Moreover, the particular transparent description on the minimum group congruence is obtained in the case of a completely regular semi group.

2.Preliminaries

Let S be a semi group and $a \in S$. As usual, E_S is the set of all idempotents of S, $< E_S >$ is the sub semigroup of S generated by E_S and N the positive integers. An element x of S is called a weak inverse of a if $xax = x$. We denote by $W(a)$ the set of all weak inverses of a in S, by $r(a)$ the least positive integer n such that a^n is a regular element. Let ρ be congruence on a semigroup S. Then ρ is called group congruence if the quotient S/ρ is a group. In particular, a congruence ρ is said to be the minimum group congruence if S/ρ is the maximum group morphic image of S. For a congruence ρ of S, the subset $\{a \in S \mid a\rho \in E(S/\rho)\}$ of S is called the kernel of ρ denoted by $\ker \rho$.

Let S be a GV-semi group and H be a subset of S. Then H is called self-conjugate if for any $a \in S$, there exists $(a^n)' \in V(a^n), r(a) = n$ such that $aHa^{n-1}(a^n)' \subseteq H, a^{n-1}(a^n)'Ha \subseteq H$. The subset $H\omega$ is called closure of H if $H\omega = \{x \in S \mid \exists h \in H, hx \in H\}$. In particular, if $H\omega = H$, then H is said to be closed. A subset

H of S is called full if $E_S \subseteq H$. Moreover, a subsemigroup K is called weak self-conjugate if for any $a \in S, a^{/} \in W(a)$, there exist $a^{/}Ka \subseteq K, aKa^{/} \in K$.

For a subset H of S, a binary relation is defined on S named σ_H as: $\sigma_H = \{(a,b) \in S \times S : \exists b^{/} \in W(b), ab^{/} \in H\}$. For a subset E_S of S, a binary relation is denoted on S named R as: $R = \{(a,b) \in S \times S : \exists x, y \in < E_S >, xa = by\}$

We give some lemmas which will be used in the sequel.

Lemma 2.1 ([2, 3]) Let S be an eventually regular semi group and ρ be a congruence on S. If $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in S such that $a\rho e$.

Remark. Since S is an eventually regular semi group and ρ is group congruence on S, $x\rho$ is an idempotent of S/ρ for all $x \in < E_S >$.

Lemma 2.2 ([6]) For an arbitrary binary relation B on a semi group S, define

$B^C = \{(xay, xby) \in SxS : \exists x, y \in S^{'}, (a,b) \in B\}$ and $B^{\#} = \bigcup_{n=1}^{\infty}[B^C \cup (B^C)^{-1} \cup 1]^n$. Then the following statements are true:

(1) B^C Is the smallest left and right compatible relation containing B;

(2) $B^{\#}$ Is the minimum congruence containing B.

Lemma 2.3 ([7]) Let S be a regular semigroup with a unique idempotent, and then S is a group.

Lemma 2.4 ([1]) Let S be a semi group and Ge be a subgroup of S with the identity e. If $a^n \in Ge$ for some positive integer n, then $ea = ae \in Ge$

3.Main Results

We begin the section with the main result of this paper.

Theorem 3.1 Let S be a GV-semi group and R be a binary relation on S defined in Preliminaries. Then the following statements are true:

(1) If $< E_S >$ satisfies self-conjugate condition, then R is the minimum group congruence on S.

(2) If $< E_S >$ does not satisfy self-conjugate condition, then $R^{\#}$ is the minimum group congruence on S.

We shall prove Theorem 3.1 preceded by a sequence of lemmas.

Lemma 3.2 Let S be a GV-semi group and $a, b \in S$. If the sub semigroup H of S is weak self-conjugate, closed and full, then the following statements are equivalent:

(1) $a\sigma_H b$;

(2) $ab^{/} \in H, ba^{/} \in H$ for $b^{/} \in W(b), a^{/} \in W(a)$;

(3) $b^{/}a \in H$ For $b^{/} \in W(b)$.

Proof. (1)$\Rightarrow$(2) Suppose $a\sigma_H b$ for $a, b \in S$, then there exists $a^{//} \in W(a)$ such that $ab^{//} \in H$, and so $ab^{//}bb^{/} \in H$ for $b^{/} \in W(b)$. For any $a^{/} \in W(a), a^{/}a \in E_S$, it follows from Lemma 2.3 that $ab^{//}b(a^{/}a)b^{/} = ab^{//}ba^{/}(ab^{/}) \in H$. Since H is weak self-conjugate, closed and full, we deduce $ab^{//}ba^{/} \in H$, so that $ab^{/} \in H$. In a

Similar way, we prove $ba^{/} \in H$ for $a^{/} \in W(a)$.

(2) ⇒ (3) Using the statement (2), we conclude that there exists $b^{/} \in W(b)$ such that $ab^{/} \in H$. Since H is weak self-conjugate, we obtain $a^{/}ab^{/}a \in H$ and $a^{/}a \in E_S \subseteq H$,

so that $b^{/}a \in H$.

(3) ⇒ (1) For $a, b \in S$, there exists $b^{/} \in W(b)$ such that $b^{/}a \in H$. From the weak self-conjugate of H , we deduce $bb^{/}ab^{/} \in H$ and $bb^{/} \in H$. And since H is closed, we have $ab^{/} \in H$, which leads to $a\sigma_H b$.

Lemma 3.3 Let S be a GV-semi group and R be a binary relation on S defined in Preliminaries. If $<E_S>$ satisfies self-conjugate condition, then R is the minimum group congruence on S .

Proof. We first show that R is equivalence on S . Let $a \in S$, then there exists $m \in N$ such that $a^m \in Ge$, where Ge is a subgroup of S with the identity e . It follows from lemma 2.4 that $a.e = e.a$, and so aRa , which leads to R is reflexive. To prove the symmetry, suppose aRb for $a, b \in S$. Then there exist $x, y \in <E_S>$ such that $xa = by$. Since $<E_S>$ is self-conjugate, together with $b^{m-1}(b^m)^{'}b \in <E_S>,\ (b^m)^{'} \in V(b^m),\ m = r(b)$,we deduce $a^{n-1}(a^n)^{'}xa \in <E_S>, (a^n)' \in V(a^n), n = r(a),$ hence $a^{n-1}(a^n)'xab^{m-1}(b^m)'b \in <E_S>$, and so $a[a^{n-1}(a^n)'xab^{m-1}(b^m)'b] = a^n(a^n)'xab^{m-1}(b^m)'b = [a^n(a^n)'byb^{m-1}(b^m)']b$,
Where $a^n(a^n)', byb^{m-1}(b^m)' \in E_S$. It follows that $a^n(a^n)'byb^{m-1}(b^m)' \in <E_S>$, and so bRa , which infers that R is symmetry. To prove the transitivity, let aRb, bRc .
Then there exist $x, y, s, t \in <E_S>$ such that $xa = by,\ sb = ct$, so that $c(ty) = sby = (sx)a,\ ty,\ sx \in <E_S>$, which leads to aRc . Hence R is transitive, and so R is an equivalence.

We now turn to showing that R is congruence. Let $aRb,\ a, b \in S$. Then there exist $x, y \in <E_S>$ such that $xa = by$. Since $<E_S>$ is self-conjugate, we conclude $c^m(c^m)' \in <E_S>$ and $bc^m(c^m)'b^{n-1}(b^n)' \in <E_S>$ where $m = r(c),\ n = r(b)$, $c \in S$. Notice that
$[bc^m(c^m)'b^{n-1}(b^n)'x]ac = bcc^{m-1}(c^m)'b^{n-1}(b^n)'byc = bc[c^{m-1}(c^m)'b^{n-1}(b^n)'byc]$.
For $(c^m)' \in V(c^m),\ (b^n)' \in V(b^n)$, we get
$b^{n-1}(b^n)'by \in <E_S>, c^{m-1}(c^m)'b^{n-1}(b^n)'byc \in <E_S>$, hence $acRbc$, and so R satisfies right compatible. On the other hand, a similar argument will show that R satisfies left compatible. Thus R is a congruence on S .

We then show that R is group congruence on S . For $a \in S$, it follows from lemma 2.4 that there exists $m \in N,\ e \in E_S$ such that $a^m \in Ge$ and $ae = ea \in Ge$, hence $ea = eea = (ea)e$, and so $a\,\mathrm{Re}\,a$. And since ea is a regular element, R is a regular congruence, which implies that S/R is a regular semigroup. Moreover, it is obvious that $e(f) = (e)f$ for $e, f \in E_S$, and so eRf . It follows from lemma 2.1 that there exists $e \in E_S$ such that $a\,\mathrm{Re}$ for $aR \in E(S/R)$, so that S/R only has an idempotent. We, By lemma 2.3, conclude that S/R is a group, so that R is a group congruence on S .

We finally show that R is the minimum group congruence on S . Let aRb for $a, b \in S$ and ρ be any group congruence on S with $e\rho$ as the unique idempotent of S/ρ . Then there exist $x, y \in <E_S>$ such that $xa = by$, and so $(xa)\rho = x\rho.a\rho = (by)\rho = b\rho.y\rho.$ It follows from lemma 2.1 and the remark that $x\rho = y\rho = e\rho$ for

$x, y \in < E_S >$, and so $(xa)\rho = (ea)\rho = a\rho = (by)\rho = b\rho e\rho = be$, so that $a\rho b$. Thus we claim $R \subseteq \rho$, and so R is the minimum group congruence on S, as required.

Lemma 3.4 Let S be a GV-semi group and R be a binary relation on S defined in Preliminaries. If $< E_S >$ does not satisfy self-conjugate condition, then $R^{\#}$ is the minimum group congruence on S.

Proof. It follows from lemma 2.2 that $R^{\#}$ is congruence on S. We now show that $R^{\#}$ is group congruence on S. For $e, f \in E_S$, we see $e(f) = (e)f$, and so eRf, so that $eR^{\#} f$. Thus we, by lemma 2.1, conclude that $S/R^{\#}$ only has an idempotent. The fact that $R^{\#}$ is a regular congruence is proved as the same technique in lemma 3.3. Therefore it follows from lemma 2.3 that $R^{\#}$ is a group congruence on S.

We then turn to showing that $R^{\#}$ is the minimum group congruence on S. Let aRb for $a, b \in S$ and ρ be any group congruence on S with $e\rho$ as the unique idempotent of S/ρ. Then there exist $x, y \in < E_S >$ such that $xa = by$. Hence $(xa)\rho = (by)\rho = xa$, $a\rho = (ea)\rho = (by)\rho = b\rho \cdot e\rho$, and so $a\rho b$, which implies $R \subseteq \rho$. It follows from lemma 2.2 that $R^{\#}$ is the minimum congruence containing R, hence $R^{\#} \subseteq \rho$, and so $R^{\#}$ is the minimum group congruence on S.

Theorem 3.1 is a direct consequence of Lemma 3.2, Lemma 3.3 and Lemma 3.4. As a specialization of Theorem 3.1, the following result is immediate.

Theorem 3.5 Let S is a completely regular semigroup. Then the following statements are true:
(1) If $< E_S >$ satisfies self-conjugate condition, then R is the minimum group congruence on S ;
(2) If $< E_S >$ does not satisfy self-conjugate condition, then $R^{\#}$ is the minimum group congruence on S.

Remark In view of the proof in lemma 3.3, we observe that $R^{\#}$ (R) is the least group congruence on S when S is a GV-semi group.

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References

1. Bogdanovic S; Semi groups with a system of Subsemigroups, Novi Sad: Novi Sad University press, 1985; 27-110.
2. Edwards PM; Eventually regular semi group, Bull. Austral, math. Soc., 1983; 28: 23-38.
3. Edwards PM; Congruence and Greens relations on eventually regular semigroups,J, Austral. Soc., 1987; 43: 64-69.
4. Hanumantha RS; Group congruences on eventually regular semigroup, J, Austral. Soc., 1988; 45: 320-325.
5. Latorre DR; Group congruences on regular semigroup. Semi group Forum, 1982; 24: 327-340.
6. Howie JM; Fundamentals of semi groups theory, Oxford: Clarendon Press, 1995; 45-200.
7. Weipoltshammer B; Certain congruences on E-inverse E-semi groups, Semi group Forum, 2002; 65: 233-248.
8. Zheng HW; Group congruences on an E-inverse semi group, Southwest Asia Bull. Math, 1997; 21: 1-8.