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On The Binary Quadratic Diophantine Equation x^2-$4xy$+y^2+$14x$=0

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Abstract: The binary quadratic equation $x^2-4xy+y^2+14x=0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.
Keywords: Binary quadratic equation, integral solutions.

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1, 2, 3, 4, 5, 6]. In the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $x^2-4xy+y^2+14x=0$.The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2-4xy+y^2+14x=0 \tag{1}$$

Note that (1) is satisfied by the following non-zero integer pairs

$(1,7),(4,-56),(3,7),(4,-14)$

However, we have other solutions for (1), which are illustrated below:

Solving (1) for y, we have

$$y=2x\pm\sqrt{3x^2-14x} \tag{2}$$

Let $\alpha^2=3x^2-14x$

Multiplying the above equation by 3 on both sides and performing a few calculations, we have

$$X^2=3\alpha^2+49 \tag{3}$$

where $X=3x-7$ (4)

The least positive integer solution of (3) is

$$\alpha_0=7, X_0=14$$

Now, to find the other solution of (3),consider the pellian equation

$$X^2=3\alpha^2+1 \tag{5}$$

whose fundamental solution is

$$\left(\tilde{\alpha}_0,\tilde{X}_0\right)=(1,2)$$

The other solutions of (5) can be derived from the relations

$$\tilde{X}_n=\frac{f_n}{2},\tilde{\alpha}_n=\frac{g_n}{2\sqrt{3}}$$

where

$$f_n = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}$$
$$g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$$

Applying the lemma of Brahmagupta between (α_0, X_0) & $(\tilde{\alpha}_n, \tilde{X}_n)$, the other solutions of (3) can be obtained from the relation

$$\alpha_{n+1} = \frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n \tag{6}$$

$$X_{n+1} = 7 f_n + \frac{21}{2\sqrt{3}} g_n \tag{7}$$

Taking positive sign on the R.H.S of (2) and using (4),(6)&(7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

$$x_{n+1} = \frac{1}{3}\left(7 f_n + \frac{21}{2\sqrt{3}} g_n + 7\right) \tag{8}$$

$$y_{n+1} = 2x_{n+1} + \left(\frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n\right), n = -1,1,3,5... \tag{9}$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$6x_{n+1} - 84x_{n+3} + 6x_{n+5} = -168$$
$$6y_{n+1} - 84y_{n+3} + 6y_{n+5} = -336$$

A few numerical examples are given in table below

Table 1: Numerical Solutions

n	x_{n+1}	y_{n+1}
-1	7	21
1	63	231
3	847	3157
5	11767	43911

Some relations satisfied by the solutions (8) & (9) are as follows

1. $x_{n+3} = -x_{n+1} + 4y_{n+1} - 14$
2. $x_{n+5} = 56y_{n+1} - 15x_{n+1} - 224$
3. $y_{n+3} = 15y_{n+1} - 4x_{n+1} - 56$
4. $y_{n+5} = 209y_{n+1} - 56x_{n+1} - 840$
5. $x_{n+1} = 15x_{n+3} - 4y_{n+3} - 14$
6. $x_{n+5} = 4y_{n+3} - x_{n+3} - 14$
7. $y_{n+1} = 4x_{n+3} - y_{n+3}$
8. $y_{n+5} = -4x_{n+3} + 15y_{n+3} - 56$
9. $x_{n+1} = 209x_{n+5} - 56y_{n+5} - 224$
10. $x_{n+3} = 15x_{n+5} - 4y_{n+5} - 14$
11. $y_{n+1} = 56x_{n+5} - 15y_{n+5} - 56$
12. $y_{n+3} = 4x_{n+5} - y_{n+5}$
13. Each of the following expressions is a nasty number

i) $24x_{2n+2} - 6y_{2n+2} - 14$

ii) $2352x_{2n+4} - 630y_{2n+4} - 2450$

REMARKABLE OBSERVATIONS

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 1) Define $X = 24x_{n+1} - 6y_{n+1} - 28,\ Y = 12y_{n+1} - 42x_{n+1} + 42$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 3X^2 - 12 \times 7^2$

Example 2) Define $X = 2352x_{n+3} - 630y_{n+3} - 2548,\ Y = 1092y_{n+3} - 4074x_{n+3} + 4410$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 3X^2 - 12 \times 49^2$

2) By considering suitable linear transformations between the solutions of (1) , one may get integer solutions for parabolas

Example 3) Define $X = 24x_{n+1} - 6y_{n+1} - 28,\ Y = 12y_{n+1} - 42x_{n+1} + 42$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 3 \times 7X - 12 \times 7^2$

Example 4) Define $X = 2352x_{n+3} - 630y_{n+3} - 2548,\ Y = 1092y_{n+3} - 4074x_{n+3} + 4410$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 49 \times 3X - 12 \times 49^2$

Solving (1) for x , we have

$$x = 2y - 7 \pm \sqrt{3y^2 - 28y + 49} \tag{10}$$

Let $\alpha^2 = 3y^2 - 28y + 49$

Multiplying the above equation by 3 on both sides and performing a few calculations, we have

$$Y^2 = 3\alpha^2 + 49 \tag{11}$$

where $Y = 3y - 14$ (12)

The least positive integer solution of (3) is

$$\alpha_0 = 7, Y_0 = 14$$

Now, to find the other solution of (11), consider the pellian equation

$$Y^2 = 3\alpha^2 + 1 \tag{13}$$

whose fundamental solution is

$$(\tilde{\alpha}_0, \tilde{Y}_0) = (1,2)$$

The other solutions of (13) can be derived from the relations

$$\tilde{Y}_n = \frac{f_n}{2}, \tilde{\alpha}_n = \frac{g_n}{2\sqrt{3}}$$

where

$$f_n = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}$$

$$g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$$

Applying the lemma of Brahmagupta between (α_0, Y_0) & $(\tilde{\alpha}_n, \tilde{Y}_n)$, the other solutions of (11) can be obtained from the relation

$$\alpha_{n+1} = \frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n \tag{14}$$

$$Y_{n+1} = 7 f_n + \frac{21}{2\sqrt{3}} g_n \tag{15}$$

Taking positive sign on the R.H.S of (10) and using (12),(14)&(15), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

$$y_{n+1} = \frac{1}{3}\left(7 f_n + \frac{21}{2\sqrt{3}} g_n + 14\right) \tag{16}$$

$$x_{n+1} = 2y_{n+1} - 7 + \left(\frac{7}{2} f_n + \frac{7}{\sqrt{3}} g_n\right), \quad n = 0,2,4,6... \tag{17}$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$6x_{n+1} - 84x_{n+3} + 6x_{n+5} = -168$$

$$6y_{n+1} - 84y_{n+3} + 6y_{n+5} = -336$$

A few numerical examples are given in table below

Table 2: Numerical Solutions

n	x_{n+1}	y_{n+1}
0	63	21
2	847	231
4	11767	3157

Some relations satisfied by the solutions (16) & (17) are as follows

1. $y_{n+3} = 4x_{n+1} - y_{n+1}$
2. $y_{n+5} = 56x_{n+1} - 15y_{n+1} - 56$
3. $x_{n+3} = 15x_{n+1} - 4y_{n+1} - 14$
4. $x_{n+5} = 209x_{n+1} - 56y_{n+1} - 224$
5. $y_{n+1} = 15y_{n+3} - 4x_{n+3} - 56$
6. $y_{n+5} = 4x_{n+3} - y_{n+3}$
7. $x_{n+1} = 4y_{n+3} - x_{n+3} - 14$
8. $x_{n+5} = -4y_{n+3} + 15x_{n+3} - 14$
9. $y_{n+1} = 209y_{n+5} - 56x_{n+5} - 840$
10. $y_{n+3} = 15y_{n+5} - 4x_{n+5} - 56$
11. $x_{n+1} = 56y_{n+5} - 15x_{n+5} - 224$
12. $x_{n+3} = 4y_{n+5} - x_{n+5} - 14$
13. Each of the following expressions is a nasty number

i) $24y_{2n+2} - 6x_{2n+2} - 84$

ii) $2352y_{n+3} - 630x_{n+3} - 9408$

REMARKABLE OBSERVATIONS

By considering suitable linear transformations between the solutions of (1), one may get integer solutions for hyperbolas

Example 5) Define $X = 24y_{n+1} - 6x_{n+1} - 98$, $Y = 12x_{n+1} - 42y_{n+1} + 168$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 3X^2 - 12 \times 7^2$

Example 6) Define $X = 2352y_{n+3} - 630x_{n+3} - 9506$, $Y = 1092x_{n+3} - 4074y_{n+3} + 16464$

Note that the pair (X, Y) satisfies the hyperbola $Y^2 = 3X^2 - 12 \times 49^2$

By considering suitable linear transformations between the solutions of (1) , one may get integer solutions for parabolas

Example 7) Define $X = 24y_{n+1} - 6x_{n+1} - 98$, $Y = 12x_{n+1} - 42y_{n+1} + 168$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 7 \times 3X - 12 \times 7^2$

Example 8) Define $X = 2352y_{n+3} - 630x_{n+3} - 9506$, $Y = 1092x_{n+3} - 4074y_{n+3} + 16464$

Note that the pair (X, Y) satisfies the parabola $Y^2 = 49 \times 3X - 12 \times 49^2$

CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

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