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A Numerical Solution for Fractional Damped Mechanical Oscillator Equation

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Abstract: In this article, we have investigated the numerical approach for solving the fractional damped mechanical oscillator equation, which has an important role in fractional calculus. Damped mechanical oscillator equation equation is solved by Bernoulli collocation method with the aid of the computer symbolic language of Maple2016. This method transforms the damped mechanical oscillator equation into matrix equations. Then, the problem has been reduced to solving linear algebraic equations.
Keywords: Damped mechanical oscillator equation, Fractional differantial equation, Collocation method, Bernoulli polynomials, approximate solution

INTRODUCTION

Fractional calculus has become the focus of interest for many researchers in different disciplines of applied science and engineering. Nowadays notable contributions have been made theory and applications of the fractional differential equations (FDEs). Several problems can be modelling with the help of the FDEs in many areas such as seismic analysis, viscous damping, viscoelastic materials and polymer physics [1-3]. Many authors have been examining the possibility of using fractional derivates in material modelling last decades [4]. Uniqueness of solutions to the FDEs and the analytic results on the existence has been investigated by many authors [5-6]. In general, most of FDEs do not have exact analytic solutions, so we need approximate solution and numerical techniques, for this reason many techniques are developed by many researchers. For example Adomian decomposition method, the homotopy-perturbation method, the variational iteration method and the homotopy analysis method [7-12].

In this study, the damped mechanical oscillator equation is defined by

$$D_*^{\alpha} y(x) + \lambda D_*^{\beta} y(x) + \nu y(x) = f(x), \quad t \in [0,1] \tag{1}$$

$$D_*^{i} y(c) = \lambda_i, \; i = 0,1,\ldots, n-1, \tag{2}$$

where $1 < \alpha \le 2, 0 < \beta \le 1, \alpha - \beta > 1$ and $f(x)$ is the forcing function.[13] According to the cases $\alpha = 2, \beta = 1$ Eq(1) can be referred to as the usual harmonic oscillator equation[14]. In this paper we use the collocation method for solving fractional damped mechanical oscillator equation[15]. We investigate the approximate solution of Eq.(1) with the fractional truncated Bernoulli series as

$$y_N(x) = \sum_{n=0}^{N} a_n \mathrm{B}_n^{\alpha}(x) \tag{3}$$

where $0 < \alpha \le 1$.

BASIC DEFINITIONS

In this section, we first give some basic definitions and then present properties of fractional calculus[2].

Definition 2.1 A real function $f(x)$, $x > 0$, is said to be in space C_μ, $\mu \in R$ if there exist a real number p ($> \mu$), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0, \infty$), and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2 The Riemann-Liouville fractional derivative of order α with respect to the variable t and with the starting point at $t = a$ is

$$ {}_aD_t^\alpha f(t) = (d/dt)^{m+1} \int_a^t (t-\tau)^{m-\alpha} f(\tau) d\tau $$

Definition 2.3 The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$$ D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau $$

for $n-1 < \alpha \le n$, $n \in$ N, $t > 0$, $f \in C_{-1}^n$. Some properties of the Caputo fractional derivative, which are needed here as follows,

$$ D^\alpha C = 0, \quad C \text{ is a constant.} $$

$$ D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathrm{N} \cup \{0\}, \beta < \lceil \alpha \rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathrm{N} \cup \{0\}, \beta \ge \lceil \alpha \rceil \text{ or } \beta \notin \mathrm{N}, \beta > \lfloor \alpha \rfloor \end{cases} $$

where the ceiling function $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal α and the floor function $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α .

Fundamental Relations

In this section, we consider the fractional differential equations

$$ \sum_{k=0}^{m} P_k(x) D_*^{k\alpha} y(x) = f(x), a \le x \le b, \; 0 \le \alpha \le 1 \tag{4} $$

with initial conditions

$$ D_*^i y(c) = \lambda_i, \; i = 0,1,\ldots, n-1, \; a \le c \le b \tag{5} $$

which $P_k(x)$ and $f(x)$ are functions defined on $a \le x \le b$, λ_i is a appropriate constant. We use the collocation method to find the truncated fractional Bernoulli series and their matrix representations for solving $m\alpha$ -th order linear fractional differential equation with constant coefficients. We first consider the solution $y(x)$ of Eq. (1) defined by a truncated fractional Bernoulli series (3). Then, we have the matrix form of the solution $y(x)$

$$ [y(x)] = \mathbf{B}^\alpha(x)\mathbf{A} \tag{6} $$

where

$$ \mathbf{B}^\alpha(x) = \begin{bmatrix} B_0^\alpha(x) & B_1^\alpha(x) & B_2^\alpha(x) & \cdots & B_N^\alpha(x) \end{bmatrix} $$

$$ \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} $$

On the other hand, fractional Bernoulli polynomials are,

$$B_N^{\alpha}(x) = \sum_{i=0}^{N} \binom{N}{i} b_{N-i} x^{i\alpha} \ , \ \alpha > 0, b_{N-i} = B_{N-i}(0) \text{ Bernoulli numbers.} \tag{7}$$

Matrix representation of Eq.(7) is,

$$\mathbf{B}^{\alpha}(x) = \mathbf{X}^{\alpha}(x)\mathbf{S} \tag{8}$$

where

$$\mathbf{X}^{\alpha}(x) = \begin{bmatrix} 1 & x^{\alpha} & x^{2\alpha} & \cdots & x^{N\alpha} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \binom{0}{0}b_0 & \binom{1}{0}b_1 & \binom{2}{0}b_2 & \cdots & \binom{N}{0}b_N \\ 0 & \binom{1}{1}b_0 & \binom{2}{1}b_1 & \cdots & \binom{N}{1}b_{N-1} \\ 0 & 0 & \binom{2}{2}b_0 & \cdots & \binom{N}{2}b_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}b_0 \end{bmatrix}$$

By substituting (6) into (8), we obtain

$$\begin{bmatrix} y \ (x) \end{bmatrix} = \mathbf{X}^{\alpha}(x)\mathbf{SA} \tag{9}$$

Similarly, the matrix representation of the function $D_*^{\alpha} y(x)$ become

$$D_*^{\alpha} y(x) = D_*^{\alpha} \mathbf{X}^{\alpha} \mathbf{SA}$$

where, we compute the $D_*^{\alpha} \mathbf{X}^{\alpha}$, then

$$\begin{aligned} D_*^{\alpha} \mathbf{X}^{\alpha} &= \begin{bmatrix} D_*^{\alpha} 1 & D_*^{\alpha} x^{\alpha} & D_*^{\alpha} x^{2\alpha} & \cdots & D_*^{\alpha} x^{N\alpha} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} x^{\alpha} & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} x^{(N-1)\alpha} \end{bmatrix} \\ &= \mathbf{X}^{\alpha} \mathbf{R}_1 \end{aligned}$$

where

$$\mathbf{R}_1 = \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

then,

$$D_*^{\alpha} y(x) = \mathbf{X}^{\alpha} \mathbf{R}_1 \mathbf{SA} \ .$$

In a similar way for any i, it can be written by

$$D_*^{k\alpha} y(x) = \mathbf{X}^{\alpha}\mathbf{R}_k\mathbf{SA} \tag{10}$$

where

$$\mathbf{R}_k = \begin{bmatrix} 0 & 0 & \dots & \frac{\Gamma(k\alpha+1)}{\Gamma(1)} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma(\alpha+1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-k)\alpha+1)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

And then, we obtain the fundemental matrix form of Eq.(1)

$$\sum_{k=0}^{m} \mathbf{P}_k\mathbf{X}^{\alpha}\mathbf{R}_k\mathbf{SA} = \mathbf{F} \tag{11}$$

Finally, we obtained the matrix representation of the condition in given Eq.(2) as

$$\mathbf{U}_i = \mathbf{X}^{\alpha}(c)\mathbf{R}_k = [u_{i0} \quad u_{i1} \quad u_{i2} \quad \cdots \quad u_{iN}] = [\lambda_i] \tag{12}$$

Method of Solutions

We can write Eq. (11) in the form

$$\mathbf{WA} = \mathbf{F} \tag{13}$$

where

$$\mathbf{W} = [w_{ij}] = \sum_{\mathbf{k=0}}^{\mathbf{m}} \mathbf{P}_k\mathbf{X}^{\alpha}\mathbf{R}_k\mathbf{S}, \quad i,j = 0.1,\dots N \ .$$

Consequently, to find the unknown Bernoulli coefficients a_k, $k = 0,1,\dots,N$, related with the approximate solution of the problem consisting of Eq. (1) and conditions (2), by replacing the m row matrices (12) by the last m rows of the matrix (13), we have augmented matrix

$$[\mathbf{W}^*;\mathbf{F}^*] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N-m0} & w_{N-m1} & \cdots & w_{N-mN} & ; & f(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{m-10} & u_{m-11} & \cdots & u_{m-1N} & ; & \lambda_{m-1} \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A} = \mathbf{F}^* \tag{14}$$

If $\det W^{*} \neq 0$, we can write Eq.(14) as

$$\mathbf{A} = (\mathbf{W}^{*})^{-1}\mathbf{F}^{*} \tag{15}$$

And the matrix $\mathbf{A}$ is uniquely determined. Therefore, the approximate solution is given by the truncated fractional Bernoulli series

$$[y(x)] = \mathbf{X}^{\alpha}\mathbf{R}_{0}\mathbf{S}\mathbf{A} \ .$$

We can easily check the accuracy of the method. Since the truncated fractional Bernoulli series (3) is an approximate solution of Eq.(1), when the solution $y(x)$ and its fractional derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is , for $x = x_q \in [a,b], \quad q = 0,1,2,...$

$$E(x_q) = \left| D_{*}^{i\alpha} y(x) - f(x) - \sum_{r=0}^{m} p_r(x) y(q_r x) \right| \cong 0$$

Examples

In this section, we give a numerical example which is presented to demonstrate the effectiveness of the proposed method.

Example 1: Let us consider the fractional damped mechanical oscillator equation

$$D_{*}^{2} y(x) + \lambda D_{*}^{1/2} y(x) + vy(x) = f(x)$$

with the initial conditions. Here is

$$y(0) = 1, y(1) = 2 \ , \ f(x) = 2\sqrt{x} + x + 1 \ , \ v = 1, \lambda = \sqrt{\pi} \ .$$

$$y_4(x) = \sum_{N=0}^{4} a_N B_N^{\alpha}(x)$$

Fundamental matrix relation of this problem is

$$\left\{ \mathbf{P}_0 \mathbf{X}^{\alpha} \mathbf{R}_0 + \mathbf{P}_1 \mathbf{X}^{\alpha} \mathbf{R}_1 + \mathbf{P}_4 \mathbf{X}^{\alpha} \mathbf{R}_4 \right\} \mathbf{S}\mathbf{A} = \mathbf{F}$$

where

$$\mathbf{P_0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{P_1} = \begin{bmatrix} \sqrt{\pi} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\pi} \end{bmatrix}, \mathbf{P_4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 \\ 2.25 \\ 2.91 \\ 3.48 \\ 4 \end{bmatrix}$$

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{R}_1 = \begin{bmatrix} 0 & \frac{\sqrt{\pi}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{\pi}} & 0 & 0 \\ 0 & 0 & 0 & \frac{3\sqrt{\pi}}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{3\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{X}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{\sqrt{4}}{4} & \frac{1}{4} & \frac{\sqrt{4}}{16} & \frac{1}{16} \\ 1 & \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ 1 & \frac{\sqrt{3}\sqrt{4}}{4} & \frac{3}{4} & \frac{3\sqrt{3}\sqrt{4}}{16} & \frac{9}{16} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{S}=\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} \\ 0 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then, we obtained

$$[\mathbf{W};\ \mathbf{F}]=\begin{bmatrix} 1 & \frac{\pi}{2}-\frac{1}{2} & -\frac{\pi}{2}+\frac{1}{6} & \frac{\pi}{4} & \frac{59}{30} & ; & 1 \\ 1 & \frac{\pi}{2}-\frac{1}{2}+\frac{\sqrt{4}}{4} & -\frac{\pi}{2}+\frac{5}{12}+\frac{\sqrt{4}}{4} & \frac{7\pi}{16}-\frac{3}{8}-\frac{9\sqrt{4}}{16} & -\frac{3\pi}{8}+\frac{547}{240}+\frac{13\sqrt{4}}{24} & ; & 2.25 \\ 1 & \frac{\pi}{2}-\frac{1}{2}+\frac{\sqrt{2}}{2} & -\frac{\pi}{2}+\frac{2}{3}+\frac{\sqrt{2}}{2} & \frac{5\pi}{8}-\sqrt{2}-\frac{3}{4} & -\frac{3\pi}{4}+\frac{163}{60}+\frac{7\sqrt{2}}{6} & ; & 2.91 \\ 1 & \frac{\pi}{2}-\frac{1}{2}+\frac{\sqrt{12}}{4} & -\frac{\pi}{2}+\frac{11}{12}+\frac{\sqrt{12}}{4} & \frac{13\pi}{16}-\frac{9}{8}-\frac{7\sqrt{12}}{16} & -\frac{9\pi}{8}+\frac{787}{240}+\frac{5\sqrt{12}}{8} & ; & 3.48 \\ 1 & \frac{\pi}{2}+\frac{1}{2} & -\frac{\pi}{2}+\frac{13}{6} & -3+\pi & \frac{199}{30}-\frac{3\pi}{2} & ; & 1 \end{bmatrix}$$

Also, we have the matrix representation of conditions

$$\begin{bmatrix} \mathbf{U}_0;\beta_0 \\ \mathbf{U}_1;\beta_1 \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & ; & 1 \\ 1 & \frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & ; & 2 \end{bmatrix}$$

Then, augmented matrix becomes

$$[\mathbf{W};\ \mathbf{F}]=\begin{bmatrix} 1 & \frac{\pi}{2}-\frac{1}{2} & -\frac{\pi}{2}+\frac{1}{6} & \frac{\pi}{4} & \frac{59}{30} & ; & 1 \\ 1 & \frac{\pi}{2}-\frac{1}{2}+\frac{\sqrt{4}}{4} & -\frac{\pi}{2}+\frac{5}{12}+\frac{\sqrt{4}}{4} & \frac{7\pi}{16}-\frac{3}{8}-\frac{9\sqrt{4}}{16} & -\frac{3\pi}{8}+\frac{547}{240}+\frac{13\sqrt{4}}{24} & ; & 2.25 \\ 1 & \frac{\pi}{2}-\frac{1}{2}+\frac{\sqrt{2}}{2} & -\frac{\pi}{2}+\frac{2}{3}+\frac{\sqrt{2}}{2} & \frac{5\pi}{8}-\sqrt{2}-\frac{3}{4} & -\frac{3\pi}{4}+\frac{163}{60}+\frac{7\sqrt{2}}{6} & ; & 2.91 \\ 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & ; & 1 \\ 1 & \frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & ; & 2 \end{bmatrix}$$

and so, solving this equation, we obtained the coefficients of the Bernoulli series

$$\mathbf{A}^{\mathbf{T}}=[1.33 \quad 1 \quad 0.99 \quad 0 \quad 0].$$

Comparison of numerical results with the exact solution is plotted in Fig.1 for various N .

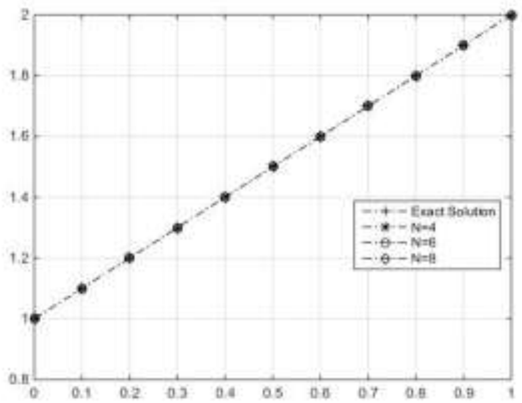

Fig-1: Comparison of approximate solutions and exact solution

CONCLUSION

In this study, we present a Bernoulli collocation method for the numerical solutions of the fractional damped mechanical oscillation equation. This method transformes the fractional damped mechanical oscillation equation into matrix equations. This paper presents a numerical solution to obtain the solution of fractional damped mechanical oscillation equation. Graphics show that this method is extremely effective and practical for this sort of approximate solutions of differential equations.

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