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Skillful application of difference equation in a mathematical puzzle

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Abstract: Unimportant and uncertain decisions can be made by flipping a coin in real-life. And apparently, toss a coin is a question of random probability. In the present paper, the author has made use of difference equation in solving a complicated probability problem of coin toss.
Keywords: flip a coin, question of probabilities, difference equation.

INTRODUCTION

There exists such a classic question of probabilities [1]: On average, how many times do we flip a coin before two continuous heads could appear firstly? It is a natural expansion to this problem. On average, how many times do we flip a coin [2-4] before m continuous heads could appear firstly? Obviously, the bigger the n-value, the more relatively complex the calculation of average times. It even evolved into some algebraic formulas that is difficult to simplify. The really interesting is that the natural conclusion is surprisingly simple. The answer is $2^{m+1}-2$. In the present paper, this mathematical puzzle is solved by using of difference equation [5-6] skillfully, and also avoids multiple Calculations.

Firstly, when the coin comes to rest head up, the tail must be underneath, and the probability that an unbiased coin will fall with the head up is $\frac{1}{2}$.

Then, defining the following variables.

P_n is the probability that m continuous heads are obtained by flipping a coin n times;

And apparently, $P_1 = P_2 = \cdots = P_{m-2} = P_{m-1} = 0, P_m = (\frac{1}{2})^m$.

$E(n)$ represents the mathematical expectation that is How many times do we flip a coin before m continuous heads could appear firstly?

$$E(n) = \sum_{n=m}^{\infty} nP_n.$$

X_{in} is the probability of having obtained $(i-1)$ continuous heads but reversed in ith times, then use $(n-i)$ more times to obtain m continuous heads;

$$X_{in} = \left(\frac{1}{2}\right)^{i-1}\left(1-\frac{1}{2}\right)P_{n-i} = \left(\frac{1}{2}\right)^{i} P_{n-i}.$$

Furthermore, we have the difference equation as following:

$$P_n = \sum_{i=1}^{m} X_{in} = \sum_{i=1}^{m}\left(\frac{1}{2}\right)^{i} P_{n-i}.$$

$$P_{n+1} = \sum_{i=1}^{m} X_{in} = \sum_{i=1}^{m}\left(\frac{1}{2}\right)^{i} P_{n-i+1}.$$

Above all, we only need to seek out $E(n)$.

2 Solving the $E(n)$ by the difference equation

Firstly, we prove the formula $\sum_{n=m}^{\infty} P_n = 1$.

We suppose the $\sum_{n=m}^{\infty} P_n = A$.

From the difference equation,

$$P_{n+1} = \sum_{i=1}^{m} X_{in} = \sum_{i=1}^{m} \left(\frac{1}{2}\right)^i P_{n-i+1} \tag{1}$$

We have,

$$\sum_{n=m}^{\infty} P_{n+1} = \frac{1}{2}\sum_{n=m}^{\infty} P_n + \left(\frac{1}{2}\right)^2 \sum_{n=m}^{\infty} P_{n-1} + \cdots + \left(\frac{1}{2}\right)^m \sum_{n=m}^{\infty} P_{n-m+1} \tag{2}$$

As the result of $P_1 = P_2 = \cdots = P_{m-2} = P_{m-1} = 0, P_m = (\frac{1}{2})^m$, we easily have

$$\sum_{n=m}^{\infty} P_{n-1} = \cdots = \sum_{n=m}^{\infty} P_{n-m+1} = \sum_{n=m}^{\infty} P_n = A \tag{3}$$

Then the formula (2) becomes

$$A - \left(\frac{1}{2}\right)^m = [\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^m]A \tag{4}$$

Therefore we solve $A = 1$.

Secondly, we find the $E(n) = \sum_{n=m}^{\infty} nP_n$, our tool is also the difference equation. The calculation procedures are as follows:

We suppose the $E(n) = \sum_{n=m}^{\infty} nP_n = B$.

From the difference equation, we have

$$\sum_{n=m}^{\infty} nP_{n+1} = \frac{1}{2}\sum_{n=m}^{\infty} nP_n + \left(\frac{1}{2}\right)^2 \sum_{n=m}^{\infty} nP_{n-1} + \cdots + \left(\frac{1}{2}\right)^m \sum_{n=m}^{\infty} nP_{n-m+1} \tag{5}$$

$$\sum_{n=m}^{\infty} (n+1)P_{n+1} - \sum_{n=m}^{\infty} P_{n+1} = \frac{1}{2}\sum_{n=m}^{\infty} nP_n + \left(\frac{1}{2}\right)^2 [\sum_{n=m}^{\infty} (n-1)P_{n-1} + \sum_{n=m}^{\infty} P_{n-1}]$$
$$+ \cdots + \left(\frac{1}{2}\right)^m [\sum_{n=m}^{\infty} (n-m+1)P_{n-m+1} + (m+1)\sum_{n=m}^{\infty} P_{n-m+1}] \tag{6}$$

Because of $P_1 = P_2 = \cdots = P_{m-2} = P_{m-1} = 0, P_m = (\frac{1}{2})^m$, we have

$$\sum_{n=m}^{\infty} P_{n-1} = \cdots = \sum_{n=m}^{\infty} P_{n-m+1} = \sum_{n=m}^{\infty} P_n = 1 \tag{7}$$

$$\sum_{n=m}^{\infty} (n-1)P_{n-1} = \cdots = \sum_{n=m}^{\infty} (n-m+1)P_{n-m+1} = \sum_{n=m}^{\infty} nP_n = B \tag{8}$$

Then the formula (6) becomes

$$B-m\left(\frac{1}{2}\right)^{m}-(1-\left(\frac{1}{2}\right)^{m})=\frac{1}{2}B+\left(\frac{1}{2}\right)^{2}(B+1)+\cdots+\left(\frac{1}{2}\right)^{m}[B+(m-1)] \quad (9)$$

Then we obtain

$$B=E(n)=2^{m+1}-2 \quad (10)$$

CONCLUSIONS

Difference equation contains Unknown function and Differences, but excludes derivatives. Difference equations are the discretization of the differential equations and also called recurrence relation.

According to this result, mathematical induction and traditional summation of sequence do not seem to play roles when solving this classic question of probabilities. Ironically, this simple difference equation works its magic and obtains the wonderful conclusion skillfully.

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