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# Lower Triangular Vector Bilinear Autoregressive Time Series Model

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**Abstract:** This work proposed a strategy for fitting a subset form of a vector bilinear model. In this case, lower triangular portion of the vector linear and non-linear components of a vector autoregressive bilinear structure was considered. The workability of the method was illustrated using a 3 – dimensional vector of time series. The models were found to fulfil the assumptions of model adequacy and each time dependent variable depended only on a subset of lagged time varying quantities under consideration. Apart from being parsimonious, the fitted models were found to perform better than the parent vector bilinear autoregressive models as revealed by some statistical comparative analysis.
**Keywords:** Lower triangular matrix, Vector bilinear autoregressive models, White noise process and residual autocorrelation

## INTRODUCTION

Recently, there has been an increasing interest in models that extend the classical time series framework developed by Box and Jenkins [1] and others. An important assumption that is made in the classical theory is that the structure of the series can be described by a linear model such as an autoregressive, moving average or mixed autoregressive moving average model. Sometimes, however, such specifications are made just for technical convenience but may not be appropriate. The interaction of effects can cause problems beyond the sum of the individual complications [3]; thus demanding an extension to non-linear models. The theory of Voltera [2] and Weiner [4] on functional series representation has provided a great stimulus to the development of non-linear models. One of such time series models allowing for non-linearity is the bilinear model introduced by Granger and Anderson [5], and extended by Tong [6], Guegan and Pham [7]. The interesting feature of a bilinear system is that though it is non-linear, its structural theory is analogous to that of linear systems [6]. According to Rao [6], a bilinear time series model $BL(p,r,m,k)$ is given by the difference equation:

$$X(t)+\sum_{j=1}^{p} a_j X(t-j)=\sum_{j=0}^{r} c_j \varepsilon(t-j)+\sum_{l=1}^{m}\sum_{l'=1}^{k} b_{ll'}X(t-j)\varepsilon(t-l') \quad (1)$$

Where $\{\varepsilon(t)\}$ is an independent white noise process and $c_0=1$. $\{X(t)\}$ is termed the bilinear process. The autoregressive moving average model $ARMA(p,r)$ is obtained from (1) by setting $b_{ll'}=0\ \forall\ l$ and $l'$.The idea put up by Rao [6] considered only a single time series variable. However, there are several situations where two or more variables could be related. In such situation, a bilinear structure that captures the interaction among the variables is required. Consequent to this, Boonchai and Eivind [8] gave the general form of multivariate bilinear time series models as:

$$X(t)=\sum A_i X(t-i)+\sum M_j\varepsilon(t-j)+\sum\sum\sum B_{d_{ij}} X(t-i)\varepsilon_d(t-j)+\varepsilon(t) \quad (2)$$

Here, the state $X(t)$ and noise $\varepsilon(t)$ are $n-$vectors and the coefficients $A_i$, $M_j$ and $B_{d_{ij}}$ are $n\times n$ matrices. If all$B_{d_{ij}}=0$, we have the class of well - known vector ARMA models. Iwok and Etuk [9] established a vector form of autoregressive moving average (VARMA) models comprising linear and non-linear components that could compete with the pure vector linear VARMA models. General bilinear vector autoregressive moving average (BIVARMA) was presented as an extension of the univariate bilinear model. The results showed that the BIVARMA model established perform best and provide better estimates than the VARMA models. Clifford [5] fitted a full Vector Bilinear Autoregressive (VEBIA) Time Series Models to three economic series. The fitted models were found to be adequately fitted to the three sets of data. However, Clifford [5] highlighted that the procedure of fitting the full VEBIA model is sometimes boring and complicated; hence a subset fit with optimum result is necessary. In line with the suggestion by Clifford [5] that a subset VEBIA model is necessary; this work proposes a strategy for reducing the number of estimated parameters such that an adequate fit of Vector Bilinear Autoregressive Time Series Models is obtained.

## METHODOLOGY

### Lower Triangular Matrix

A lower triangular matrix $\underline{L}$ of an $m\times n$ matrix is a matrix of the form:

$$\underline{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ l_{m1} & l_{m2} & \cdots & l_{mn} \end{bmatrix} \tag{3}$$

where $l_{ij} = 0 \quad \forall \ j > i$.

**Vector Bilinear Autoregressive (VEBIA) Time Series Models**

Consider an $r-$ dimensional time series vector

$$\underline{X_t'} = [\, X_{1t} \quad X_{2t} \quad \cdots \quad X_{rt} \,] \,. \tag{4}$$

Let $P$ be the set of all autoregressive (AR) orders of the elements of the vector $\underline{X_t}$ .

Then, $p_i \in P$ and $P = \{p_1, p_2, \dots, p_r\}$ and the VEBIA model is of the form:

$$\underline{X_t} = \sum_{k=1}^{\max p_i} \underline{\gamma_k} \ \underline{X_{t-k}} + \sum_{k=1}^{\max p_i} \underline{X_{t-k}^{(D)}} \left( \underline{\beta_k} \ \ \underline{\varepsilon_{t-0}} \right) + \underline{u_t} \tag{5}$$

Where,

1. All underlined alphabets represent vectors and matrices.
2. $\underline{\gamma_k}$ is a vector of parameters of the linear part of the model
3. $\underline{\beta_k}$ is a vector of parameters of the non-linear component of the model.
4. $\underline{\varepsilon_{t-0}}$ is a vector of white noise innovation
5. $\underline{u_t}$ is a vector of residuals obtained from the model (5).
6. $\underline{X_{t-k}^{(D)}}$ is a diagonal matrix and $k$ is the lag.

The model (5) can explicitly be expressed with the orders of the matrices indicated as shown:

$$\{X_{it}\}_{n\times 1} = \sum_{k=1}^{\max p_i} \{\gamma_{k\cdot ij}\}_{r\times r} \ \{X_{it-k}\}_{r\times 1} + \sum_{k=1}^{\max p_i} \{X_{it-k}\}_{r\times r}^{(D)} \left[ \{\beta_{k0\cdot ij}\}_{r\times r} \ \{\varepsilon_{it-0}\}_{r\times 1} \right] + \{u_{it}\}_{r\times 1} \tag{6}$$

where,

1. $\gamma_{k\cdot ij} \neq 0 \ \ if \ \exists \, X_{jt-k}$ And $\gamma_{k\cdot ij} = 0 \ \ if \ \nexists \, X_{jt-k}$
2. $\beta_{k0\cdot ij} \neq 0 \ \ if \ \exists \, X_{it-k}$ and$\beta_{k0\cdot ij} = 0 \ \ if \ \nexists \, X_{it-k}$.

**Lower Triangular Vector Bilinear Autoregressive (LOTVEBIA) Time Series Models**

From the definition of lower triangular matrix; the lower triangular VEBIA model can only capture the lower triangular parameters of the VEBIA model and all other parameters are set equals to zero. That is, to obtain the LOTVEBIA time series model, we must set

$$\gamma_{k\cdot ij} = 0 \quad \text{And} \quad \beta_{k0\cdot ij} = 0 \quad \forall \ j > i.$$

**White Noise Process**

A process $\{\varepsilon_t\}$ is said to be a white noise process with mean 0 and variance $\sigma_\varepsilon^2$ written $\{\varepsilon_t\} \sim WN(0, \sigma_\varepsilon^2)$, if it is a sequence of uncorrelated random variables from a fixed distribution. The residual of an adequate model is expected to follow a white noise process as a fulfilment of the assumption of model adequacy.

**Diagnostic Checks of the Models**

The diagnostic checks shall be based on the analysis of the residuals. Under the assumption of model adequacy, the residuals obtained from each model are examined whether they are uncorrelated at the various lags. This is easily detected by the Autocorrelation Function (ACF) plot of the residuals. If the Autocorrelation Function (ACF) plot does not show any spike above or below the 95% confidence interval, then the residuals of each model are uncorrelated at various lags and the fitted model is adequate. This simply means that the residuals follow a white noise process. Also, the performance of the fitted models shall be assessed by examining the combined plots of the actual values of the series and the predicted values. If the models are adequate, the actual and the predicted values at each time point will be closed to each other; and this can easily and clearly be detected if the two superimposed plots move in the same direction. For comparative purpose, the performance of models will be based on mean absolute error (MAE) or residual variance given as

$$MSE = \frac{1}{N} \sum_{i=1}^{N} \left( \mathrm{X}_i - \hat{X}_i \right)^2 \tag{7}$$

and the mean absolute percentage error (MAPE) given as

$$MAPE = \left[ \frac{1}{N} \sum_{i-1}^{N} \left| \frac{\mathrm{X}_i - \hat{X}_i}{\mathrm{X}_i} \right| \right] \times 100 \tag{8}$$

where $\mathrm{X}_i$ are the observed values and $\hat{X}_i$ are the estimated values. Of course, the model with the smallest error is chosen as the best model.

## ANALYSIS AND RESULTS

### The VEBIA Models

The data used for this work is the revenue series used by Iwok and Etuk [9]. The orders identified for each series were: AR (3) for $X_{1t}$, AR (2) for $X_{2t}$ and AR (1) for $X_{3t}$. As noted in Iwok [10], fitting a full vector bilinear autoregressive time series model described in section 2.2 of the methodology results in the following vector bilinear autoregressive models:

$$\begin{aligned} X_{1t} &= 0.266X_{1t-1} + 0.061X_{2t-1} + 0.0322X_{3t-1} + 0.397X_{1t-2} + 0.163X_{2t-2} + 0.178X_{1t-3} \\ &\quad + 0.00158\varepsilon_{1t-0}X_{1t-1} + 0.000724\varepsilon_{2t-0}X_{1t-1} - 0.000253\varepsilon_{3t-0}X_{1t-1} \\ &\quad + 0.00196\varepsilon_{1t-0}X_{1t-2} + 0.000291\varepsilon_{2t-0}X_{1t-2} - 0.00132\varepsilon_{3t-0}X_{1t-2} \\ &\quad + 0.000432\varepsilon_{1t-0}X_{1t-3} + 0.000918\varepsilon_{2t-0}X_{1t-3} + 0.00135\varepsilon_{3t-0}X_{1t-3} \end{aligned} \tag{9}$$

$$\begin{aligned} X_{2t} &= 0.087X_{1t-1} + 0.433X_{2t-1} - 0.177X_{3t-1} + 0.0903X_{1t-2} + 0.497X_{2t-2} - 0.0609X_{1t-3} \\ &\quad + 0.000036\varepsilon_{1t-0}X_{2t-1} + 0.00149\varepsilon_{2t-0}X_{2t-1} + 0.000549\varepsilon_{3t-0}X_{2t-1} \\ &\quad + 0.000304\varepsilon_{1t-0}X_{2t-2} + 0.00260\varepsilon_{2t-0}X_{2t-2} - 0.00114\varepsilon_{3t-0}X_{2t-2} \end{aligned} \tag{10}$$

$$\begin{aligned} X_{3t} &= 0.0922X_{1t-1} - 0.0544X_{2t-1} + 0.651X_{3t-1} + 0.0073X_{1t-2} + 0.0079X_{2t-2} + 0.0435X_{1t-3} \\ &\quad + 0.0000\varepsilon_{1t-0}X_{3t-1} - 0.000374\varepsilon_{2t-0}X_{3t-1} + 0.0104\varepsilon_{3t-0}X_{3t-2} \end{aligned} \tag{11}$$

The residual variances obtained in fitting vector bilinear models in equations (9), (10) and (11) were 16.36 for $X_{1t}$, 22.90 for $X_{2t}$ and 21.96 for $X_{3t}$ respectively.

### The LOTVEBIA Models

From matrix expression (6) and the 'orders' identified for each series, the Lower Triangular Vector Bilinear Autoregressive (LOTVEBIA) Time Series Models can be expressed explicitly as:

$$\begin{aligned} \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} &= \begin{pmatrix} \gamma_{1.11} & 0 & 0 \\ \gamma_{1.21} & \gamma_{1.22} & 0 \\ \gamma_{1.31} & \gamma_{1.32} & \gamma_{1.33} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \\ X_{3t-1} \end{pmatrix} + \begin{pmatrix} \gamma_{2.11} & 0 & 0 \\ \gamma_{2.21} & \gamma_{2.22} & 0 \\ \gamma_{2.31} & \gamma_{2.32} & 0 \end{pmatrix} \begin{pmatrix} X_{1t-2} \\ X_{2t-2} \\ X_{3t-2} \end{pmatrix} \\ &\quad + \begin{pmatrix} \gamma_{3.11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1t-3} \\ X_{2t-3} \\ X_{3t-3} \end{pmatrix} + \begin{pmatrix} X_{1t-1} & 0 & 0 \\ 0 & X_{2t-1} & 0 \\ 0 & 0 & X_{3t-1} \end{pmatrix} \left\{ \begin{pmatrix} \beta_{10.11} & 0 & 0 \\ \beta_{10.21} & \beta_{10.22} & 0 \\ \beta_{10.31} & \beta_{10.32} & \beta_{10.33} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-0} \\ \varepsilon_{2t-0} \\ \varepsilon_{3t-0} \end{pmatrix} \right\} \\ &\quad + \begin{pmatrix} X_{1t-2} & 0 & 0 \\ 0 & X_{2t-2} & 0 \\ 0 & 0 & X_{3t-2} \end{pmatrix} \left\{ \begin{pmatrix} \beta_{20.11} & 0 & 0 \\ \beta_{20.21} & \beta_{20.22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-0} \\ \varepsilon_{2t-0} \\ \varepsilon_{3t-0} \end{pmatrix} \right\} \\ &\quad + \begin{pmatrix} X_{1t-3} & 0 & 0 \\ 0 & X_{2t-3} & 0 \\ 0 & 0 & X_{3t-3} \end{pmatrix} \left\{ \begin{pmatrix} \beta_{20.11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-0} \\ \varepsilon_{2t-0} \\ \varepsilon_{3t-0} \end{pmatrix} \right\} + \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} \end{aligned}$$

This can further be expressed as:

$$\begin{aligned} \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} &= \begin{pmatrix} \gamma_{1.11}X_{1t-1} \\ \gamma_{1.21}X_{1t-1} + \gamma_{1.22}X_{2t-1} \\ \gamma_{1.31}X_{1t-1} + \gamma_{1.32}X_{2t-1} + \gamma_{1.33}X_{3t-1} \end{pmatrix} + \begin{pmatrix} \gamma_{2.11}X_{1t-2} \\ \gamma_{2.21}X_{1t-2} + \gamma_{2.22}X_{2t-2} \\ \gamma_{2.31}X_{1t-2} + \gamma_{2.32}X_{2t-2} \end{pmatrix} + \begin{pmatrix} \gamma_{3.11}X_{1t-3} \\ 0 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \beta_{10.11}X_{1t-1}\varepsilon_{1t-0} \\ \beta_{10.21}X_{2t-1}\varepsilon_{1t-0} + \beta_{10.22}X_{2t-1}\varepsilon_{2t-0} \\ \beta_{10.31}X_{3t-1}\varepsilon_{1t-0} + \beta_{10.32}X_{3t-1}\varepsilon_{2t-0} + \beta_{10.33}X_{3t-1}\varepsilon_{3t-0} \end{pmatrix} \\ &\quad + \begin{pmatrix} \beta_{20.11}X_{1t-2}\varepsilon_{1t-0} \\ \beta_{20.21}X_{2t-2}\varepsilon_{1t-0} + \beta_{20.22}X_{2t-2}\varepsilon_{2t-0} \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \beta_{20.11}X_{1t-3}\varepsilon_{1t-0} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} \end{aligned}$$

The above expanded matrix can be compressed as:

$$X_{it} = \sum_{r=1}^{3} \sum_{k=1}^{3} \gamma_{k.ir}\, X_{rt-k} + \sum_{r=1}^{3} \sum_{k=1}^{3} \beta_{k0.ir} X_{it-k}\, \varepsilon_{rt-0} + u_{it} \tag{12}$$

$$\gamma_{k\cdot ij} = 0 \quad \text{and} \quad \beta_{k0\cdot ij} = 0 \quad \forall\ j > i.$$

### Estimates of the LOTVEBIA Models

Fitting the above LOTVEBIA time series model (12) to the three series generated the following parametric equations:

$$\begin{aligned} X_{1t} &= 0.021X_{1t-1} + 0.324X_{1t-2} - 0.42X_{1t-3} + 0.33X_{1t-1}\varepsilon_{1t-0} + 0.45X_{1t-2}\varepsilon_{1t-0} \\ &\quad - 0.081X_{1t-3}\varepsilon_{1t-0} \end{aligned} \tag{13}$$

$$\begin{aligned} X_{2t} &= 0.43X_{1t-1} - 0.21X_{2t-1} + 0.55X_{1t-2} - 0.11X_{2t-2} + 0.61X_{2t-1}\varepsilon_{1t-0} + 0.03X_{2t-1}\varepsilon_{2t-0} \\ &\quad + 0.63X_{2t-2}\varepsilon_{1t-0} + 0.32X_{2t-2}\varepsilon_{2t-0} \end{aligned} \tag{14}$$

$$X_{3t} = 0.22X_{1t-1} + 0.44X_{2t-1} + 0.12X_{3t-1} + 0.51X_{1t-2} - 0.33X_{2t-2} + 0.66X_{3t-1}\varepsilon_{1t-0}$$

$$+0.54X_{3t-1}\varepsilon_{2t-0} - 0.13X_{3t-1}\varepsilon_{3t-0} \quad (15)$$

**Diagnostic Checks**

**Residual Autocorrelation (ACF)**

The autocorrelation functions (ACF) of the residuals obtained by fitting the LOTVEBIA models (13), (14) and (15) are displayed in figures 1, 2 and 3 of the appendix. As clearly seen in the figures, there is no significant spike at any lag. This is an indication that the residuals are uncorrelated and resembles a white noise process. Hence, the models are adequately fitted.

**Actual and Predicted Values Plots**

The actual and predicted values plots of the fitted LOTVEBIA models (13), (14) and (15) are displayed for each series in figures 4, 5 and 6 of the appendix. Each figure contains two graphs (the actual and its estimates plots). Examination of this plots shows that there is no significant difference between the actual values and the predicted values. The two plots on each figure are intertwined and move in the same direction. This shows that the estimates provided by each model are good. Hence the models are adequate.

**Comparative Analysis**

Using the mean absolute error (MAE) and the mean absolute percentage error (MAPE) described in section 3; the values of each test for the VEBIA and the LOTVEBIA models are displayed in table-1 below:

**Table-1: Comparative Test Values**

| Models | SERIES | MAE | MAPE |
|---|---|---|---|
| VEBIA | $X_{1t}$ | 16.36 | 31.11 |
| | $X_{2t}$ | 22.90 | 34.42 |
| | $X_{3t}$ | 21.96 | 33.77 |
| LOTVEBIA | $X_{1t}$ | 11.43 | 21.33 |
| | $X_{2t}$ | 14.16 | 23.12 |
| | $X_{3t}$ | 13.77 | 22.58 |

The above table-1 clearly shows that the errors incurred by the VEBIA models are larger than those incurred by the LOTVEBIA model. In simple terms, the LOTVEBIA models perform better than the VEBIA models.

**DISCUSSION AND CONCLUSION**

The Lower Triangular Vector Bilinear Autoregressive (LOTVEBIA) Time Series Model is more or less a subset fit of the full Vector Bilinear Autoregressive (VEBIA) Time Series Model. As noted in the review, Clifford [5] highlighted that the procedure of fitting the full VEBIA model is sometimes boring and complicated; hence a subset fit with optimum result is necessary. In line with this demand, this work has considered fitting a part (lower triangular portion) of the VERBIA model which is a form of subset VERBIA fit described by Clifford [5]. One of the assumptions of model adequacy is that residuals are expected to follow a white noise process. The proposed LOTVEBIA model has fulfilled this assumption and therefore be taken as one of the methods of addressing bilinear concepts. Comparative analysis carried out has shown that the LOTVEBIA models, despite its simplicity, outperform other bilinear forms. It is therefore pertinent to conclude that in most cases, a subset form of a vector bilinear structure can be more adequately fitted to a data than the complete bilinear form.

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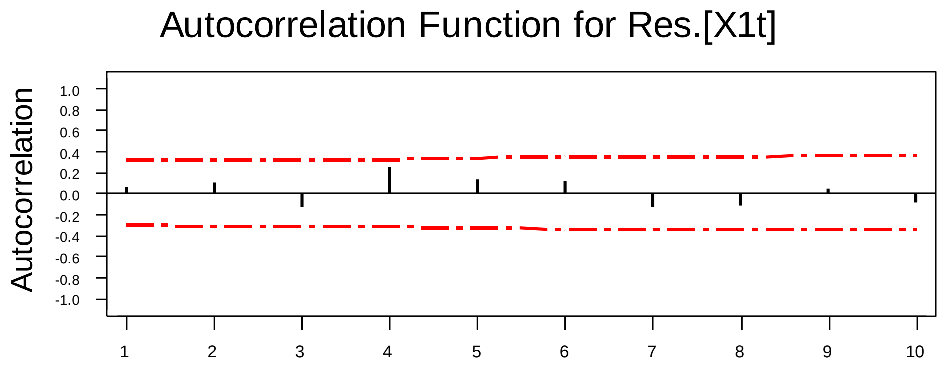
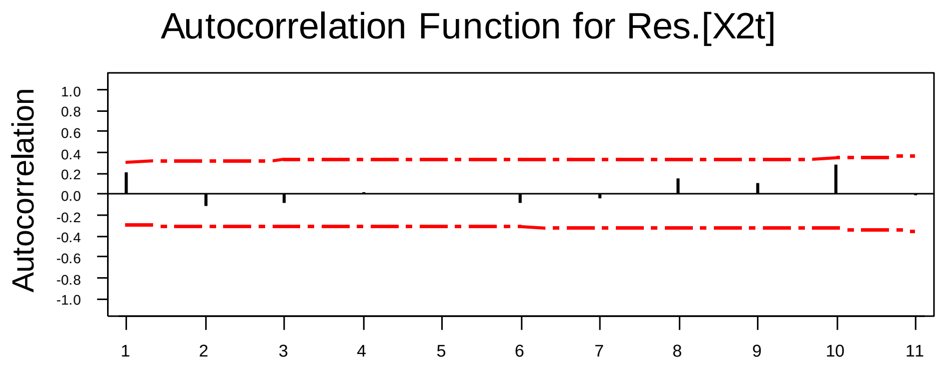
**Appendix**



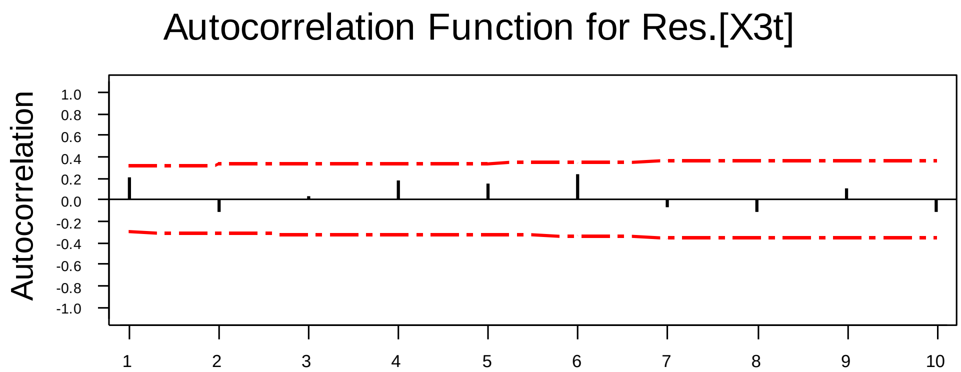

| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.06 | 0.42 | 0.19 | 8 | -0.12 | -0.72 | 7.70 |
| 2 | 0.10 | 0.65 | 0.66 | 9 | 0.04 | 0.25 | 7.80 |
| 3 | -0.13 | -0.81 | 1.42 | 10 | -0.09 | -0.52 | 8.30 |
| 4 | 0.25 | 1.58 | 4.46 | | | | |
| 5 | 0.13 | 0.77 | 5.29 | | | | |
| 6 | 0.11 | 0.67 | 5.96 | | | | |
| 7 | -0.13 | -0.76 | 6.85 | | | | |

**Fig-1: Autocorrelation function for Residuals from fitting $X_{1t}$**



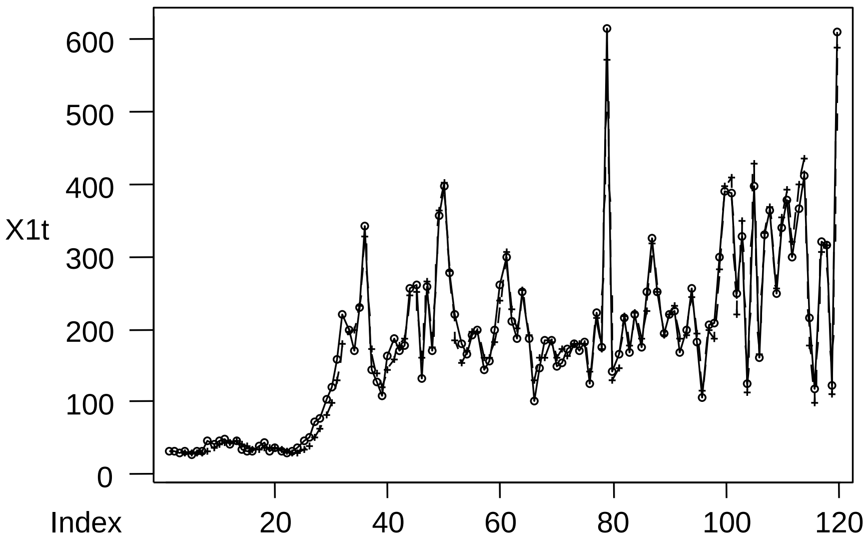

| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.20 | 1.35 | 1.94 | 8 | 0.15 | 0.91 | 4.80 |
| 2 | -0.12 | -0.78 | 2.65 | 9 | 0.11 | 0.65 | 5.46 |
| 3 | -0.09 | -0.54 | 3.01 | 10 | 0.27 | 1.64 | 9.90 |
| 4 | 0.02 | 0.11 | 3.03 | 11 | -0.02 | -0.09 | 9.92 |
| 5 | 0.00 | 0.00 | 3.03 | | | | |
| 6 | -0.09 | -0.58 | 3.48 | | | | |
| 7 | -0.04 | -0.27 | 3.59 | | | | |

**Fig-2: Autocorrelation function for Residuals from fitting $X_{2t}$**



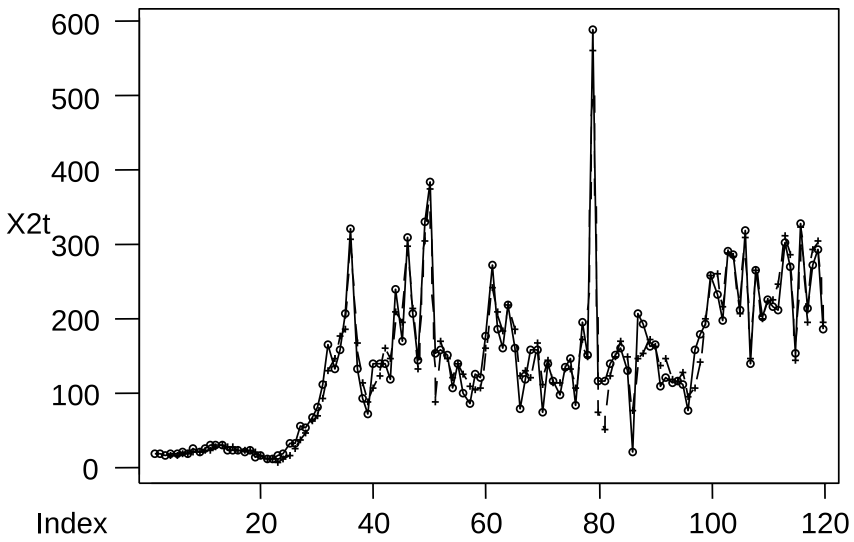

| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.21 | 1.35 | 1.95 | 8 | -0.12 | -0.68 | 9.27 |
| 2 | -0.12 | -0.75 | 2.63 | 9 | 0.10 | 0.55 | 9.81 |
| 3 | 0.03 | 0.19 | 2.68 | 10 | -0.11 | -0.62 | 10.55 |
| 4 | 0.17 | 1.07 | 4.16 | | | | |
| 5 | 0.15 | 0.92 | 5.33 | | | | |
| 6 | 0.23 | 1.37 | 8.14 | | | | |
| 7 | -0.08 | -0.45 | 8.48 | | | | |

**Fig-3: Autocorrelation function for Residuals from fitting $X_{3t}$**



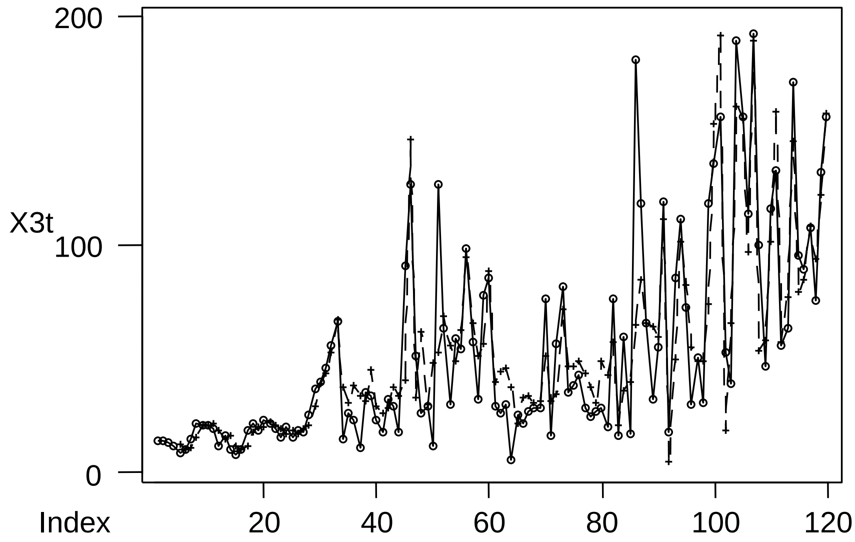

*Key*: o (actual plot) and + (estimates plot)

**Fig-4: LOTVEBIA plots of actual and estimates of $X_{1t}$**



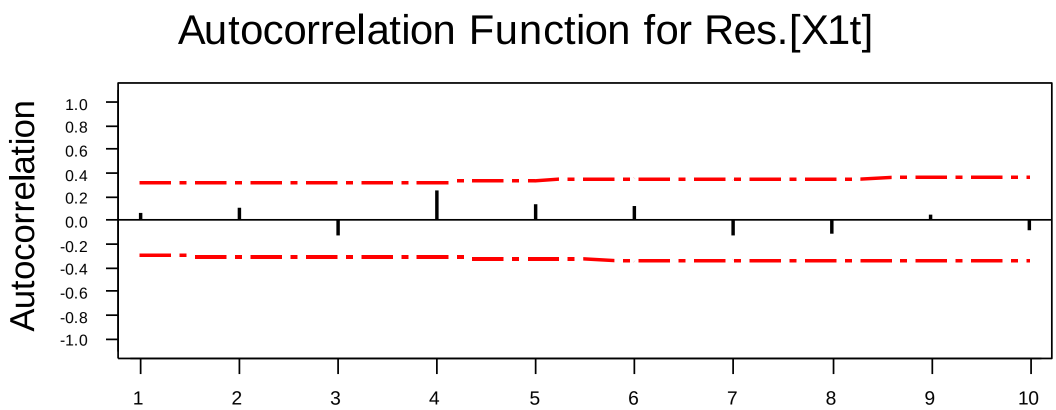

*Key*: o (actual plot) and + (estimates plot)

**Fig-5: LOTVEBIA plots of actual and estimates of $X_{2t}$**

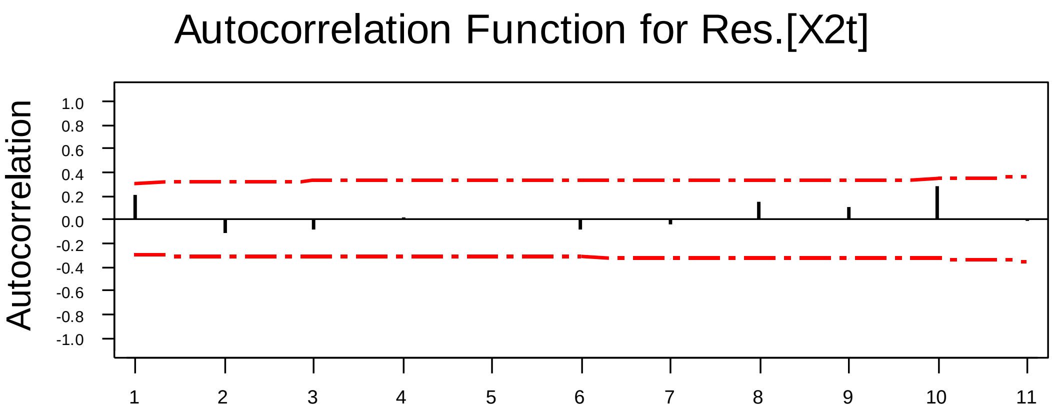



*Key*: o (actual plot) and + (estimates plot)

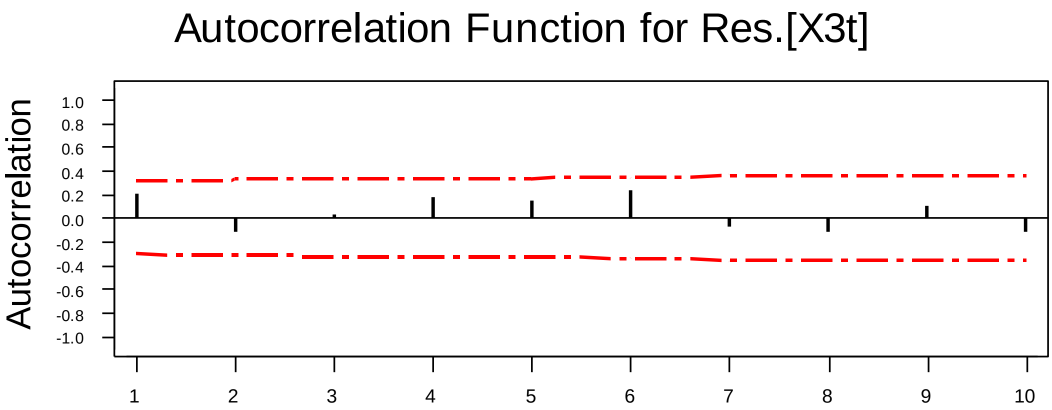
**Fig-6: LOTVEBIA plots of actual and estimates of $X_{3t}$**




| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.06 | 0.42 | 0.19 | 8 | -0.12 | -0.72 | 7.70 |
| 2 | 0.10 | 0.65 | 0.66 | 9 | 0.04 | 0.25 | 7.80 |
| 3 | -0.13 | -0.81 | 1.42 | 10 | -0.09 | -0.52 | 8.30 |
| 4 | 0.25 | 1.58 | 4.46 | | | | |
| 5 | 0.13 | 0.77 | 5.29 | | | | |
| 6 | 0.11 | 0.67 | 5.96 | | | | |
| 7 | -0.13 | -0.76 | 6.85 | | | | |




| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.20 | 1.35 | 1.94 | 8 | 0.15 | 0.91 | 4.80 |
| 2 | -0.12 | -0.78 | 2.65 | 9 | 0.11 | 0.65 | 5.46 |
| 3 | -0.09 | -0.54 | 3.01 | 10 | 0.27 | 1.64 | 9.90 |
| 4 | 0.02 | 0.11 | 3.03 | 11 | -0.02 | -0.09 | 9.92 |
| 5 | 0.00 | 0.00 | 3.03 | | | | |
| 6 | -0.09 | -0.58 | 3.48 | | | | |
| 7 | -0.04 | -0.27 | 3.59 | | | | |




| Lag | Corr | T | LBQ | Lag | Corr | T | LBQ |
|---|---|---|---|---|---|---|---|
| 1 | 0.21 | 1.35 | 1.95 | 8 | -0.12 | -0.68 | 9.27 |
| 2 | -0.12 | -0.75 | 2.63 | 9 | 0.10 | 0.55 | 9.81 |
| 3 | 0.03 | 0.19 | 2.68 | 10 | -0.11 | -0.62 | 10.55 |
| 4 | 0.17 | 1.07 | 4.16 | | | | |
| 5 | 0.15 | 0.92 | 5.33 | | | | |
| 6 | 0.23 | 1.37 | 8.14 | | | | |
| 7 | -0.08 | -0.45 | 8.48 | | | | |