

## Coincidence point theorems under F-contraction in Ultrametric Space

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**Abstract:** In this paper, we establish some results on coincidence points for a pair of single valued maps with a pair of multivalued maps of an Ultrametric space which satisfy F-contraction. Our theorems generalize and extend the theorem of Wang and Song [13], thereby generalizing some known results in the literature.

**Keywords:** Coincidence point, coincidentally commuting, fixed point, F-contraction and spherically complete

### INTRODUCTION AND PRELIMINARIES

Roovij [12] introduced the concept of ultrametric space. Gajic studied the fixed point theorems of contractive type maps on spherically complete ultrametric spaces which are generalizations of the Banach fixed point theorems. In 2007 Rao *et al.* obtained coincidence point theorems for three and four self maps in ultrametric space. Kubiacyk and Mostafa [5] extended to the set-valued maps. In 2013, Wang and Song obtained some results on coincidence and common fixed point for a pair of single valued and a pair of multivalued maps. Many researchers took interest in generalizing and improving fixed point theorems, recently Wardowski [14] gave a generalization by introducing a new contractive map called F-contraction, which was generalized and improved [4, 7, 8, 11].

The aim of this paper is to establish the existence of coincidence and common fixed point for a pair of single valued maps and a pair of multivalued maps in ultrametric using F-contraction.

**Definition 1.1.** [12] Let  $(X, d)$  be a metric space. If the metric  $d$  satisfies strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \text{ for all } x, y, z \text{ in } X$$

then  $d$  is called an ultrametric on  $X$  and  $(X, d)$  is called an ultrametric space.

**Definition 1.2.** [12] An ultrametric space is said to be spherically complete if every shrinking collection of balls in  $X$ , has a non empty intersection.

**Definition 1.3.** An element  $x \in X$  is said to be a coincidence point of  $f: X \rightarrow C(X)$  and  $T: X \rightarrow X$ , if  $Tx \in fx$ .

**Definition 1.4.** [9] Let  $(X, d)$  be an ultrametric space and  $C(X)$  denote the class of all non empty compact subsets of  $X$ , a multivalued map  $f: X \rightarrow C(X)$  and a self map  $T: X \rightarrow X$  are said to be coincidentally commuting at  $z \in X$  if  $Tz \in fz$  implies  $Tfz \subseteq fTz$ .

**Definition 1.5.** [1] An element  $x \in X$  is a common fixed point of  $f, g: X \rightarrow C(X)$  and  $T: X \rightarrow X$  if,  $x = Tx \in fx \cap gx$ .

Let  $F$  be the set of all functions  $F: (0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

(a)  $F$  is strictly increasing, that is, for all  $\alpha, \beta \in (0, \infty)$  if  $\alpha < \beta$  then  $F(\alpha) < F(\beta)$ .

(b) For each sequence  $\{\alpha_n\}$  of positive numbers, the following holds:

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} F(\alpha_n) = -\infty.$$

(c) There exist  $k \in (0, \infty)$  such that  $\lim_{n \rightarrow 0^+} (\alpha^k F(\alpha)) = 0$ .

**Definition 1.6.** [14] Let  $(X, d)$  be a metric space. A self map  $T$  on  $X$  is an F-contraction, if  $F \in F$  and there exist  $\tau > 0$  such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \quad (1)$$

for all  $x, y \in X$  with  $Tx \neq Ty$

**MAIN RESULT**

In this section, we prove the existence of coincidence point for pair single valued maps with a pair of multi-valued maps.

The Hausdorff metric is defined as  $H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$  where  $d(x, A) = \inf\{d(x, a) : a \in A\}$

**Theorem 2.1.** Let  $(X, d)$  be an Ultra metric space. If  $S, T : X \rightarrow C(X)$  be a pair of multi valued maps and  $f, g : X \rightarrow X$  be a pair of single valued maps satisfying,

- (a)  $fgX$  is spherically complete.
- (b)  $\tau + F(H(Sx, Ty)) < F(\max\{d(Sx, fx), d(Ty, gy), d(fx, gy)\})$ , for all  $x, y \in X, Sx \neq Ty, fx \neq gy$
- (c)  $fS=Sf, fg=gf, fT=Tf, gS=Kg, gT=Tg, ST=TS$ .
- (d)  $SX \subseteq fX, TX \subseteq gX$ .

Then there exists points  $u, v$  in  $X$  such that  $fu \in Su, gv \in Tv, fu=gv, Su=Tv$ .

**Proof.** For  $\alpha \in X$ , let  $B_\alpha = B(fg\alpha, \max\{d(fg\alpha, Sg\alpha), d(fg\alpha, Tfa)\})$  denote the closed ball with center at  $fg\alpha$  and radius  $\max\{d(fg\alpha, Sg\alpha), d(fg\alpha, Tfa)\}$ .

Let  $A$  be the collection of these balls for all  $\alpha \in fgX$ . Then the relation  $B_\alpha \leq B_\beta$  if  $B_\beta \subseteq B_\alpha$  is a partial order on  $A$ .

Now, consider a totally ordered subfamily  $A_I$  of  $A$ . Since  $fgX$  is spherically complete we have that  $\bigcap_{B_\alpha \in A_I} B_\alpha = B \neq \phi$ , Let  $fg\beta \in B = \bigcap B_\alpha$  where  $\beta \in fgX$  and  $B_\alpha \in A_I$ .

Then  $fg\beta \in B_\alpha$ . Hence

$$d(fg\beta, fg\alpha) \leq \max\{d(fg\alpha, Sg\alpha), d(fg\alpha, Tfa)\} \tag{2}$$

If  $\alpha = \beta$  then  $B_\alpha = B_\beta$ .

Let  $\alpha \neq \beta$  and  $x \in B_\beta$ . Then  $d(x, fg\beta) \leq \max\{d(fg\beta, Sg\beta), d(fg\beta, Tfb)\}$ . Since  $Sg\beta$  is non empty compact set, there exist  $u \in Sg\alpha$  such that  $d(fg\alpha, u) = d(fg\alpha, Sg\alpha)$ .

And since  $Tfa$  is non empty compact set, there exist  $v \in Tfa$  such that  $d(fg\alpha, v) = d(fg\alpha, Tfa)$

$$\begin{aligned} \text{Consider } \tau + F(H(Tfa, Sg\beta)) &\leq F(\max\{d(Sg\beta, fg\beta), d(Tfa, gfa), d(fg\beta, gfa)\}) \\ F(H(Tfa, Sg\beta)) &\leq F(\max\{d(Sg\beta, fg\beta), d(Tfa, gfa), d(fg\beta, gfa)\}) - \tau \\ &< F(\max\{d(Sg\beta, fg\beta), d(Tfa, gfa), d(fg\beta, gfa)\}) \end{aligned}$$

Since  $F$  is increasing,

$$H(Tfa, Sg\beta) < \max\{d(Sg\beta, fg\beta), d(Tfa, gfa), d(fg\beta, gfa)\}g \tag{3}$$

And

$$\begin{aligned} \tau + F(H(Sga, Tfb)) &\leq F(\max\{d(Sga, fga), d(Tfb, gfb), d(fga, gfb)\}) \\ F(H(Sga, Tfb)) &\leq F(\max\{d(Sga, fga), d(Tfb, gfb), d(fga, gfb)\}) - \tau \\ &< F(\max\{d(Sga, fga), d(Tfb, gfb), d(fga, gfb)\}) \end{aligned}$$

Since  $F$  is increasing,

$$H(Sga, Tfa) < \max\{d(Sga, fga), d(Tfb, gfb), d(fga, gfb)\} \tag{4}$$

Now

$$\begin{aligned} \max\{d(fg\beta, Sg\beta), d(fg\beta, Tfb)\}g &= \max\{\inf_{c \in Sg\beta} d(fg\beta, c), \inf_{l \in Sg\beta} d(fg\beta, l)\} \\ &\leq \max\{d(fg\beta, fga), d(fga, v), \inf_{l \in Sg\beta} d(v, c), d(fg\beta, fga), \\ &\quad d(fga, u), \inf_{d \in Tfb} d(u, d)\} \\ &\leq \max\{d(fg\beta, fga), d(fga, Tfa), H(Tfa, Sg\beta), \\ &\quad d(fg\beta, fga), d(fga, Sga), H(Sga, Tfb)\}g \\ &< \max\{d(fg\beta, fga), d(fga, Tfa), d(fga, Sga), \\ &\quad \max\{d(Sg\beta, fg\beta), d(Tfa, gfa), d(fg\beta, gfa)\}, \\ &\quad \max\{d(Sga, fga), d(Tfb, gfb), d(fga, fg\beta)\}\} \\ &= \max\{d(Sga, fga), d(fga, Tfa)\} \end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned} d(x, fga) &\leq \max\{d(x, fg\beta), d(fg\beta, fga)\} \\ &\leq \max\{d(fga, Sga), d(fga, Tfa)\} \end{aligned} \tag{6}$$

Thus,  $x \in B_\alpha$ . Hence  $B\beta \subseteq B_\alpha$  for any  $B_\alpha$  in  $A_I$ . Thus  $B\beta$  is the upper bound for the family  $A_I$

in A and hence by Zorn's lemma A has a maximum element say  $B_z, z \in fgX$ . There exists  $w \in X$  such that  $z = fgw$ .

Now to prove that  $f(gfgw) \in S(gfgw)$  and  $f(ffgw) \in T(ffgw)$ . Suppose that  $f(gfgw) \notin S(gfgw)$  and  $f(ffgw) \notin T(ffgw)$ .

Since  $Sgfgw, Tffgw$  are non empty compact sets, there exist  $k \in Sgfgw, t \in Tffgw$ , such that  $d(fgfgw, Sgfgw) = d(fgfgw, k), d(fgfgw, Tffgw) = d(fgfgw, t)$

We have,

$$\begin{aligned} d(Sgfgw, TSfgw) &= \inf_{e \in TSfgw} d(Sgfgw, e) \\ &\leq \max\{d(Sgfgw, fgfgw), d(fgfgw, k), \inf_{e \in TSfgw} d(k, e)\} \\ &\leq \max\{d(Sgfgw, fgfgw), d(fgfgw, Sgfgw), \inf_{e \in TSfgw} d(k, e)\} \\ &\leq \max\{d(Sgfgw, fgfgw), H(Sgfgw, TSfgw)\} \\ &< \max\{d(Sgfgw, fgfgw), \max\{d(Sgfgw, fgfgw), d(TSfgw, gSfgw), d(fgfgw, gSfgw)\}\} \\ &= \max\{d(Sgfgw, fgfgw), \max\{d(Sgfgw, fgfgw), d(TSfgw, Sgfgw), d(fgfgw, Sgfgw)\}\} \\ &= d(Sgfgw, fgfgw) \end{aligned} \tag{7}$$

And also we have

$$\begin{aligned} d(Tffgw, STfgw) &= \inf_{h \in STfgw} d(Tffgw, h) \\ &\leq \max\{d(Tffgw, fgfgw), d(fgfgw, t), \inf_{h \in TSfgw} d(k, e)\} \\ &\leq \max\{d(Sgfgw, fgfgw), d(fgfgw, Sgfgw), \inf_{e \in TSfgw} d(k, e)\} \\ &\leq \max\{d(Sgfgw, fgfgw), H(Sgfgw, TSfgw)\} \\ &< \max\{d(Sgfgw, fgfgw), \max\{d(Sgfgw, fgfgw), d(TSfgw, gSfgw), d(fgfgw, gSfgw)\}\} \\ &= \max\{d(Sgfgw, fgfgw), \max\{d(Sgfgw, fgfgw), d(TSfgw, Sgfgw), d(fgfgw, Sgfgw)\}\} \\ &= d(Sgfgw, fgfgw) \end{aligned} \tag{8}$$

And

$$\begin{aligned} d(fggSw, SggSw) &= \inf_{m \in SggSw} d(fggSw, m) \\ &\leq \max\{d(fggSw, TSfgw), d(TSfgw, Tffgw), d(Tffgw, fgfgw), d(fgfgw, t), \inf_{m \in SggSw} d(t, m)\} \\ &\leq \max\{d(gSfgw, TSfgw), d(Tffgw, fgfgw), H(Tffgw, SggSw)\} \\ &< \max\{d(Sgfgw, fgfgw), d(Tffgw, fgfgw), \max\{d(SggSw, fgfgw), d(Tffgw, gffTw), d(fggSw, gffTw)\}\} \\ &= \max\{d(Sgfgw, fgfgw), d(Tffgw, fgfgw)\} \end{aligned} \tag{9}$$

Also

$$\begin{aligned} d(fgfTw, TffTw) &= \inf_{n \in TffTw} d(fgfTw, n) \\ &\leq \max\{d(fgfTw, STfgw), d(STfgw, Sgfgw), d(Sgfgw, fgfgw), d(fgfgw, k), \inf_{m \in TffTw} d(k, n)\} \\ &\leq \max\{d(ffTgw, STfgw), d(fgfgw, Sgfgw), H(Sgfgw, TffTw)\} \\ &< \max\{d(Tffgw, fgfgw), d(fgfgw, Sgfgw), \max\{d(Sgfgw, fgfgw), d(TffTw, gffTw), d(fgfgw, gffTw)\}\} \\ &= \max\{d(Tffgw, fgfgw), d(fgfgw, Sgfgw)\} \end{aligned} \tag{10}$$

From the equation (7) and (9)

$$\begin{aligned} \max\{d(Sgfgw, TSfgw), d(fggSw, SggSw)\} &< \max\{d(fgfgw, Sgfgw), \max\{d(fgfgw, Sgfgw), d(fgfgw, Tffgw)\}\} \\ &= \max\{d(fgfgw, Sgfgw), d(fgfgw, Tffgw)\} \end{aligned} \tag{11}$$

From the equation (8)

$$\begin{aligned} \max\{d(STfgw, Tffgw), d(fgfTw, TffTw)\} &< \max\{d(Tffgw, fgfgw), \max\{d(fgfgw, Tffgw), d(fgfgw, Sgfgw)\}\} \\ &= \max\{d(Tffgw, fgfgw), d(fgfgw, Sgfgw)\} \end{aligned} \tag{12}$$

Case:(i) If  $\max\{d(fgfgw, Sgfgw), d(fgfgw, Tffgw)\} = d(fgfgw, Sgfgw)$

Then from (11)  $fgfgw \notin B_{gSw}$  which implies  $fgz \notin B_{gSw}$ . Therefore  $B_z \subsetneq B_{gSw}$ . It is a contradiction to the maximality of  $B_z$  in A. Hence  $gSw \subsetneq gfX = fgX$ .

Case:(ii) If  $\max\{d(fgfgw, Sgfgw), d(fgfgw, Tffgw)\} = d(fgfgw, Tffgw)$ .

Then from (12)  $fgfgw \notin B_{fTw}$  which implies  $fgz \notin B_{fTw}$ . Hence  $B_z \subsetneq B_{fTw}$

$$fgfgw \in Sgfgw, gffgw \in Tffgw \tag{13}$$

and  $f(gfgw) = fgfgw = gffgw = g(ffgw)$ . Using (b), (c) and equation (13), we have

$$\begin{aligned} \tau + F(H(Sgfgw, Tffgw)) &< F(\max\{d(Sgfgw, fgfgw), d(Tffgw, gffgw), d(fgfgw, gffgw)\}) \\ F(H(Sgfgw, Tffgw)) &< F(\max\{d(Sgfgw, fgfgw), d(Tffgw, gffgw), d(fgfgw, gffgw)\}) - \tau \\ &< F(\max\{d(Sgfgw, fgfgw), d(Tffgw, gffgw), d(fgfgw, gffgw)\}) \\ &= 0 \end{aligned}$$

Hence  $Sgfgw = Tffgw$  which implies If  $u = gfgw, v = ffgw$  then  $Su = Tv, fu = gv, fu \in Su$ ,

$gv \in Tv$ . Hence the proof.

**Corollary 2.2.** Theorem 2.1 holds if the condition (b) is replaced by

$$\tau + F(H(Sx, Ty)) < F(\max\{d(Sx, fx), d(Ty, gy), d(fx, gy), d(gy, Sx), d(Ty, fx)\}) \quad (14)$$

for all  $x, y \in X, Sx \neq Ty, fx \neq fy$ .

**Proof.** By strong triangle inequality,  $d(Sx, gy) \leq \max\{d(Sx, fx), d(fx, gy)\}$  and  $d(Ty, fx) \leq \max\{d(Ty, gy), d(gy, fx)\}$  it follows that (14) implies condition (b) of Theorem 2.1.

**Theorem 2.3.** Let  $(X, d)$  be an Ultra metric space. If  $S, T : X \rightarrow C(X)$  be a pair of multi-valued maps and  $f : X \rightarrow X$  be a single valued map satisfying,

- (a)  $fX$  is spherically complete.
- (b)  $\tau + F(H(Sx, Ty)) \leq F(\max\{d(Sx, fx), d(Ty, fy), d(fx, fy)\})$ , for all  $x, y \in X, Sx \neq Ty, fx \neq fy$
- (c)  $fS = Sf, fT = Tf, ST = TS$ .
- (d)  $SX \in fX, TX \in fX$ .

Then  $f, S$  and  $T$  have a coincidence point in  $X$ . If  $f$  and  $S, f$  and  $T$  are coincidentally commuting at  $z \in C(f, T)$  and  $ffz = fz$  then  $f, S$  and  $T$  have a common fixed point in  $X$ .

**Proof.** If  $f = g$  in the theorem 2.1 then we have the points  $u$  and  $v$  such that  $fu \in Su, fv \in Tv, fu = fv, Su = Tv$ .

As  $u \in C(f, S), f$  and  $S$  are coincidentally commuting at  $u$  and  $ffu = fu$ . Let  $w = fu$ , then  $w \in Su, w \in Tv$  implies  $w \in Su \cap Tv$ . Therefore  $fw = w$  and  $w = fw \in fSu \subseteq Sfu = Sw$ . Hence  $w = fw \in Sw$ .

Also, since  $u \in C(f, T), f$  and  $T$  are coincidentally commuting at  $u$  and  $ffu = fu$ . Take  $w = fu$ , then  $w \in Su, w \in Tv$ . Then we have  $fw = w$  and  $w = fw \in f(Tu) \subseteq Tfu = Sw$ .

Now, since also  $u \in C(f, T), f$  and  $T$  are coincidentally commuting at  $u$  and  $ffu = fu$ , We have  $w = fw \in f(Tv) \subseteq Tfv = Tw$ . Thus, we have proved that  $w = fw \in Sw \cap Tw$ , that is,  $w$  is the common fixed point of  $f, S$  and  $T$ .

**Corollary 2.4.** Theorem 2.3 holds if the condition (b) is replaced by

$$\tau + F(H(Sx, Ty)) < F(\max\{d(Sx, fx), d(Ty, fy), d(fx, gy), d(fy, Sx), d(Ty, fx)\}) \quad (15)$$

for all  $x, y \in X, Sx \neq Ty, fx \neq fy$ .

**Proof.** By strong triangle inequality,  $d(Sx, fy) \leq \max\{d(Sx, fx), d(fx, fy)\}$  and  $d(Ty, fx) \leq \max\{d(Ty, fy), d(fy, fx)\}$  it follows that (15) implies condition (b) of Theorem 2.3.

**Corollary 2.5.** Let  $(X, d)$  be a spherically complete ultra metric space. Let  $T, S : X \rightarrow C(X)$  be a pair of multi-valued maps satisfying:

- (a)  $\tau + F(H(Sx, Ty)) < F(\max\{d(x, y), d(x, Sx), d(y, Ty)\})$  for all  $x, y \in X, Sx \neq Ty, x \neq y$ .
- (b)  $ST = TS$ .

Then, there exist a point  $z$  in  $X$  such that  $z \in Sz \cap Tz$  and  $Sz = Tz$ .

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