Scholars Journal of Physics, Mathematics and Statistics **ISSN 2393-8056 (Print)**
Sch. J. Phys. Math. Stat. 2017; 4(4):145-161 **ISSN 2393-8064 (Online)**

On the Exact Solutions and the Approximate Solutions by Adomian Decomposition Method of the Second-Order Linear Fuzzy İnitial Value Problems Using the Generalized Differentiability

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Article History
Received: 20.09.2017
Accepted: 18.10.2017
Published: 30.10.2017

DOI:
10.21276/sjpms.2017.4.4.1

Abstract: In this paper investigates the exact solutions and the approximate solutions by Adomian decomposition method of the second-order linear fuzzy initial value problems with positive and negative constant coefficients using the generalized differentiability. Thus, comparisons results is given.
Keywords: Fuzzy initial value problem, second-order fuzzy differential equation, Generalized differentiability, Adomian decomposition method.
2010 AMS Classification. 03E72; 34A07, 65L05

INTRODUCTION

Many researhers study fuzzy differential equations. Fuzzy differential equations can be solved with several approach. The first approach is using the Hukuhara differentiability. For this, mainly the existence and uniqueness of the solution of a fuzzy differential equation is studied [13, 19]. The existence and uniqueness of solutions of two-point fuzzy boundary value problems for second-order fuzzy differential equations under the approach of Hukuhara differentiability have been investigated by Gültekin and Altınışık [10]. Also, Gültekin Çitil and Altınışık [11] have defined the fuzzy Sturm-Liouville equation under the approach of the Hukuhara differentiability. The second approach is using the generalized differentiability. New solutions for some fuzzy boundary value problems using the generalized differentiability have been found by Khastan and Nieto [16].

Also, Khastan *at al.* [15] present a generalized concept of higher-order differentiability to solve nth-order fuzzy differential equations. The third approach generate the fuzzy solution from the crips solution [5, 6, 12, 8]. But, many fuzzy initial and boundary value problems can not be solved analyitically . Some numeric methods are introduced in [1,2, 4, 7, 14] . Adomian decomposition method was introduced by Adomian [3]. Guo *at al.* [9] have found the approximate solution of a class of second-order linear differential equation with fuzzy boundary value conditions by the undetermined fuzzy coefficients method.

In this paper we investigate the exact solutions and the approximate solutions by Adomian decomposition method of the second-order linear fuzzy initial value problems with positive and negative constant coefficients using the generalized differentiability . Thus, we give comparisons results.

Preliminaries

Definition 1 [17] A fuzzy number is a function $u:\mathbb{R}\to[0,1]$ satisfying the following properties:

u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$, $cl\{x\in\mathbb{R}\mid u(x)>0\}$ is compact where cl denotes the closure of a subset.
Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition-2 [16] Let $u \in \mathbb{R}_F$. The α -level set of u, denoted, $[u]^\alpha$, $0 < \alpha \le 1$, is $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \ge 0\}$. If $\alpha = 0$, the support of u is defined $[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. The notation, denotes explicitly the α -level set of u. The notation, $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ denotes explicitly the α -level set of u .We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ is a valid α -level set.

Remark-1 [16] The sufficient and necessary conditions for $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ to define the parametric form of a fuzzy number as follows:

$\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

$\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

$\underline{u}_\alpha \le \overline{u}_\alpha$, $0 \le \alpha \le 1$.

Definition-3 [17] If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the α – level sets of $[A]^\alpha$ is $[A]^\alpha = \left[\underline{a} + \left(\dfrac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\dfrac{\overline{a} - \underline{a}}{2}\right)\alpha\right]$.

Definition-4 [18, 9, 16] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w = u \underset{H}{-} v$.

Definition 5 [16] Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is (1)-differentiable at t_0, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small near to 0, exist $f(t_0 + h) - f(t_0)$, $f(t_0) - f(t_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) - f(t_0 - h)}{h} = f'(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0) - f(t_0 + h)$, $f(t_0 - h) - f(t_0)$ and the limits

$$\lim_{h \to 0} \frac{f(t_0) - f(t_0 + h)}{-h} = \lim_{h \to 0} \frac{f(t_0 - h) - f(t_0)}{-h} = f'(t_0),$$

Theorem-1 [15] Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$.

(i) If f is (1)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)\right]$,

(ii) If f is (2)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)\right]$.

Theorem-2 [15] Let $f' : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$, f is (1)-differentiable or (2)-differentiable.

(i) If f and f' are (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$,

(ii) If f is (1)-differentiable and f' is (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iii) If f is (2)-differentiable and f' is (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iv) If f and f' are (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$,

Second-order fuzzy linear initial value problems

The case of positive constant coefficient

Consider the fuzzy boundary value problem

$$y''(t) = \lambda y(t),\ y(t_0) = A,\ y'(t_0) = B, \tag{3.1}$$

where $\lambda > 0$, $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right]$, $[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right]$ are symmetric triangular fuzzy numbers. Here, (i,j) solution means that y is (i) differentiable and y' is (j) differentiable, i,j=1,2.

The Exact Solution By Generalized Differentiability

From the fuzzy differential equation in (3.1), for the (1,1) solution and (2,2) solution we have differential equations

$$\underline{Y}''_\alpha(t) = \lambda \underline{Y}_\alpha(t), \quad \overline{Y}''_\alpha(t) = \lambda \overline{Y}_\alpha(t)$$

by using the generalized differentiability. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.1) are obtained as

$$\underline{Y}_\alpha(t) = \underline{c}_1(\alpha)e^{\sqrt{\lambda}t} + \underline{c}_2(\alpha)e^{-\sqrt{\lambda}t}, \quad \overline{Y}_\alpha(t) = \overline{c}_1(\alpha)e^{\sqrt{\lambda}t} + \overline{c}_2(\alpha)e^{-\sqrt{\lambda}t}.$$

Using the initial conditions, coefficients $\underline{c}_1(\alpha)$, $\underline{c}_2(\alpha)$, $\overline{c}_1(\alpha)$, $\overline{c}_2(\alpha)$ are solved as

$$\underline{c}_1(\alpha) = \frac{\sqrt{\lambda}\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}e^{\sqrt{\lambda}t_0}}, \quad \underline{c}_2(\alpha) = \frac{\sqrt{\lambda}\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) - \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}e^{-\sqrt{\lambda}t_0}},$$

$$\overline{c}_1(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}e^{\sqrt{\lambda}t_0}}, \quad \overline{c}_2(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) - \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}e^{-\sqrt{\lambda}t_0}}.$$

Similarly, for the (1,2) solution and (2,1) solution we have differential equations

$$\underline{Y}''_\alpha(t) = \lambda\overline{Y}_\alpha(t), \quad \overline{Y}''_\alpha(t) = \lambda\underline{Y}_\alpha(t)$$

by using the generalized differentiability. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.1) are obtained as

$$\underline{Y}_\alpha(t) = c_1(\alpha)e^{\sqrt{\lambda}t} + c_2(\alpha)e^{-\sqrt{\lambda}t} - c_3(\alpha)\sin\left(\sqrt{\lambda}t\right) - c_4(\alpha)\cos\left(\sqrt{\lambda}t\right),$$

$$\overline{Y}_\alpha(t) = c_1(\alpha)e^{\sqrt{\lambda}t} + c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\sin\left(\sqrt{\lambda}t\right) + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right).$$

Using the initial conditions, coefficients $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are solved as

$$c_1(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a}+\underline{a}\right) + \left(\overline{b}+\underline{b}\right)}{4\sqrt{\lambda}e^{\sqrt{\lambda}t_0}}, \quad c_2(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a}+\underline{a}\right) - \left(\overline{b}+\underline{b}\right)}{4\sqrt{\lambda}e^{\sqrt{\lambda}t_0}},$$

$$c_3(\alpha) = \frac{(1-\alpha)\left[\left(\overline{a}-\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}t_0\right) + \left(\overline{b}-\underline{b}\right)\cos\left(\sqrt{\lambda}t_0\right)\right]}{2\sqrt{\lambda}},$$

$$c_4(\alpha) = \frac{(1-\alpha)\left[\left(\overline{a}-\underline{a}\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}t_0\right) - \left(\overline{b}-\underline{b}\right)\sin\left(\sqrt{\lambda}t_0\right)\right]}{2\sqrt{\lambda}}.$$

For the (1,1) solution and (2,2) solution, the equation (3.1) is written as

$$\underline{y}''_\alpha(t) = \lambda\underline{y}_\alpha(t), \quad \overline{y}''_\alpha(t) = \lambda\overline{y}_\alpha(t) \tag{3.2}$$

by using the generalized differentiability.In the operator form, the first equation in (3.2) becomes $L\underline{y}_\alpha = \lambda \underline{y}_\alpha$, where the differential operator L is given by $L = \frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equation and using the initial conditions we obtain

$$\underline{y}_\alpha(t) = \underline{y}_\alpha(t_0) + t\,\underline{y}_\alpha'(t_0) + L^{-1}\left(\lambda \underline{y}_\alpha\right),$$

$$\underline{y}_\alpha(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda L^{-1}\left(\underline{y}_\alpha\right).$$

Let take

$$\underline{y}_\alpha(t) = \sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t).$$

Then

$$\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda L^{-1}\left(\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t)\right)$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\underline{y}_{1\alpha}(t) = \lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right) \Rightarrow \underline{y}_{1\alpha}(t) = \lambda\left\{\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right\},$$

$$\underline{y}_{2\alpha}(t) = \lambda L^{-1}\left(\underline{y}_{1\alpha}(t)\right) \Rightarrow \underline{y}_{2\alpha}(t) = \lambda\left\{\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right\},\ldots$$

are obtained. Then, the lower approximate solution by the Adomian decomposition method of the problem (3.1) for the (1,1) solution and (2,2) solution becomes

$$\begin{aligned}\underline{y}_\alpha(t) = {} & \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda\left\{\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right\} + \\ & + \lambda\left\{\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right\} + \ldots\end{aligned}$$

Similarly, upper approximate solution by the Adomian decomposition method of the problem (3.1) for the (1,1) solution and (2,2) solution becomes

$$\overline{y}_\alpha(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda\left\{\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right\} +$$

$$+\lambda\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6}+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right)+...$$

For the (1,2) solution and (2,1) solution, the equation (3.1) is written as

$$\underline{y}_\alpha''(t)=\lambda\overline{y}_\alpha(t),\ \overline{y}_\alpha''(t)=\lambda\underline{y}_\alpha(t) \tag{3.3}$$

by using the generalized differentiability and $-\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=\left[-\overline{y}_\alpha,-\underline{y}_\alpha\right]$ .In the operator form, the first equation in (3.3) becomes $L\underline{y}_\alpha=-\lambda\overline{y}_\alpha$, where the differential operator L is given by $L=\frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equations and using the initial conditions we obtain

$$\underline{y}_\alpha(t)=\underline{y}_\alpha(t_0)+\underline{y}_\alpha'(t_0)t+L^{-1}\left(\lambda\overline{y}_\alpha\right),\overline{y}_\alpha(t)=\overline{y}_\alpha(t_0)+\overline{y}_\alpha'(t_0)t+L^{-1}\left(\lambda\underline{y}_\alpha\right),$$

$$\underline{y}_\alpha(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t+\lambda L^{-1}\left(\overline{y}_\alpha\right),$$

$$\overline{y}_\alpha(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t+\lambda L^{-1}\left(\underline{y}_\alpha\right).$$

Let take

$$\underline{y}_\alpha(t)=\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t),\overline{y}_\alpha(t)=\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t).$$

Then

$$\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t+\lambda L^{-1}\left(\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t)\right),$$

$$\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t+\lambda L^{-1}\left(\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t)\right).$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\overline{y}_{0\alpha}(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\underline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\overline{y}_{0\alpha}(t)\right) \Rightarrow \underline{y}_{1\alpha}(t) = \lambda\left(\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),...$$

$$\overline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right) \Rightarrow \overline{y}_{1\alpha}(t) = \lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),...$$

are obtained. Then, the lower approximate solution by the Adomian decomposition method of the problem (3.1) for the (1,2) solution and (2,1) solution becomes

$$\underline{y}_{\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda\left(\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) + ...$$

Similarly, upper approximate solution by the Adomian decomposition method of the problem (3.1) for the (1,2) solution and (2,1) solution becomes

$$\overline{y}_{\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) + ...$$

Example-1 Consider the fuzzy boundary value problem

$$y''(t) = y(t),\ t > 0, \tag{3.4}$$

$$y(0) = [-1+\alpha, 1-\alpha],\ y'(0) = [1+\alpha, 3-\alpha]. \tag{3.5}$$

For the (1,1) solution and (2,2) solution using the generalized differentiability, the lower exact solution and the upper exact solution of the fuzzy initial value problem (3.4)-(3.5) are obtained

$$\underline{Y}_{\alpha}(t) = \alpha e^{t} - e^{-t},\ \overline{Y}_{\alpha}(t) = (2-\alpha)e^{t} - e^{-t}. \tag{3.6}$$

For the (1,1) solution and (2,2) solution by the Adomian decomposition method, we obtain the solution of (3.4)-(3.5)

$$\underline{y}_{\alpha}(t) = (-1+\alpha) + (1+\alpha)t + (-1+\alpha)\frac{t^2}{2} + (1+\alpha)\frac{t^3}{6},$$

$$\overline{y}_{\alpha}(t) = (1-\alpha) + (3-\alpha)t + (1-\alpha)\frac{t^2}{2} + (3-\alpha)\frac{t^3}{6}.$$

The exact lower and upper solution and the approximate lower and upper solution for $t = 0.01$ are

$$\underline{Y}_{\alpha}(t) = -0.99004983374 + 1.01005016708\alpha,\ \overline{Y}_{\alpha}(t) = 1{,}03005050042 - 1.01005016708\alpha,$$

$$\underline{y}_{\alpha}(t) = -0{,}99004983334 + 1{,}01005016667\alpha,\ \overline{y}_{\alpha}(t) = 1{,}0300505 - 1{,}01005016667\alpha.$$

Comparison results of the lower exact and approximate solutions for (1,1) solution and (2,2) solution

α	$\underline{Y}_{\alpha}(t)$	$\underline{y}_{\alpha}(t)$	$Error$
0	$-0.99004983\ 374$	$-0.99004983\ 334$	$4.0000003 \times 10^{-10}$
0.1	$-0.88904481\ 703$	$-0.88904481\ 667$	$3.6000003 \times 10^{-10}$
0.2	$-0.78803980\ 032$	-0.7880398	$3.2000003 \times 10^{-10}$
0.3	$-0.68703478\ 361$	$-0.68703478\ 333$	$2.8000002 \times 10^{-10}$
0.4	$-0.58602976\ 69$	$-0.58602976\ 667$	$2.3000002 \times 10^{-10}$
0.5	$-0.48502475\ 02$	-0.48502475	$2.0000002 \times 10^{-10}$
0.6	$-0.38401973\ 349$	$-0.38401973\ 333$	$1.6000001 \times 10^{-10}$
0.7	$-0.28301471\ 678$	$-0.28301471\ 667$	$1.1000001 \times 10^{-10}$
0.8	$-0.18200970\ 007$	-0.1820097	$7.0000006 \times 10^{-11}$
0.9	$-0.08100468\ 336$	$-0.08100468\ 333$	$3.0000002 \times 10^{-11}$
1	$0.02000033\ 334$	$0.02000033\ 333$	$9.9999974 \times 10^{-12}$

Comparison results of the upper exact and approximate solutions for (1,1) solution and (2,2) solution

α	$\overline{Y}_{\alpha}(t)$	$\overline{y}_{\alpha}(t)$	$Error$
0	$1.03005050\ 042$	1.0300505	$4.2000003 \times 10^{-10}$
0.1	$0.92904548\ 371$	$0.92904548\ 333$	$3.7999992 \times 10^{-10}$
0.2	$0.82804046\ 7$	$0.82804046\ 666$	$3.4000003 \times 10^{-10}$
0.3	$0.72703545\ 029$	$0.72703544\ 999$	$3.0000002 \times 10^{-10}$
0.4	$0.62603043\ 358$	$0.62603043\ 333$	$2.5000002 \times 10^{-10}$
0.5	$0.52502541\ 688$	$0.52502541\ 666$	$2.2000002 \times 10^{-10}$
0.6	$0.42402040\ 017$	$0.42402039\ 999$	$1.7999996 \times 10^{-10}$
0.7	$0.32301538\ 346$	$0.32301538\ 333$	$1.3000001 \times 10^{-10}$
0.8	$0.22201036\ 675$	$0.22201036\ 666$	8.999998×10^{-11}
0.9	$0.12100535\ 004$	$0.12100534\ 999$	$5.0000004 \times 10^{-11}$
1	$0.02000033\ 334$	$0.02000033\ 333$	$9.9999974 \times 10^{-12}$

For the (1,2) solution and (2,1) solution using the differentiability, the lower exact solution and the upper exact solution of the fuzzy initial value problem (3.4)-(3.5) are obtained as

$$\underline{Y}_{\alpha}(t) = e^{t} - e^{-t} - (1-\alpha)\sin(t) - (1-\alpha)\cos(t),$$

$$\overline{Y}_{\alpha}(t) = e^{t} - e^{-t} + (1-\alpha)\sin(t) + (1-\alpha)\cos(t).$$

For the (1,2) solution and (2,1) solution by the Adomian decomposition method, we obtain the solution of (3.4)-(3.5) as

$$\underline{y}_{\alpha}(t) = (-1+\alpha) + (1+\alpha)t + (1-\alpha)\frac{t^{2}}{2} + (3-\alpha)\frac{t^{3}}{6},$$

$$\overline{y}_{\alpha}(t) = (1-\alpha) + (3-\alpha)t + (-1+\alpha)\frac{t^2}{2} + (1+\alpha)\frac{t^3}{6}.$$

The exact lower and upper solution and the approximate lower and upper solution for $t = 0.01$ are

$$\underline{Y}_{\alpha}(t) = -0{,}98017418\ \ 435 + 1{,}00017451\ \ 769\ \alpha,\ \overline{Y}_{\alpha}(t) = 1{,}02017485\ \ 103 - 1{,}00017451\ \ 769\ \alpha,$$

$$\underline{y}_{\alpha}(t) = -0{,}9899495 + 1{,}00994983\ \ 334\ \alpha,\ \overline{y}_{\alpha}(t) = 1{,}02995016\ \ 667 - 1{,}00994983\ \ 334\ \alpha.$$

Comparison results of the lower exact and approximate solutions for (1,2) solution and (2,1) solution

α	$\underline{Y}_{\alpha}(t)$	$\underline{y}_{\alpha}(t)$	$Error$
0	− 0.98017418 435	− 0.9899495	0.00977531 565
0.1	− 0.88015673 258	− 0.88895451 666	0.00879778 408
0.2	− 0.78013928 081	− 0.78795953 333	0.00782025 252
0.3	− 0.68012182 904	− 0.68696454 999	0.00684272 095
0.4	− 0.58010437 727	− 0.58596956 666	0.00586518 939
0.5	− 0.48008692 55	− 0.48497458 333	0.00488765 783
0.6	− 0.38006947 373	− 0.38397959 999	0.00391012 626
0.7	− 0.28005202 196	− 0.28298461 666	0.00293259 47
0.8	− 0.18003457 019	− 0.18198963 332	0.00195506 313
0.9	− 0.08001711 842	− 0.08099464 999	0.00097753 157
1	0.02000033 334	0.02000033 334	0

Comparison results of the upper exact and approximate solutions for (1,2) solution and (2,1) solution

α	$\overline{Y}_{\alpha}(t)$	$\overline{y}_{\alpha}(t)$	$Error$
0	1.02017485 103	1.02995016 667	0.00977531 564
0.1	0.92015739 926	0.92895518 333	0.00879778 407
0.2	0.82013994 749	0.8279602	0.00782025 251
0.3	0.72012249 572	0.72696521 666	0.00684272 094
0.4	0.62010504 395	0.62597023 333	0.00586518 938
0.5	0.52008759 218	0.52497525	0.00488765 782
0.6	0.42007014 041	0.42398026 666	0.00391012 625
0.7	0.32005268 864	0.32298528 333	0.00293259 469
0.8	0.22003523 687	0.22199029 999	0.00195506 312
0.9	0.12001778 51	0.12099531 666	0.00097753 156
1	0.02000033 334	0.02000033 333	$9.9999974 \times 10^{-12}$

The case of negative constant coefficient

Consider the fuzzy boundary value problem

$$y''(t) = -\lambda y(t),\ y(t_0) = A,\ y'(t_0) = B, \tag{3.7}$$

where $\lambda > 0$, $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right], [B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right]$ are symmetric triangular fuzzy numbers. Here, (i,j) solution means that y is (i) differentiable and y' is (j) differentiable, i,j=1,2.

The Exact Solution By Generalized Differentiability

For (1,1) solution and (2,2) solution from the fuzzy differential equation in (3.7), we have differential equations

$$\underline{Y}''_\alpha(t) = -\lambda \overline{Y}_\alpha(t),\ \overline{Y}''_\alpha(t) = -\lambda \underline{Y}_\alpha(t)$$

by using the generalized differentiability and $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.7) for (1,1) solution and (2,2) solution are

$$\underline{Y}_\alpha(t) = -c_1(\alpha)e^{\sqrt{\lambda}t} - c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\sin\left(\sqrt{\lambda}t\right) + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right),$$

$$\overline{Y}_\alpha(t) = c_1(\alpha)e^{\sqrt{\lambda}t} + c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\sin\left(\sqrt{\lambda}t\right) + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right).$$

Using the initial conditions, the coefficient $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are obtained as

$$c_1(\alpha) = \frac{(1-\alpha)\left[\left(\overline{a} - \underline{a}\right)\sqrt{\lambda} + \left(\overline{b} - \underline{b}\right)\right]}{4\sqrt{\lambda}\, e^{\sqrt{\lambda}t_0}},\ c_2(\alpha) = \frac{(1-\alpha)\left[\left(\overline{a} - \underline{a}\right)\sqrt{\lambda} - \left(\overline{b} - \underline{b}\right)\right]}{4\sqrt{\lambda}\, e^{\sqrt{\lambda}t_0}},$$

$$c_3(\alpha) = \frac{\left(\overline{a} + \underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}t_0\right) + \left(\overline{b} + \underline{b}\right)\cos\left(\sqrt{\lambda}t_0\right)}{2\sqrt{\lambda}},$$

$$c_4(\alpha) = \frac{\left(\overline{a} + \underline{a}\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}t_0\right) - \left(\overline{b} + \underline{b}\right)\sin\left(\sqrt{\lambda}t_0\right)}{2\sqrt{\lambda}}.$$

For (1,2) solution and (2,1) solution from the fuzzy differential equation in (3.7), we have differential equations

$$\underline{Y}''_\alpha(t) = -\lambda \underline{Y}_\alpha(t),\ \overline{Y}''_\alpha(t) = -\lambda \underline{Y}_\alpha(t)$$

by using the generalized differentiability and $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.7) for (1,2) solution and (2,1) solution are

$$\underline{Y}_\alpha(t) = \underline{a}_1(\alpha)\cos\left(\sqrt{\lambda}t\right) + \underline{a}_2(\alpha)\sin\left(\sqrt{\lambda}t\right),$$

$$\overline{Y}_\alpha(t) = \overline{a}_1(\alpha)\cos\left(\sqrt{\lambda}t\right) + \overline{a}_2(\alpha)\sin\left(\sqrt{\lambda}t\right),$$

Using the initial conditions, the coefficient $\underline{a}_1(\alpha)$, $\underline{a}_2(\alpha)$, $\overline{a}_1(\alpha)$, $\overline{a}_2(\alpha)$ are obtained as

$$\underline{a}_1(\alpha)=\frac{\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}t_0\right)-\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\sin\left(\sqrt{\lambda}t_0\right)}{\sqrt{\lambda}},$$

$$\underline{a}_2(\alpha)=\frac{\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}t_0\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\cos\left(\sqrt{\lambda}t_0\right)}{\sqrt{\lambda}},$$

$$\overline{a}_1(\alpha)=\frac{\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}t_0\right)-\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\sin\left(\sqrt{\lambda}t_0\right)}{\sqrt{\lambda}},$$

$$\overline{a}_2(\alpha)=\frac{\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}t_0\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\cos\left(\sqrt{\lambda}t_0\right)}{\sqrt{\lambda}}.$$

The Approximate Solution By The Adomian Decomposition Method

For (1,1) solution and (2,2) solution, the equation in (3.7) is written as

$$\underline{y}_\alpha''(t)=-\lambda\,\overline{y}_\alpha(t),\ \overline{y}_\alpha''(t)=-\lambda\,\underline{y}_\alpha(t) \tag{3.8}$$

by using the generalized differentiability and $-\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=\left[-\overline{y}_\alpha,-\underline{y}_\alpha\right]$. In the operator form, the first equation in (3.8) becomes $L\underline{y}_\alpha=-\lambda\overline{y}_\alpha$, where the differential operator L is given by $L=\frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equations and using the initial conditions we obtain

$$\underline{y}_\alpha(t)=\underline{y}_\alpha(t_0)+\underline{y}_\alpha'(t_0)t+L^{-1}\left(-\lambda\overline{y}_\alpha\right),\ \overline{y}_\alpha(t)=\overline{y}_\alpha(t_0)+\overline{y}_\alpha'(t_0)t+L^{-1}\left(-\lambda\underline{y}_\alpha\right),$$

$$\underline{y}_\alpha(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t-\lambda L^{-1}\left(\overline{y}_\alpha\right),$$

$$\overline{y}_\alpha(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t-\lambda L^{-1}\left(\underline{y}_\alpha\right).$$

Let take

$$\underline{y}_\alpha(t)=\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t),\ \overline{y}_\alpha(t)=\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t).$$

Then

$$\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\sum_{n=0}^{\infty} \overline{y}_{n\alpha}(t)\right),$$

$$\sum_{n=0}^{\infty} \overline{y}_{n\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t)\right)$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\overline{y}_{0\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\underline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\overline{y}_{0\alpha}(t)\right) \Rightarrow \underline{y}_{1\alpha}(t) = -\lambda\left(\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right), \ldots$$

$$\overline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right) \Rightarrow \overline{y}_{1\alpha}(t) = -\lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right), \ldots$$

are obtained. Then, the lower approximate solution by the Adomian decomposition method of the problem (3.7) for (1,1) solution and (2,2) solution becomes

$$\underline{y}_{\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda\left(\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) + \ldots$$

Similarly, upper approximate solution by the Adomian decomposition method of the problem (3.7) for (1,1) solution and (2,2) solution becomes

$$\overline{y}_{\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) + \ldots$$

For (1,2) solution and (2,1) solution, the equation in (3.7) is written as

$$\underline{y}''_{\alpha}(t) = -\lambda\,\underline{y}_{\alpha}(t),\ \overline{y}''_{\alpha}(t) = -\lambda\,\overline{y}_{\alpha}(t) \tag{3.9}$$

by using the generalized differentiability and $-\left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right] = \left[-\overline{y}_{\alpha}, -\underline{y}_{\alpha}\right]$. In the operator form, the first equation in (3.9) becomes $L\underline{y}_{\alpha} = -\lambda\,\underline{y}_{\alpha}$, where the differential operator L is given by $L = \frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equations and using the initial conditions we obtain

$$\underline{y}_\alpha(t) = \underline{y}_\alpha(t_0) + \underline{y}_\alpha'(t_0)t + L^{-1}\left(-\lambda \underline{y}_\alpha\right), \overline{y}_\alpha(t) = \overline{y}_\alpha(t_0) + \overline{y}_\alpha'(t_0)t + L^{-1}\left(-\lambda \overline{y}_\alpha\right),$$

$$\underline{y}_\alpha(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\underline{y}_\alpha\right),$$

$$\overline{y}_\alpha(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\overline{y}_\alpha\right).$$

Let take

$$\underline{y}_\alpha(t) = \sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t), \overline{y}_\alpha(t) = \sum_{n=0}^{\infty} \overline{y}_{n\alpha}(t).$$

Then

$$\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t)\right),$$

$$\sum_{n=0}^{\infty} \overline{y}_{n\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda L^{-1}\left(\sum_{n=0}^{\infty} \overline{y}_{n\alpha}(t)\right)$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\overline{y}_{0\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\underline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right) \Rightarrow \underline{y}_{1\alpha}(t) = -\lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),\ldots$$

$$\overline{y}_{1\alpha}(t) = -\lambda L^{-1}\left(\overline{y}_{0\alpha}(t)\right) \Rightarrow \overline{y}_{1\alpha}(t) = -\lambda\left(\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),\ldots$$

are obtained. Then, the lower approximate solution by the Adomian decomposition method of the problem (3.7) for (1,2) solution and (2,1) solution becomes

$$\underline{y}_\alpha(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t - \lambda\left(\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) + \ldots$$

Similarly, the upper approximate solution by the Adomian decomposition method of the problem (3.7) for (1,2) solution and (2,1) solution becomes

$$\overline{y}_{\alpha}(t) = \left(\overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right)t - \lambda\left(\left(\overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right)\frac{t^2}{2} + \left(\overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right)...$$

Example-2 Consider the fuzzy boundary value problem

$$y''(t) = -y(t),\ t > 0, \tag{3.10}$$

$$y(0) = [-1+\alpha, 1-\alpha],\ y'(0) = [1+\alpha, 3-\alpha]. \tag{3.11}$$

For (1,1) solution and (2,2) solution, using the generalized differentiability and using $-\left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right] = \left[-\overline{y}_{\alpha}, -\underline{y}_{\alpha}\right]$ the lower exact solution and the upper exact solution of the fuzzy initial value problem (3.10)-(3.11) are obtained as

$$\underline{Y}_{\alpha}(t) = (-1+\alpha)e^{t} + 2\sin(t),$$

$$\overline{Y}_{\alpha}(t) = (1-\alpha)e^{t} + 2\sin(t).$$

For (1,1) solution and (2,2) solution, by the Adomian decomposition method, we obtain the approximate solution of $(3.10) - (3.11)$ as

$$\underline{y}_{\alpha}(t) = (-1+\alpha) + (1+\alpha)t - (1-\alpha)\frac{t^2}{2} - (3-\alpha)\frac{t^3}{6} - ...$$

$$\overline{y}_{\alpha}(t) = (1-\alpha) + (3-\alpha)t - (-1+\alpha)\frac{t^2}{2} - (1+\alpha)\frac{t^3}{6} - ...$$

The exact lower and upper solution and the approximate lower and upper solution for $t = 0.01$ are

$$\underline{Y}_{\alpha}(t) = -1{,}00970110\ \ 124 + 1{,}01005016\ \ 708\ \alpha,\ \overline{Y}_{\alpha}(t) = 1{,}01039923\ \ 292 - 1{,}01005016\ \ 708\ \alpha,$$

$$\underline{y}_{\alpha}(t) = -0{,}9900505\ \ + 1{,}01005016\ \ 666\ \alpha,\ \overline{y}_{\alpha}(t) = 1{,}03005016\ \ 666 - 1{,}01005016\ \ 666\ \alpha.$$

Comparison results of the lower exact and approximate solutions for (1,1) solution and (2,2) solution

α	$\underline{Y}_{\alpha}(t)$	$\underline{y}_{\alpha}(t)$	$Error$
0	−1.00970110 124	−0.9900505	0.01965060 124
0.1	−0.90869608 453	−0.88904548 333	0.01965060 12
0.2	−0.80769106 782	−0.78804046 666	0.01965060 116
0.3	−0.70668605 111	−0.68703545	0.01965060 111
0.4	−0.60568103 44	−0.58603043 333	0.01965060 107
0.5	−0.50467601 77	−0.48502541 667	0.01965060 103
0.6	−0.40367100 099	−0.3840204	0.01965060 099
0.7	−0.30266598 428	−0.28301538 333	0.01965060 095
0.8	−0.20166096 757	−0.18201036 667	0.01965060 09
0.9	−0.10065595 086	−0.08100535	0.01965060 086
1	0.00034906 584	0.01999966 666	0.01965060 082

Comparison results of the upper exact and approximate solutions for (1,1) solution and (2,2) solution

α	$\overline{Y}_{\alpha}(t)$	$\overline{y}_{\alpha}(t)$	$Error$
0	1.01039923 292	1.03005016 666	0.01965093 374
0.1	0.90939421 621	0.92904514 999	0.01965093 378
0.2	0.80838919 95	0.82804013 332	0.01965093 382
0.3	0.70738418 279	0.72703511 666	0.01965093 387
0.4	0.60637916 608	0.62603009 999	0.01965093 391
0.5	0.50537414 938	0.52502508 333	0.01965093 395
0.6	0.40436913 267	0.42402006 666	0.01965093 399
0.7	0.30336411 596	0.32301504 999	0.01965093 403
0.8	0.20235909 925	0.22201003 333	0.01965093 408
0.9	0.10135408 254	0.12100501 666	0.01965093 412
1	0.00034906 584	0.02	0.01965093 416

For (1,2) solution and (2,1) solution, the lower exact solution and the upper exact solution of the fuzzy initial value problem (3.10)-(3.11) are obtained as

$$\underline{Y}_{\alpha}(t) = (-1+\alpha)\cos(t) + (1+\alpha)\sin(t),$$

$$\overline{Y}_{\alpha}(t) = (1-\alpha)\cos(t) + (3-\alpha)\sin(t),$$

For the (1,2) solution and (2,1) solution by the Adomian decomposition method, we obtain the solution of (3.10)-(3.11) as

$$\underline{y}_{\alpha}(t) = (-1+\alpha) + (1+\alpha)t - (-1+\alpha)\frac{t^2}{2} - (1+\alpha)\frac{t^3}{6} - \ldots$$

$$\overline{y}_{\alpha}(t) = (1-\alpha)+(3-\alpha)t-(1-\alpha)\frac{t^{2}}{2}-(3-\alpha)\frac{t^{3}}{6}-...$$

The exact lower and upper solution and the approximate lower and upper solution for $t = 0.01$ are

$$\underline{Y}_{\alpha}(t) = -0.99982545\ 184 + 1.00017451\ 769\ \alpha, \overline{Y}_{\alpha}(t) = 1.00052358\ 353 - 1.00017451\ 769\ \alpha.$$

$$\underline{y}_{\alpha}(t) = -0{,}98995016\ 667 + 1{,}00994983\ 334\ \alpha, \overline{y}_{\alpha}(t) = 1{,}0299495 - 1{,}00994983\ 334\ \alpha$$

Comparison results of the lower exact and approximate solutions for (1,2) solution and (2,1) solution

α	$\underline{Y}_{\alpha}(t)$	$\underline{y}_{\alpha}(t)$	$Error$
0	$-0.99982545\ 184$	$-0.98995016\ 667$	$0.00987528\ 517$
0.1	$-0.89980800\ 007$	$-0.88895518\ 333$	$0.01085281\ 674$
0.2	$-0.79979054\ 83$	-0.7879602	$0.01183034\ 83$
0.3	$-0.69977309\ 653$	$-0.68696521\ 666$	$0.01280787\ 987$
0.4	$-0.59975564\ 476$	$-0.58597023\ 333$	$0.01378541\ 143$
0.5	$-0.49973819\ 299$	-0.48497525	$0.01476294\ 299$
0.6	$-0.39972074\ 122$	$-0.38398026\ 666$	$0.01574047\ 456$
0.7	$-0.29970328\ 945$	$-0.28298528\ 333$	$0.01671800\ 612$
0.8	$-0.19968583\ 768$	$-0.18199029\ 999$	$0.01769553\ 769$
0.9	$-0.09966838\ 591$	$-0.08099531\ 666$	$0.01867306\ 925$
1	$0.00034906\ 585$	$0.01999966\ 667$	$0.01965060\ 082$

Comparison results of the upper exact and approximate solutions for (1,2) solution and (2,1) solution

α	$\overline{Y}_{\alpha}(t)$	$\overline{y}_{\alpha}(t)$	$Error$
0	$1.00052358\ 353$	1.0299495	$0.02942591\ 647$
0.1	$0.90050613\ 176$	$0.92895451\ 666$	$0.02844838\ 49$
0.2	$0.80048867\ 999$	$0.82795953\ 333$	$0.02747085\ 334$
0.3	$0.70047122\ 822$	$0.72696454\ 999$	$0.02649332\ 177$
0.4	$0.60045377\ 645$	$0.62596956\ 666$	$0.02551579\ 021$
0.5	$0.50043632\ 468$	$0.52497458\ 333$	$0.02453825\ 865$
0.6	$0.40041887\ 291$	$0.42397959\ 999$	$0.02356072\ 708$
0.7	$0.30040142\ 114$	$0.32298461\ 666$	$0.02258319\ 552$
0.8	$0.20038396\ 937$	$0.22198963\ 332$	$0.02160566\ 395$
0.9	$0.10036651\ 76$	$0.12099464\ 999$	$0.02062813\ 239$
1	$0.00034906\ 584$	$0.01999966\ 666$	$0.01965060\ 082$

4-Conclusions

In this paper investigates the exact solutions and the approximate solutions by Adomian decomposition method of the second-order linear fuzzy initial value problems with positive and negative constant coefficients using the generalized differentiability. The values of the exact solutions and the approximate solutions for each $\alpha = 0,\ 0.1,\ 0.2,\ 0{,}3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1$ are computed. Consequently, the errors of lower and upper solutions are reduced for (1,1), (2,2), (1,2) and (2,1) solutions in the case of positive constant coefficient. But while the error of lower solution is reduced, the error of upper solution increases for (1,1) and (2,2) solutions in the case of negative constant coefficient. Also, while the error of lower solution increases, the error of upper solution is reduced for (1,2) and (2,1) solutions in the case of negative constant coefficient.

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