DOI: 10.21276/sjpms.2016.3.3.2

Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2016; 3(3):106-109 **ISSN 2393-8064 (Online)**
©Scholars Academic and Scientific Publishers (SAS Publishers)
(An International Publisher for Academic and Scientific Resources)

A Specific Formula to Compute the Determinant of One Matrix of Order n

Ber-Lin Yu
Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an, Jiangsu, 223003, P. R. China

***Corresponding Author:**
Ber-Lin Yu
Email: berlinyu@163.com

Abstract: Let $A=[\alpha_{ij}]$ be an $n\times n$ matrix, where $\alpha_{ij}=\dfrac{1}{a_i+b_j}$, $i,\ j=1,2,\ldots,n.$ In this paper, we establish a specific formula to calculate the determinant of matrix A.
Keywords: Determinant; Matrix; Laplace Theorem.

Introduction

Determinants occur throughout mathematics. For example, a matrix is often used to represent the coefficients in a system of linear equations, and the determinant can be used to solve those equations, although more efficient techniques are actually used, some of which are determinant-revealing and consist of computationally effective ways of computing the determinant itself. For an $n\times n$ matrix A, its determinant is defined as

$$|A|=\sum_{\sigma} sign(\sigma)\prod_{i=1}^{n}\alpha_{i\sigma(i)},$$

where the sum runs over all $n!$ permutations σ of the n items $1,2,\ldots,n$ and the $sign$ of a permutation σ, $sign(\sigma)$ is $+1$ or -1, according to whether the minimum number of transpositions, or pair-wise interchanges, necessary to achieve it starting from $1,2,\ldots,n$ is even or odd. Thus, each product

$$\prod_{i=1}^{n}\alpha_{i\sigma(i)}$$

enters into the determinant with a $+$ sign if the permutation σ is even or a $-$ sign if it is odd. The most fundamental and naive method of implementing an algorithm to compute the determinant is to use Laplace's formula [1] for expansion by cofactors, i.e.,

$$|A|=\sum_{j=1}^{n}\alpha_{ij}A_{ij},\ i=1,2,\ldots,n,$$

where A_{ij} which is called the cofactor of α_{ij}, is a product of $(-1)^{i+j}$ and the minor resulting from the deletion of row i and column j. This approach is extremely inefficient in general, however, as it is of order $n!$ for an $n\times n$ matrix. Consequently, those determinants which have special constructers are investigated. There are a series of literatures about this topic, such as the referenced [2-6] and the references therein.

In this paper, we focus on an $n\times n$ matrix $A=[\alpha_{ij}]_{n\times n}$, where $\alpha_{ij}=\dfrac{1}{a_i+b_j}$, $i,\ j=1,2,\ldots,n.$ One specific formula to calculate the determinant of matrix A is established.

Main result and its proof

To state clearly, let D_n be the determinant of the $n\times n$ matrix $A=[\alpha_{ij}]$. For $n=1$, the conclusion is trival. In general, assume that $n\geq 2$. Our main result is to establish a specific formula to compute D_n.

Theorem 1. *For* $n \geq 2$,

$$D_n = \frac{\prod_{n\geq i>j\geq 1}(a_j - a_i)(b_j - b_i)}{\prod_{j=1}^{n}\prod_{i=1}^{n}(a_i + b_j)}.$$

Proof. We complete the proof by induction on the order n of matrix A. For $n = 2$, we obtain

$$D_2 = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} \end{vmatrix}$$

$$= \frac{1}{a_1+b_1}\times\frac{1}{a_2+b_2} - \frac{1}{a_1+b_2}\times\frac{1}{a_2+b_1}$$

$$= \frac{(a_1+b_2)(a_2+b_1)-(a_1+b_1)(a_2+b_2)}{(a_1+b_1)(a_2+b_2)(a_1+b_2)(a_2+b_1)}$$

$$= \frac{a_1b_1 + a_2b_2 - a_1b_2 - b_1a_2}{(a_1+b_1)(a_2+b_2)(a_1+b_2)(a_2+b_1)}$$

$$= \frac{(a_1-a_2)(b_1-b_2)}{(a_1+b_1)(a_2+b_2)(a_1+b_2)(a_2+b_1)}.$$

It follows that Thoerm 1 holds when $n = 2$.

Now, we assume that Thoerm 1 holds when $n = k,$ where $k \geq 2$. That is to say,

$$D_k = \frac{\prod_{k\geq i>j\geq 1}(a_j - a_i)(b_j - b_i)}{\prod_{j=1}^{k}\prod_{i=1}^{k}(a_i + b_j)}.$$

Then when $n = k+1$,

$$D_{k+1} = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_{k+1}} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{k+1}+b_1} & \frac{1}{a_{k+1}+b_2} & \cdots & \frac{1}{a_{k+1}+b_{k+1}} \end{vmatrix}.$$

By adding column 1 multiplied by a scalar -1 to column j, $j = 2, 3, \quad \ldots, k+1$, we obtain that

$$D_{k+1} = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{b_1-b_2}{(a_1+b_2)(a_1+b_1)} & \cdots & \frac{b_1-b_{k+1}}{(a_1+b_{k+1})(a_1+b_1)} \\ \frac{1}{a_2+b_1} & \frac{b_1-b_2}{(a_2+b_2)(a_2+b_1)} & \cdots & \frac{b_1-b_{k+1}}{(a_2+b_{k+1})(a_2+b_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{k+1}+b_1} & \frac{b_1-b_2}{(a_{k+1}+b_2)(a_{k+1}+b_1)} & \cdots & \frac{b_1-b_{k+1}}{(a_{k+1}+b_{k+1})(a_{k+1}+b_1)} \end{vmatrix}$$

$$= \frac{\prod_{i=2}^{k+1}(b_1-b_i)}{\prod_{i=1}^{k+1}(a_i+b_1)} \begin{vmatrix} 1 & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_{k+1}} \\ 1 & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{a_{k+1}+b_2} & \cdots & \frac{1}{a_{k+1}+b_{k+1}} \end{vmatrix}.$$

By adding row 1 multiplied by a scalar -1 to row j, $j = 2, 3, \ldots, k+1$, we obtain that

$$D_{k+1}$$

$$= \frac{\prod_{i=2}^{k+1}(b_1-b_i)}{\prod_{i=1}^{k+1}(a_i+b_1)} \begin{vmatrix} 1 & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_{k+1}} \\ 0 & \frac{a_1-a_2}{(a_2+b_2)(a_1+b_2)} & \cdots & \frac{a_1-a_2}{(a_2+b_{k+1})(a_1+b_{k+1})} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{a_1-a_{k+1}}{(a_{k+1}+b_2)(a_1+b_2)} & \cdots & \frac{a_1-a_{k+1}}{(a_{k+1}+b_{k+1})(a_1+b_{k+1})} \end{vmatrix}.$$

By Laplacian Theorem and row-multiplying transformations, we have

$$D_{k+1} = \frac{\prod_{i=2}^{k+1}(b_1-b_i)(a_1-a_i)}{(a_1+b_1)\prod_{i=2}^{k+1}(a_i+b_1)(a_1+b_i)} \begin{vmatrix} \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{a_{k+1}+b_2} & \cdots & \frac{1}{a_{k+1}+b_{k+1}} \end{vmatrix}.$$

It is clear that

$$\begin{vmatrix} \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{a_{k+1}+b_2} & \cdots & \frac{1}{a_{k+1}+b_{k+1}} \end{vmatrix}$$

is a determinant of order k. By the assumption, we have

$$\begin{vmatrix} \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{a_{k+1}+b_2} & \cdots & \frac{1}{a_{k+1}+b_{k+1}} \end{vmatrix} = \frac{\prod\limits_{k+1\ge i>j\ge 2}(a_j-a_i)(b_j-b_i)}{\prod\limits_{j=2}^{k+1}\prod\limits_{i=2}^{k+1}(a_i+b_j)}.$$

Consequently,

$$D_{k+1} = \frac{\prod\limits_{i=2}^{k+1}(b_1-b_i)(a_1-a_i)}{(a_1+b_1)\prod\limits_{i=2}^{k+1}(a_i+b_1)(a_1+b_i)} \frac{\prod\limits_{k+1\ge i>j\ge 2}(a_j-a_i)(b_j-b_i)}{\prod\limits_{j=2}^{k+1}\prod\limits_{i=2}^{k+1}(a_i+b_j)}.$$

Simplying the above equality leads to

$$D_{k+1} = \frac{\prod\limits_{k+1\ge i>j\ge 1}(a_j-a_i)(b_j-b_i)}{\prod\limits_{j=1}^{k+1}\prod\limits_{i=1}^{k+1}(a_i+b_j)}.$$

By induction, we obtain that for $n \ge 2$,

$$D_n = \frac{\prod\limits_{n\ge i>j\ge 1}(a_j-a_i)(b_j-b_i)}{\prod\limits_{j=1}^{n}\prod\limits_{i=1}^{n}(a_i+b_j)}.$$

Acknowledgement

This research is supported by Sci. & Tech. Research Fund of Huaiyin Institute of Technology.

References

1. Leon SJ; Linear algebra and its application. Beijing: China Machine Press, 2007.
2. Huang Z; Generalization of vandermonde determinant. Journal of Jimei University (Natural Science), 2009; 12: 88-91.
3. Yu Y, Cai L; On the smarandache prime part sequences and its determinant. Pure and Applied Mathematics, 2008; 4: 747-751.
4. Li H, Zeng L; Properties of boolean determinant. Journal of Jishou University (Natural Science Edition), 2009; 30: 6-10.
5. Liu X, Hao X; One type cyclic matrix determinant calculation and one of its application. Journal of Yanan University (Natural Science Edition), 2009; 28:17-18.
6. Yu B; Calculation of the determinant with a claw or double claws structure. Far East Journal of Mathematical Sciences, 2010; 43:145-151.