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A Note on Existence of Positive Solutions of One-Dimensional p-Laplacian Equations

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Abstract: In this paper, we proved some new results on the existence of positive solutions of one-dimensional p-Laplacian equations under sublinear conditions. Our assumptions are weaker than the existing that, some recent results are improved essentially.
Keywords: positive solutions, one-dimensional p-Laplacian, sublinear conditions, negative values, fixed Point

INTRODUCTION

We investigate existence results of positive (classical) solutions for one-dimensional p-Laplacian equations of the form

$$\begin{cases} -\Delta_p z(x) = f(x, z(x)) \text{ for a.e. } x \in (0,1) \\ z(0) = z(1) = 0 \end{cases} \tag{1.1}$$

under the sublinear conditions, where $p \in (1, \infty)$, $-\Delta_p z(x) = \left(|z'(x)|^{p-2} z'(x)\right)' := \left(\phi_p(z'(x))\right)'$, $z'(x)$ denotes the usual derivative of the function z at x, and $\phi_p : R \to R$ is defined by $\phi_p(s) = |s|^{p-2} s$.

Existence of nonzero nonnegative positive solution of (1.1) has been studied by many authors, for example, by Dai and Ma [1], where the nonlinearity is of the form $g(x) f(u)$, under the following condition:

$$\lim_{u\to\infty} f(x,u)/\phi_p(u) \geq 0 \quad \text{for } x \in (0,1),$$

and by Lan and Yang [2], where $p \neq 2$, under the following sublinear condition :

$$\overline{f}(u) = \sup_{x\in[0,1]\setminus E} f(x,u), \quad \lim_{u\to\infty} \sup \overline{f}(u)/\phi_p(u) \geq 0.$$

When the first eigenvalue μ_n with $n=1$, Ćwiszewski and Maciejewski [3] use the Granas fixed point index to study the existence of positive weak solutions of p-Laplacian equations under the following sublinear condition:

$$\lim_{u\to\infty} f(x,u)/\phi_p(u) \in L^\infty.$$

We refer to [1], [4], [5], [6] for the study of the existence and uniqueness of systems of p-Laplacian equa-tions under superliner or sublinear conditions. As mentioned above, there have been many papers studying the existence of solutions of one-dimensional p-Laplacian equation (1.1), but to the best of our knowledge, then essential condition f_p^∞ is required to be bounded below(numeral or functional lower bounds) such as Theorem 4.1 [1], Theorem 2.2 [2], Theorem 1.1 [3]. In this paper, we do not need this assumption, and our assumeptions on (1.1) are weaker than usual that. The obtained results can be not derived by the existing results and some recent results are improved fundamentally. Regarding the study of (1.1), one may obtain more results referring to [1, 2, 3, 7] and their references .

New results of positive solutions for (1.1)

In this paper, we always assume the following conditions hold.

(C_1) $f:[0,1]\times R_+\left(R_+=[0,\infty)\right)\to R$ is a Carathéodory function, that is, $f(\cdot,u)$ is measurable for each fixed $u\in R_+$, $f(x,\cdot)$ is continuous for almost every (a.e.) $x\in[0,1]$.

(C_2) For each r > 0, there exists $g_r\in L^1_+[0,1]$ such that $|f(x,u)|\le g_r(x)$ for a.e. $x\in[0,1]$ and all $u\in[0,r]$.

(C_3) $f(x,0)\ge 0$ for a.e. $x\in[0,1]$.

Remark 2.1 The conditions (C_1)-(C_2) have been used usually, for example in [2, 3]. The condition (C_3) is weaker than usual assumptions, see Remark 2.1 [8, 9].

We denote by $AC[0,1]$ the space of all the absolutely continuous functions defined on $[0,1]$. A function $z:[0,1]\to R$ is said to be a (classical) solution of (1.1) if $z\in C^1[0,1]$, $\phi_p(z')\in AC[0,1]$ and z satisfies (1.1) [7], to be nonnegative if $z(x)\ge 0$ for $x\in[0,1]$ and to be positive if $z(x)>0$ for $x\in(0,1)$.

We denote by $W_0^{1,p}(0,1)$ the standard Sobolev space with norm [10]

$$\|u\|_{W_0^{1,p}}=\left(\int_0^1|u'(x)|^p dx\right)^{1/p}:=\|u'\|_{L_p}.$$

We need to recall some facts and prove several Lemmas. First, the following facts

$$W_0^{1,p}(0,1)\subseteq C[0,1] \text{ and } \|u\|_{C[0,1]}\le c_0\|u\|_{W_0^{1,p}} \text{ for } u\in W_0^{1,p} \tag{2.1}$$

([11],Lemma A.9 (ii), Page 56) will be used to study (1.1), where $c_0>0$ is a constant.

Lemma 2.1 *([12]) For every $w\in L^1(0,1)$, the quasilinear boundary value problem*

$$\begin{cases}-\Delta_p u(x)=w(x) \ \text{ for a. e. } x\in(0,1)\\ u(0)=u(1)=0\end{cases} \tag{2.2}$$

has a unique solution u in $D(\Delta_p)$,where

$$D(\Delta_p)=\{u\in C_0^1[0,1]:\phi_p(u')\in AC[0,1]\}$$

and $C_0^1[0,1]=\{u\in C_0^1[0,1]:u(0)=u(1)=0\}$ is a Banach space with the norm $\|u\|_{C^1[0,1]}=\|u\|_{C[0,1]}+\|u'\|_{C[0,1]}$.

We denote by T the inverse of $-\Delta_p$. Then $T:L^1(0,1)\to D(\Delta_p)$ is defined by

$$Tw=u \tag{2.3}$$

where u is the unique solution of (2.2) in $D(\Delta_p)$.

It is easy to see that the inverse operator T has the following property [2].

$$T(tw)=t^{\frac{1}{p-1}}T(w) \text{ for } w\in L^1(0,1) \text{ and } t\ge 0. \tag{2.4}$$

Lemma 2.2 *The map $T:L^1(0,1)\to D(\Delta_p)$ is increasing, that is, $w_1,w_2\in L^1(0,1)$, $w_1\le w_2$ implies $Tw_1\le Tw_2$ and $Tw\ge 0$ for $w\in L^1_+(0,1)$.*

Proof. By lemma 2.1, we may assume that $u_i\in D(\Delta_p)(i=1,2)$ such that $Tw_i=u_i\ (i=1,2)$. Then

$$-\Delta_p u_i(x)=w_i(x),\ i=1,2 \tag{2.5}$$

Since $u_i(0)=0=u_i(1)$, we have $\xi\in(0,1)$ such that $u_1'(\xi)=u_2'(\xi)$. By (2.5), we have

$$u_i(x)\begin{cases}\int_0^x\phi_p^{-1}\left[\int_s^\xi w_i(\tau)d\tau+\phi_p(u_i'(\xi))\right]ds, & 0\le x\le\xi\\ \int_0^x\phi_p^{-1}\left[\int_\xi^s w_i(\tau)d\tau-\phi_p(u_i'(\xi))\right]ds, & \xi<x\le 1\end{cases} \tag{2.6}$$

Since ϕ_p^{-1}, the inverse function of ϕ_p is an increasing function, then $w_1 \le w_2$ implies that $u_1 \le u_2$ and $Tw_1 \le Tw_2$. Let $w_1 = 0$ and $-\Delta_p u = w \in L_+^1(0,1)$. Then $u \ge u_1 = 0$ and $Tw = u \ge 0$. □

Lemma 2.3 ([13]) *For each* $g \in L_+^1(0,1)$ *with* $\int_0^1 g(x)ds > 0$, *there exists* $u_g > 0$ *and* $\varphi_g \in C_0^1[0,1] \cap (P \setminus \{0\})$ *satisfying*

$$\begin{cases} -\Delta_p \varphi_g(x) = \mu_g g(x) \varphi_g^{p-1}(x) & \text{for a.e. } x \in (0,1), \\ \varphi_g(0) = \varphi_g(1) = 0. \end{cases} \tag{2.7}$$

The positive value μ_g is called the first eigenvalue of (2.7), φ_g is called the eigenfunction corresponding to the eigenvalue μ_g and we see that for $g \in L_+^1(0,1) \setminus \{0\}$,

$$\mu_g = \inf\left\{ \int_0^1 |u'(x)|^p dx / \int_0^1 g(x)|u(x)|^p dx : u \in W_0^{1,p}(0,1) \setminus \{0\} \right\}, \tag{2.8}$$

where $\int_0^1 |u'(x)|^p dx / \int_0^1 g(x)|u(x)|^p dx = \infty$ if $\int_0^1 g(x)|u(x)|^p dx = 0$. It is given in [7] that the frst eigenvalue μ_g with $g \equiv 1$ equals

$$\mu_1(p) := \left\{ 2\int_0^{(p-1)^{\frac{1}{p}}} \left[1 - s^p (p-1)^{-1}\right]^{-\frac{1}{p}} ds \right\}^p. \text{ Specially, } \mu_1(2) = \pi^2. \tag{2.9}$$

Define a map A from $W_0^{1,p}$ to $D(\Delta_p)$ by

$$Az(x) = (TFz)(x) \tag{2.10}$$

where T is given in (2.3) and the Nemytskii operator $F : C[0,1] \to L^1(0,1)$ is defined by

$$Fz(x) = f(x, rz(x)), \tag{2.11}$$

where $rz(x) = \max\{z(x), 0\}$.

Lemma 2.4 *Assume that* $z \not\equiv 0$ *and is (classical) solution of following problem*

$$\begin{cases} -\Delta_p z(x) = \nu f(x, rz(x)) & \text{for a.e. } x \in (0,1) \\ z(0) = z(1) = 0 \end{cases} \tag{2.12}$$

where $\nu > 0$. *Then the following assertions hold.*

(i) $z(x) \ge 0$ *for* $x \in (0,1)$

(ii) *If there exists* $\rho > 0$ *such that* $f(x,u) \ge 0$ *for a.e.* $(x,u) \in [0,1] \times [0,\rho]$.

Then $z(x) > 0$ *for* $x \in (0,1)$.

Proof. (i) If there exists $x_0 \in (0,1)$ such that $z(x_0) < 0$, by $z(0) = z(1) = 0$, we may choose $[a,b] \subseteq [0,1]$ such that $x_0 \in (a,b)$, $z(x) < 0$ on (a,b) and $z(a) = z(b) = 0$.

Let $\xi \in (a,b)$ such that $z(\xi) = \min\{z(x) : x \in [a,b]\} < 0$, then $z'(\xi) = 0$.

Integrating (2.12) from x to ξ and noticing $rz(x) = 0$ on $[a,b]$, we have

$$\Phi_p(z'(x)) = \nu \int_x^\xi f(t, rz(t))dt = \nu \int_x^\xi f(t,0)dt \ge 0 \text{ for } x \in [a,\xi]$$

and

$$\Phi_p(z'(x)) = \nu \int_x^\xi f(t,0)dt \ge 0 \text{ for } x \in [a,\xi].$$

Then, we obtain $z'(x) \ge \Phi_p^{-1}(0) = 0$ on $[a,\xi]$, that is, $z(x)$ is increasing on $[a,\xi]$ and $z(\xi) \ge z(a)$, a contradiction. Hence $z(x) \ge 0$ on $[0,1]$.

(ii) If there exists $x \in (0,1)$ such that $z(x) = 0$,without loss of generality, we may assume by $z(x) \ne 0$ and $z(x) \ge 0$ on $[0,1]$ that there exists $a \in (0, x_0)$ such that $z(a) > 0$, $z(x_0) = 0$ and $z(x) \le \rho$ on $[a, x_0]$.

Integrating (2.12) from x to x_0 and noticing $z'(x_0)=0$. We obtain easily

$$\Phi_p\left(z'(x)\right)=\nu\int_x^{x_0} f(t,0)dt \ge 0 \text{ for } x\in[a,x_0].$$

From this we have $z'(x)\ge \Phi_p^{-1}(0)=0$ on $[a,x_0]$ and $z(x_0)\ge z(a)>0$, it is a contradiction. Hence $z(x)>0$ for $x\in(0,1)$. □

Lemma 2.5 Under (C_1)-(C_3), *the following assertions hold.*

(i) *The map* A *defined in* (2.10) *maps* $W_0^{1,p}(0,1)$ *into* $W_0^{1,p}(0,1)$ *and is compact.*

(ii) $z\in W_0^{1,p}(0,1)$ *is a fixed point of* A *id and only if* z *is a nonnegative solution of* (1.1).

Proof. (i) The proof is very similar to that of Theorem 2.1 (i) in [2] and is omitted.

(ii) If $z\in W_0^{1,p}(0,1)$ is fixed point of A ,then $z(x)\ge 0$ for $[0,1]$ by Lemma 2.4 (i). Since T maps $L^1(0,1)$ into $C_0^1[0,1]$, and thus, by (2.3) , $z\in C^1[0,1]$ and is a nonnegative solution of (1.1).

The inverse is true obviously.

We shall use the following known result(for example [14], which can be proved by using Leray-Schauder degree theory for compact maps in Banach spaces).

Lemma 2.6 *Let* E *be a Banach space,* Ω_1 *and* Ω_2 *be two bounded open sets of* E , $\theta\in\Omega_1\subset\Omega_2$, *where* θ *is zero element of* E . *Assume that* $A:\Omega_2\setminus\Omega_1\to E$ *is compact and satisfies*

(1) *There exists* $y_0\in E\setminus\{0\}$ *such that* $x\ne Ax+ty_0$ *for* $x\in\partial\Omega_1$ *and* $t\ge 0$.

(2) $x\ne tAx$ *for* $x\in\partial\Omega_2$ *and* $0<t\le 1$.

Then A *has a fixed point in* $\Omega_2\setminus\overline{\Omega}_1$.

Now, we state and prove new results of positive solutions for (1.1).

Theorem 2.1 Assume *that* (C_1)-(C_3) *and the following conditions hold.*

(H_1) *There exists* $\rho_0>0$, $\varepsilon_1>0$ *and* $\psi_{\rho_0}\in L_+^1(0,1)\setminus\{0\}$ *such that*

$$f(x,u)\ge\left(\mu_{\psi_{\rho_0}}+\varepsilon_1\right)\psi_{\rho_0}(x)u^{p-1} \text{ for a.e. } x\in[0,1] \text{ and all } u\in[0,\rho_0].$$

(H_2) *There exist* $r_0>0$, $\varepsilon_2\in\left(0,\mu_{\phi_{r_0}}\right)$ *and* $\phi_{r_0}\in L_+^1(0,1)\setminus\{0\}$ *such that*

$$f(x,u)\le\left(\mu_{\phi_{r_0}}-\varepsilon_2\right)\phi_{r_0}(x)u^{p-1} \text{ for a.e. } x\in[0,1] \text{ and all } u\in[r_0,\infty].$$

Then (1.1) *has a positive solution* z *in* $C_0^1[0,1]$, *that is,* $z\in C_0^1[0,1]$ *satisfies* $z(x)>0$ *for* $x\in(0,1)$.

Proof. Let $\rho=c_0^{-1}\rho_0$ and $\Omega_\rho=\{z:z\in W_0^{1,p}(0,1),\|z\|<\rho\}$. If there exists $z\in\partial\Omega_\rho$ such that $z=T(Fz)$, then Lemma 2.4 (ii) (see H_1) implies that the results of Theorem 2.1 holds. Hence, we assume that $z\ne T(Fz)$ for $z\in\partial\Omega_\rho$ and prove that

$$z\ne T(Fz)+te \text{ for } z\in\partial\Omega_\rho \text{ and } t>0, \tag{2.13}$$

where e is the eigenfunction corresponding to the eigenvalue $\mu_{\psi_{\rho_0}}$, that is,

$$\begin{cases}-\Delta_p e(x)=\mu_{\psi_{\rho_0}}\psi_{\rho_0}(x)e^{p-1}(x) \text{ for a.e. } x\in(0,1),\\ e(0)=e(1)=0.\end{cases}$$

In fact, if not, there exists $z\in\partial\Omega_\rho$ and $t>0$ such that $z=T(Fz)+te$. By (2.1) ,we have for $z\in\partial\Omega_\rho$,

$$|z(x)|\le\|z\|_{C[0,1]}\le c_0\|z\|_{W_0^{1,p}}=c_0\rho=\rho_0 \text{ for } x\in[0,1] \tag{2.14}$$

Hence $0 \le rz(x) \le \rho_0$, by (H_1), and $Fz(x) = F(x, rz(x)) \ge 0$ for $x \in [0,1]$. From this and Lemma 2.2 we know $z \ge te$. Let $\tau = \sup\{v : z \ge ve\}$. Then $0 < \tau < \infty$ and $z \ge \tau e$.

It follows from (H_1) and (2.14) that for a.e. $x \in [0,1]$

$$f(x, z(x)) \ge \left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)\psi_{\rho_0}(x) z^{p-1}(x) \ge \left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)\tau^{p-1}\psi_{\rho_0}(x) e^{p-1}(x). \tag{2.15}$$

By Lemma 2.2, (2.4) and (2.15), we have

$$z(x) \ge TFz(x) \ge T\left[\left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)\tau^{p-1}\psi_{\rho_0}(x) e^{p-1}(x)\right] = \tau\left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)^{\frac{1}{p-1}} T\left[\psi_{\rho_0}(x) e^{p-1}(x)\right].$$

Noticing that $\mu_{\psi_{\rho_0}}^{-\frac{1}{p-1}} e(x) = T\left[\psi_{\rho_0}(x) e^{p-1}(x)\right]$, we have

$$z(x) \ge \tau\left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)^{\frac{1}{p-1}} \mu_{\psi_{\rho_0}}^{-\frac{1}{p-1}} e(x).$$

The definition of τ implies $\tau \ge \tau\left(\mu_{\psi_{\rho_0}} + \varepsilon_1\right)^{\frac{1}{p-1}} \mu_{\psi_{\rho_0}}^{-\frac{1}{p-1}} > \tau$, it is a contradiction. Hence (2.13) holds.

By (C_2), there exists $g_{r_0} \in L_+^1[0,1]$ such that $|f(x,u)| \le g_{r_0}(x)$ for a.e. $x \in [0,1]$ and all $u \in [0, r_0]$. This together with (H_2), implies, for a.e. $x \in [0,1]$ and all $u \in R_+$,

$$f(x,u) \le g_{r_0}(x) + \left(\mu_{\phi_{r_0}} - \varepsilon_2\right)\phi_{r_0}(x) u^{p-1} . \tag{2.16}$$

Let $r_1 = \left(\varepsilon_2^{-1} c_0 \mu_{\phi_{r_0}} \|g_{r_0}\|_{L^1}\right)^{\frac{1}{p-1}}$, $r > \max\{r_1, c_0^{-1}\rho_0\}$ and $\Omega_r = \{z : z \in W_0^{1,p}(0,1), \|z\| < r\}$.

Clearly, $r > c_0^{-1}\rho_0 = \rho$. We prove that

$$z \ne tAz \quad \text{for} \quad z \in \partial\Omega_r \text{ and } t \in [0,1] \tag{2.17}$$

If (2.17) is false, then there exists $z \in \partial\Omega_r$ and $t \in [0,1]$ such that $z = tAz$. It follows from (2.4) and (2.3) that $z(x) = T(t^{p-1}Fz)(x)$ and $-\Delta_p z(x) = t^{p-1} f(x, rz(x))$ for a.e. $x \in [0,1]$. By Lemma 2.4(i), we have $z(x) \ge 0$ for $x \in (0,1)$. Hence

$$-\Delta_p z(x) = t^{p-1} f(x, z(x)) \quad \text{for a.e. } x \in [0,1]. \tag{2.18}$$

By (2.18), (2.16), (2.8) with $g = \phi_{r_0}$ and (2.1), we have

$$\begin{aligned}
\|z\|_{W_0^{1,p}}^p &= (-\Delta_p z, z) = t^{p-1}\int_0^1 f(x, z(x)) z(x)\,dx \le \int_0^1 f(x, z(x)) z(x)\,dx \\
&\le \int_0^1 \left[g_{r_0}(x) + \left(\mu_{\phi_{r_0}} - \varepsilon_2\right)\phi_{r_0}(x) z^{p-1}(x)\right] z(x)\,dx \\
&\le \|z\|_{C[0,1]} \|g_{r_0}\|_{L^1} + \left(\mu_{\phi_{r_0}} - \varepsilon_2\right) u_{\phi_{r_0}}^{-1} \|z\|_{W_0^{1,p}}^p \\
&\le c_0 \|z\|_{W_0^{1,p}} \|g_{r_0}\|_{L^1} + \left(1 - \varepsilon_2 u_{\phi_{r_0}}^{-1}\right)\|z\|_{W_0^{1,p}}^p
\end{aligned}$$

This implies that $\|z\|_{W_0^{1,p}}^{p-1} \le \varepsilon_2^{-1} c_0 u_{\phi_{r_0}} \|g_{r_0}\|_{L^1}$. Hence, we have

$$r_1 < r = \|z\|_{W_0^{1,p}} \le \left(\varepsilon_2^{-1} c_0 u_{\phi_{r_0}} \|g_{r_0}\|_{L^1}\right)^{\frac{1}{p-1}} = r_1,$$

a contradiction. Hence (2.17) holds.

By Lemma 2.5 (i) and Lemma 2.6, there exists $z \in \Omega_r \setminus \bar{\Omega}_\rho$ such that $z = Az$ and by Lemma 2.5 (ii), z is a nonnegative solution of (1.1). It follows from (H_1) and Lemma 2.4(ii) that $z(x) > 0$ for $x \in (0,1)$, that is, z is a positive solution of (1.1).

Let E be a fixed subset of $[0,1]$ of measure zero and

$$\overline{f}(u)=\sup_{x\in[0,1]\setminus E} f(x,u),\quad \underline{f}(u)=\inf_{x\in[0,1]\setminus E} f(x,u).$$

Notation:

$$(f_p)_0=\liminf_{z\to 0^+}\inf_{t\in[0,1]}\frac{\underline{f}(u)}{\phi_p(u)},\quad f_p^\infty=\limsup_{z\to\infty}\sup_{t\in[0,1]}\frac{\overline{f}(u)}{\phi_p(u)}.$$

Corollary 2.1 *Assume that* (C_1) - (C_3) *and the following conditions hold.*

$$f_p^\infty<\mu_1(p)<(f_p)_0\le\infty, \tag{2.19}$$

where $\mu_1(p)$ *is the same as in* (2.9).

Then (1.1) *has a positive solution* z *in* $C_0^1[0,1]$.

Proof. By (2.19), (H_1) with $\phi_{\rho_0}=1$ and (H₂) with $\psi_{r_0}\equiv 1$ hold for some $\varepsilon_i>0(i=1,2)$, $\rho_0>0$ and $r_0>0$. The result follows from Theorem 2.1.

Example 2.1 Let $f(u)=\lambda u^p(1-p)(\lambda>\mu_1(p))$, where $\mu_1(p)$ is the same as in (2.9). Then (1.1) has a positive solution in $C_0^1[0,1]$.

Proof. $f_0=\lambda$ and $f^\infty=-\infty$, the result follows from Corollary 2.1.

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