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Characterization of the Zero-One Inflated Poisson Distribution

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Abstract: In this paper, we give a characterization of the zero-one inflated Poisson distributions through a linear differential equation of its probability generating function.
Keywords: Poisson distribution, Zero-One Inflated Poisson Distribution, Probability Generating Function, Linear Differential Equation.

INTRODUCTION

The Poisson distribution (PD) is a well-known discrete distribution that has so many empirical applications. Its Inflated model was studied by several researchers recently. In particular, Hassanzadeh and Kazemi[1] extends regression modeling of positive count data to deal with excessive proportion of one counts and proposed one-inflated positive Poisson and negative binomial regression models and present some of their properties. Zhang et al [2] studied zero-one Inflated Poisson distribution (ZOIPD), obtained maximum likelihood estimates of parameters using both the Fisher scoring and expectation-maximization algorithms, and provided Bootstrap confidence intervals for parameters of interest and testing hypotheses under large sample sizes. Nanjundan and Pasha [3] characterized the zero-inflated Poisson distribution through a linear differential equation. Alshkaki [4] studied some structural properties of the ZOIPD, as well as estimation of its parameters using the methods of moments and maximum likelihood estimators, and show that the ZOIPD model is better that the zero-inflate PD using the three set of real empirical data.

In this paper, we introduce, in Section 2, the definition of the PD and its zero-one inflated form with their probability generating function (pgf), followed in Section 3 we characterize the zero and ZOIPD through a linear equation of its pgf.

THE POISSON DISTRIBUTION

Let $\theta \epsilon \Omega = \{\theta; 0 < \theta < \omega\}$, where ω is the radius of convergence of e^{θ}, then the discrete random variable (rv) X having probability mass function (pmf);

$$P(X = x) = \frac{\theta^x}{x!} e^{-\theta} \qquad x = 0, 1, 2, \qquad (1)$$

is said to have a PD with parameter θ, and we will denote that by writing $X \sim PD(\theta)$. For further details on PD see Johnson et al [5].

Let $X \sim PD(\theta)$as given in (1), let $\alpha \in (0,1)$ be an extra proportion added to the proportion of zero of the rv X, and let$\beta \in (0,1)$ be an extra proportion added to the proportion of ones of the rv X, such that $0 < \alpha + \beta < 1$, then the rv Z defined by;

$$P(Z = z) = \begin{cases} \alpha + (1 - \alpha - \beta)e^{-\theta}, & z = 0 \\ \beta + (1 - \alpha - \beta)\theta e^{-\theta}, & z = 1 \\ (1 - \alpha - \beta)\frac{\theta^z}{z!} e^{-\theta}, & z = 2, 3, 4, \dots \\ 0 & otherwise, \end{cases} \qquad (2)$$

is said to have a ZOIPD, and will denote that by writing$Z \sim ZOIPD(\theta; \alpha, \beta)$.

Note that, if $\beta \to 0$, then (2) reduces to the form of the zero-inflated Poisson distribution (ZIPD). Similarly, the case with $\alpha \to 0$ and $\beta \to 0$, reduces to the standard case of PD.

The pgf of the rv X, $G_X(t)$, is given by;

$$G_X(t) = E(t^X) = e^{\theta(t-1)},$$

while the pgf of the rv Z, can be shown to be;

$$G_Z(t) = \alpha + \beta t + (1-\alpha-\beta)e^{\theta(t-1)} \qquad (3)$$

3. Characterization of the Poisson Distribution

We give below the main result of this paper.

Theorem: The discrete rv Z taking non-negative integer values, has a ZOIPD if its pgf, $G(t)$, satisfies for some arbitrary number b and non-zeros numbers a, c and d, that;

$$a\frac{\partial}{\partial t}G(t) = b + ct + dG(t) \qquad (4)$$

Proof: Assume first that $b \neq 0$.

Without loss of generality, let us assume that $d = 1$, hence (4) can be written as;

$$\frac{\frac{\partial}{\partial t}G(t)}{e^{\frac{t}{a}}} - \frac{G(t)}{ae^{\frac{t}{a}}} = \frac{b+ct}{ae^{\frac{t}{a}}} \qquad (5)$$

Using the fact that$\frac{\partial}{\partial t}\left[e^{\frac{-t}{a}}\right] = -\frac{e^{\frac{-t}{a}}}{a}$, (5) can rewritten as;

$$\frac{\partial}{\partial t}\left[G(t).e^{\frac{-t}{a}}\right] = \frac{1}{a}(b+ct)e^{\frac{-t}{a}}$$

It follows that;

$$\int \frac{\partial}{\partial t}\left[G(t).e^{\frac{-t}{a}}\right]dt = \int \frac{1}{a}(b+ct)e^{\frac{-t}{a}}\,dt$$

Thus;

$$G(t).e^{\frac{-t}{a}} = -be^{\frac{-t}{a}} + \frac{c}{a}\int te^{\frac{-t}{a}}\,dt \qquad (6)$$

Using integration by parts to evaluate the form $\int te^{wt}\,dt$, (6) reduces to;

$$G(t).e^{\frac{-t}{a}} = -be^{\frac{-t}{a}} + (-ac-ct)e^{\frac{-t}{a}} + k$$

where k is a an arbitrary constant. Thus;

$$G(t) = -b - ac - ct + ke^{\frac{t}{a}}$$

Since $1 = G(1)$;

$$1 = -b - ac - c + ke^{\frac{1}{a}}$$

and hence;

$$k = (1+b+ac+c)e^{\frac{-1}{a}}$$

Therefore;

$$G(t) = -b - ac - ct + (1+b+ac+c)e^{\frac{-1}{a}}.e^{\frac{t}{a}},$$

or equivalently, as in the form given by (3);

$$G(t) = \alpha + \beta t + (1 - \alpha - \beta)e^{\theta(t-1)} \qquad (7)$$

where;

$$\theta = \frac{1}{a} \qquad (8)$$

$$\alpha = -b - ac \qquad (9)$$

and

$$\beta = -c \qquad (10)$$

Let us consider possible values of θ, α and β, given by (8), (9) and (10), respectively. Now if; $a > 0$, then $\theta > 0$. If; $-1 < c < 0$, then $0 < \beta < 1$.If b satisfies that $-(ac + 1) < b < -ac$, then $0 < -b - ac < 1$, that is $0 < \alpha < 1$.In order forα and β to satisfy that $0 < \alpha + \beta < 1$, b has to satisfies that $-1 - c(a + 1) < b < -c(a + 1)$, therefore, if b takes values such that $-1 - c(a + 1) < b < -ac$, that is, in the intersection of $(-(ac + 1), -ac)$ and $(-1 - c(a + 1), -c(a + 1))$, then $0 < \alpha < 1$ and $0 < \alpha + \beta < 1$.

Finally, if $b = 0$, then we arrive simply, either by solving (4) directly or by letting $b \to 0$ in the above proof, which is straightforward, to the same conclusion with θ and β are given by (8) and (10), respectively, and that $\alpha = -ac$. Hence, if $a > 0$, then $\theta > 0$. If; $-\frac{1}{a} < c < 0$, then $0 < \alpha < 1$. If $-1 < c < 0$, then $0 < \beta < 1$. If $-\frac{1}{a+1} < c < 0$ then $0 < \alpha + \beta < 1$. Therefore, if c satisfies that max(*-1*, $-\frac{1}{a+1}$)$< c < 0$, then $0 < \alpha < 1, 0 < \beta < 1$ and $0 < \alpha + \beta < 1$. This completes the proof.

Theorem 1 gives the following conclusion.

Theorem 2: Let Z be a discrete rv taking non-negative integer values, then $Z \sim ZOIPD(\theta; \alpha, \beta)$, for some non-zero θ, β and αif and only if its pgf satisfying (4) for some arbitrary number b and non-zeros numbers a, c and d, that.

Proof: If $Z \sim ZOIPD(\theta; \alpha, \beta)$ for some non-zero θ, α and β, then it easy to see find its pgf,$G(t)$, is given by (3), and hence its pgf satisfies (4) with $a = \frac{1}{\theta}$, $b = \frac{\beta}{\theta} - \alpha$, $c = -\beta$ and $d = 1$, or in the equivalent form;$a = 1$, $b = \beta - \alpha\theta$, $c = -\beta\theta$ and $d = \theta$, hence the proof is complete using Theorem 1.

Theorem 2 leads to the following conclusion about the zero-inflated PD.

Theorem 3: Let Z be a discrete rv taking non-negative integer values, then $Z \sim ZIPD(\theta; \alpha)$, for some non-zero θ and αif and only if its pgf satisfying for non-zeros numbers a and d, that;

$$a\frac{\partial}{\partial t}G(t) = b + dG(t).$$

Proof: Just let $c \to 0$ in Theorem 2.

We note that Nanjundan and Pasha [3] obtained the same result of Theorem 3 with $d = 1$.

CONCLUSIONS

We introduced a characterization of the zero-one inflated Poisson distributions through a linear differential equation of its probability generating function. We would propose an extension of these results to other forms and distributions.

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