Scholars Journal of Physics, Mathematics and Statistics
Abbreviated Key Title: Sch. J. Phys. Math. Stat.
ISSN 2393-8056 (Print)
ISSN 2393-8064 (Online)

Arens Regularity of Bilinear Mapping and Reflexivity

Abotaleb Sheikhali, Nader Kanzi
E-mail address: Abotaleb.sheikhali.20@gmail.com, Nad.kanzi@gmail.com

Depatment of Mathematics, Payame Noor University (PNU), Tehran, Iran

***Corresponding author**
Abotaleb Sheikhali

Article History
Received: 20.12.2017
Accepted: 20.01.2018
Published: 30.01.2018

DOI:
10.21276/sjpms.2018.5.1.3

Abstract: Let X, Y and Z be normed spaces. In this article we give a tool to investigate Arens regularity of a bounded bilinear map $f : X \times Y \to Z$. Also, under some assumptions on f^{****} and f^{r****r}, we give some new results to determine reflexivity of the spaces.
Keywords: Arens regular, bilinear map, topological center, factor, second dual.
2010 Mathematics Subject Classification. 46H20, 46H25

INTRODUCTION AND PRELIMINARIES

Arens showed in [1] that a bounded bilinear map $f : X \times Y \to Z$ on normed spaces, has two natural different extensions f^{***}, f^{r***r} from $X^{**} \times Y^{**}$ into Z^{**}. When these extensions are equal, f is said to be Arens regular. Throughout the article, we identify a normed space with its canonical image in the second dual.

We denote by X^* the topological dual of a normed space X. We write X^{**} for $(X^*)^*$ and so on. Let X, Y and Z be normed spaces and $f : X \times Y \to Z$ be a bounded bilinear mapping. The natural extensions of f are as following:

(i) $f^* : Z^* \times X \to Y^*$, give by $< f^*(z^*, x), y > = < z^*, f(x, y) >$ where $x \in X$, $y \in Y$, $z^* \in Z^*$ (f^* is said the adjoint of f).
(ii) $f^{**} = (f^*)^* : Y^{**} \times Z^* \to X^*$, give by $< f^{**}(y^{**}, z^*), x > = < y^{**}, f^*(z^*, x) >$ where $x \in X$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.
(iii) $f^{***} = (f^{**})^* : X^{**} \times Y^{**} \to Z^{**}$, give by $< f^{***}(x^{**}, y^{**}), z^* > = < x^{**}, f^{**}(y^{**}, z^*) >$ where $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.

Let now $f^r : Y \times X \to Z$ be the flip of f defined by $f^r(y, x) = f(x, y)$, for every $x \in X$ and $y \in Y$. Then f^r is a bounded bilinear map and it may extends as above to $f^{r***} : Y^{**} \times X^{**} \to Z^{**}$. In general, the mapping $f^{r***r} : X^{**} \times Y^{**} \to Z^{**}$ is not equal to f^{***}. When these extensions are equal, then f is Arens regular.

One may define similarly the mappings $f^{****} : Z^{***} \times X^{**} \to Y^{***}$ and $f^{*****} : Y^{****} \times Z^{***} \to X^{***}$ and the higher rank adjoints. Consider the nets $(x_\alpha) \subseteq X$ and $(y_\beta) \subseteq Y$ converge to $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ in the $weak^*$ −topologies, respectively, then

$$f^{***}(x^{**}, y^{**}) = w^* - \lim_\alpha w^* - \lim_\beta f(x_\alpha, y_\beta) \quad \text{and} \quad f^{r***r}(x^{**}, y^{**}) = w^* - \lim_\beta w^* - \lim_\alpha f(x_\alpha, y_\beta)$$

So Arens regularity of f is equivalent to the following

$$\lim_\alpha \lim_\beta < z^*, f(x_\alpha, y_\beta) > = \lim_\beta \lim_\alpha < z^*, f(x_\alpha, y_\beta) >$$

If the limits exit for each $z^* \in Z^*$. The map f^{***} is the unique extension of f such that $x^{**} \to f^{***}(x^{**}, y^{**}) : X^{**} \to Z^{**}$ is $weak^* - weak^*$ continuous for each $y^{**} \in Y^{**}$ and $y^{**} \to f^{***}(x, y^{**}) : Y^{**} \to Z^{**}$ is $weak^* - weak^*$ continuous for each $x \in X$. The left topological center of f is defined by

$$Z_l(f) = \{x^{**} \in X^{**} : y^{**} \to f^{***}(x^{**}, y^{**}) : Y^{**} \to Z^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

Since $f^{r***r} : X^{**} \times Y^{**} \to Z^{**}$ is the unique extension of f such that the map $y^{**} \to f^{r***r}(x^{**}, y^{**}) : Y^{**} \to Z^{**}$ is $weak^* - weak^*$ continuous for each $x^{**} \in X^{**}$, we can set

$$Z_l(f) = \{x^{**} \in X^{**} : f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**}), (y^{**} \in Y^{**})\}.$$

The right topological center of f may therefore be defined as

$$Z_r(f) = \{y^{**} \in Y^{**} : x^{**} \to f^{r***r}(x^{**}, y^{**}) : X^{**} \to Z^{**} \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

Again since the map $x^{**} \to f^{***}(x^{**}, y^{**}) : X^{**} \to Z^{**}$ is $weak^* - weak^*$ continuous for each $y^{**} \in Y^{**}$, we have

$$Z_r(f) = \{y^{**} \in Y^{**} : f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**}), (x^{**} \in X^{**})\}.$$

A bounded bilinear mapping f is Arens regular if and only if $Z_l(f) = X^{**}$ or equivalently $Z_r(f) = Y^{**}$. It is clear that $X \subseteq Z_l(f)$. If $X = Z_l(f)$ then the map f is said to be left strongly irregular. Also $Y \subseteq Z_r(f)$ and if $Y = Z_r(f)$ then the map f is said to be right strongly irregular. A bounded bilinear mapping $f: X \times Y \to Z$ is said to factor if it is onto.

Investigate Arens regularity of bounded bilinear maps

S. Mohammadzadeh and Vishki H.R proved in [6] acriterion concerning to the regularity of a bounded bilinear map. They showed that f is Arens regular if and only if $f^{****}(Z^*, X^{**}) \subseteq Y^*$. In the section we provide the same conditions of Arens regularity. First, we give a similar lemma to the [6, $Theorem$ 2.1].

Lemma 2.1. *For a bounded bilinear map f from $X \times Y$ into Z the following statements are equivalent:*

(i) *f is Arens regular;*
(ii) $f^{r*****r} = f^{*****}$;
(iii) $f^{****} = f^{r*****r}$.

Proof. If (i) hold then f^r is Arens regular. Therefor $f^{r***} = f^{***r}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{***} \in Z^{***}$ we have

$$\begin{aligned} < f^{r****r}(y^{**}, z^{***}), x^{**} > &=< f^{r****}(z^{***}, y^{**}), x^{**} > =< z^{***}, f^{r***}(y^{**}, x^{**}) > \\ &=< z^{***}, f^{***r}(y^{**}, x^{**}) > =< z^{***}, f^{***}(x^{**}, y^{**}) > \\ &=< f^{****}(z^{***}, x^{**}), y^{**} > =< f^{*****}(y^{**}, z^{***}), x^{**} > \end{aligned}$$

Therefore $f^{r****r} = f^{*****}$.

$(ii) \Rightarrow (iii)$ Let $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{***} \in Z^{***}$ we have

$$\begin{aligned} < f^{r*****r}(z^{***}, x^{**}), y^{**} > &=< f^{r*****}(x^{**}, z^{***}), y^{**} > =< x^{**}, f^{r****}(z^{***}, y^{**}) > \\ &=< x^{**}, f^{r****r}(y^{**}, z^{***}) > =< x^{**}, f^{*****}(y^{**}, z^{***}) > \\ &=< f^{****}(z^{***}, x^{**}), y^{**} > \end{aligned}$$

$(iii) \Rightarrow (i)$ Let $f^{****} = f^{r*****r}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^* \in Z^*$ we have

$$\begin{aligned} < f^{r***r}(x^{**}, y^{**}), z^* > &=< f^{r***}(y^{**}, x^{**}), z^* > =< f^{r****}(z^*, y^{**}), x^{**} > \\ &=< f^{r*****}(x^{**}, z^*), y^{**} > =< f^{r*****r}(z^*, x^{**}), y^{**} > \\ &=< f^{****}(z^*, x^{**}), y^{**} > =< f^{***}(x^{**}, y^{**}), z^* >. \end{aligned}$$

It follows that f is Arens regular and this completes the proof ■

Theorem 2.2. *Bounded bilinear map f from $X \times Y$ into Z is Arens regular if and only if $f^{r****r}(Y^{**}, Z^*) \subseteq X^*$.*

Proof. Let $y^{**} \in Y^{**}$ and $z^* \in Z^*$ be arbitrary. If f is Arens regular Then $f^{r****r} = f^{*****}$ Therefore

$$f^{r****r}(y^{**}, z^*) = f^{*****}(y^{**}, z^*) = f^{*****}|_{Y^{**} \times Z^*}(y^{**}, z^*) = f^{**}(y^{**}, z^*) \in X^*.$$

Conversely, suppose $f^{r****r}(Y^{**}, Z^*) \subseteq X^*$ and let $(x_\alpha) \subseteq X$ and $(y_\beta) \subseteq Y$ be two nets that are converge to x^{**} and y^{**} in the weak*-topologies, respectivety. Then

$$\begin{aligned} < f^{r***r}(x^{**}, y^{**}), z^* > &=< f^{r***}(y^{**}, x^{**}), z^* > =< f^{r****}(z^*, y^{**}), x^{**} > \\ &= \lim_\alpha < f^{r****}(z^*, y^{**}), x_\alpha > = \lim_\alpha < z^*, f^{r***}(y^{**}, x_\alpha) > \\ &= \lim_\alpha < y^{**}, f^{r**}(x_\alpha, z^*) > = \lim_\alpha \lim_\beta < f^{r**}(x_\alpha, z^*), y_\beta > \\ &= \lim_\alpha \lim_\beta < x_\alpha, f^{r*}(z^*, y_\beta) > = \lim_\alpha \lim_\beta < z^*, f^r(y_\beta, x_\alpha) > \\ &= \lim_\alpha \lim_\beta < f(x_\alpha, y_\beta), z^* > = < f^{***}(x^{**}, y^{**}), z^* > \end{aligned}$$

Therefore f is Arens regular and this completes the proof ■

Corollary 2.3. For a bounded bilinear map $f: X \times Y \to Z$, the following statements are equivalent:

(i) $f^{r****r}(Y^{**}, Z^{***}) \subseteq X^*$;
(ii) f and f^* are Arens regular;
(iii) $f^{*r***r} = f^{r***r*}$.

Proof. The implication (i) ⇒ (ii) follows from the fact that $f^{r****r}(Y^{**}, Z^*) \subseteq f^{r****r}(Y^{**}, Z^{***}) \subseteq X^*$. Now Theorem 2.2 implies the Arens regularity of f, or equivalenty $f^{*****r} = f^{r****}$. From which

$$(f^{*r})^{r****r}(Z^{***}, Y^{**}) = f^{*****r}(Z^{***}, Y^{**}) = f^{r****}(Z^{***}, Y^{**}) = f^{r****r}(Y^{**}, Z^{***}) \subseteq X^*$$

Therefore the Arens regularity of f^{*r} follows again by Theorem 2.2. Thus f^* is Arens regular.
(ii) ⇒ (iii) If f is Arens regular. Then

$$f^{***} = f^{r***r} \Rightarrow f^{****} = f^{r***r*} \qquad (2-1)$$

Now if f^* is Arens regular. Then we have

$$f^{****} = f^{*r***r} \qquad (2-2)$$

The equalities (2-1) and (2-2) together establish the assertion.
(iii) ⇒ (i) First we show that $f^{r****r} = f^{*r**r}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{***} \in Z^{***}$

$$\begin{aligned}< f^{r****r}(y^{**}, z^{***}), x^{**} > &= < f^{r****}(z^{***}, y^{**}), x^{**} > = < z^{***}, f^{r***}(y^{**}, x^{**}) > \\ &= < z^{***}, f^{r***r}(x^{**}, y^{**}) > = < f^{r***r*}(z^{***}, x^{**}), y^{**} > \\ &= < f^{*r***r}(z^{***}, x^{**}), y^{**} > = < f^{*r***}(x^{**}, z^{***}), y^{**} > \\ &= < x^{**}, f^{*r**}(z^{***}, y^{**}) > = < f^{*r**r}(y^{**}, z^{***}), x^{**} >\end{aligned}$$

As $f^{*r**r}(Y^{**}, Z^{***})$ lies in X^* thus $f^{r****r}(Y^{**}, Z^{***}) \subseteq X^*$ and the proof ■

Theorem 2.4. Let X and A be normed spaces and $g : X \times A \longrightarrow X$ is a bounded bilinear map. If $g^{***}: X^{**} \times A^{**} \to X^{**}$ factor and g^* is Arens regular. Then g is Arens regular.

Proof. Let g^{***} factor. Thus for every $x^{**} \in X^{**}$ there exists $y^{**} \in X^{**}$ and $b^{**} \in A^{**}$ such that $x^{**} = g^{***}(y^{**}, b^{**})$. Suppose that $a^{**} \in A^{**}$and $(a_\alpha) \subseteq A$, $(b_\beta) \subseteq A$ and $(y_\gamma) \subseteq X$ be bounded nets $weak^*$−converging to a^{**}, b^{**} and y^{**} respectively. For every $x^* \in X^*$ we have

$$\begin{aligned}< g^{r***r}(x^{**}, a^{**}), x^* > &= < g^{r***}(a^{**}, x^{**}), x^* > = < g^{r****}(x^*, a^{**}), x^{**} > \\ &= < g^{r****}(x^*, a^{**}), g^{***}(y^{**}, b^{**}). > = < g^{****}(g^{r****}(x^*, a^{**}), y^{**}), b^{**} > \\ &= < g^{*r***r}(g^{r****}(x^*, a^{**}), y^{**}), b^{**} > = \lim_\beta < g^{*r***r}(g^{r****}(x^*, a^{**}), y^{**}), b_\beta > \\ &= \lim_\beta < g^{*r***}\big(y^{**}, g^{r****}(x^*, a^{**})\big), b_\beta > = \lim_\beta < y^{**}, g^{*r**}(g^{r****}(x^*, a^{**}), b_\beta) > \\ &= \lim_\beta \lim_\gamma < g^{*r**}\big(g^{r****}(x^*, a^{**}), b_\beta\big), y_\gamma > = \lim_\beta \lim_\gamma < g^{r****}(x^*, a^{**}), g^{*r*}(b_\beta, y_\gamma) > \\ &= \lim_\beta \lim_\gamma < x^*, g^{r***}(a^{**}, g^{*r*}(b_\beta, y_\gamma)) > = \lim_\beta \lim_\gamma < a^{**}, g^{r**}\big(g^{*r*}(b_\beta, y_\gamma), x^*\big) > \\ &= \lim_\beta \lim_\gamma \lim_\alpha < g^{r**}\big(g^{*r*}(b_\beta, y_\gamma), x^*\big), a_\alpha > = \lim_\beta \lim_\gamma \lim_\alpha < g^{*r*}\big(b_\beta, y_\gamma\big), g^{r*}(x^*, a_\alpha) > \\ &= \lim_\beta \lim_\gamma \lim_\alpha < b_\beta, g^{*r}(y_\gamma, g^{r*}(x^*, a_\alpha)) > = \lim_\beta \lim_\gamma \lim_\alpha < g^*\big(g^{r*}(x^*, a_\alpha), y_\gamma\big), b_\beta > \\ &= \lim_\beta \lim_\gamma \lim_\alpha < g^{r*}(x^*, a_\alpha), g(y_\gamma, b_\beta) > = \lim_\beta \lim_\gamma \lim_\alpha < x^*, g^r(a_\alpha, g(y_\gamma, b_\beta)) > \\ &= \lim_\beta \lim_\gamma \lim_\alpha < x^*, g(g(y_\gamma, b_\beta), a_\alpha) > = \lim_\beta \lim_\gamma \lim_\alpha < g^*\Big(x^*, g\big(y_\gamma, b_\beta\big)\Big), a_\alpha > \\ &= \lim_\beta \lim_\gamma < a^{**}, g^*\Big(x^*, g\big(y_\gamma, b_\beta\big)\Big) > = \lim_\beta \lim_\gamma < g^{**}(a^{**}, x^*), g\big(y_\gamma, b_\beta\big) > \\ &= \lim_\beta \lim_\gamma < g^*\big(g^{**}(a^{**}, x^*), y_\gamma\big), b_\beta > = \lim_\beta \lim_\gamma < g^{**}\Big(b_\beta, g^{**}(a^{**}, x^*)\Big), y_\gamma > \\ &= \lim_\beta < y^{**}, g^{**}\Big(b_\beta, g^{**}(a^{**}, x^*)\Big) > = \lim_\beta < g^{***}\big(y^{**}, b_\beta\big), g^{**}(a^{**}, x^*) > \\ &= \lim_\beta < g^{**}(a^{**}, x^*), g^{***}\big(y^{**}, b_\beta\big) > = \lim_\beta < g^{****}(g^{**}(a^{**}, x^*), y^{**}), b_\beta > \\ &= < g^{****}(g^{**}(a^{**}, x^*), y^{**}), b^{**} > = < g^{**}(a^{**}, x^*), g^{***}(y^{**}, b^{**}) > \\ &= < g^{**}(a^{**}, x^*), x^{**} > = < g^{***}(x^{**}, a^{**}), x^* >\end{aligned}$$

It follows that g is Arens regular ■

As an cosequnce of this theorem we have the following result:

Corollary 2.5. Let X and A be normed spaces and $g : A \times X \longrightarrow X$ is a bounded bilinear map. If $g^{r***r}: A^{**} \times X^{**} \to X^{**}$ factor and g^{r*} is Arens regular. Then g is Arens regular.

Arens regularity and reflexivity

In this section, we show that with which assumptions left strongly irregular property is equivalent to the right strongly irregular property**.**

Theorem 3.1. For a bounded bilinear map $f: X \times Y \to Z$,

(i) If f^{****} factor then both f and f^{r*} are Arens regular if and only if Y is reflexive.
(ii) If f^{r****r} factor then both f and f^* are Arens regular if and only if X is reflexive.

Proof. We only give a proof for (ii), A similar proof applies for (i). Let f and f^* are Arens regular by Corollary 2.3 $f^{r****r}(Y^{**}, Z^{***}) \subseteq X^*$. On the other hand f^{r****r} factors, So $f^{r****r}(Y^{**} \times Z^{***}) = X^{***}$. Therefore $X^{***} \subseteq X^*$. Conversely, using [8,2.3] is obvious ■

As an immediate consequnce of Theorem 3.1 and [8,2.4], we have the next Corollary.

Corollary 3.2. If one of the two following statement is assumed:

(i) f and f^* are Arens regular and f^{r****r} factor;
(ii) f and f^{r*} are Arens regular and f^{****} factor;
Then every adjoint map and every flip map of f is Arens regular.

Corollary 3.3. Let f and f^* are Arens regular and f^{r****r} factor. Then f is left strongly irregular if and only it is right strongly irregular.

Proof. The follows by applying Theorem 3.1 and $[8, Theorem\ 2.5]$ ■

If X is reflexive. Then obviously bounded bilinear map f from $X \times Y$ into Z is Arens regular. But from Arens regularity f does not imply the reflexivity of X. The next Theorem, we use the Theorem 2.2 and show that if $f^{r*}(z^*, Y) = X^*$. Then X is reflexive.

Theorem 3.4. Let bounded bilinear map f from $X \times Y$ into Z is Arens regular and let Y is a Banach space. If $f^{r*}(z^*, Y) = X^*$ for some $z^* \in Z^*$. Then X is reflexive.

Proof. Let $h: Y \rightarrow X^*$ define by $h(y) = f^{r*}(z^*, y)$ for every $y \in Y$. Obviously $h^*(x^{**}) = f^{r**}(x^{**}, z^*)$ for every $x^{**} \in X^{**}$. We have

$$< h^{**}(y^{**}), x^{**} > = < y^{**}, h^*(x^{**}) > = < y^{**}, f^{r**}(x^{**}, z^*) >$$
$$= < f^{r***}(y^{**}, x^{**}), z^* > = < f^{r****}(z^*, y^{**}), x^{**} > = < f^{r****r}(y^{**}, z^*), x^{**} >.$$

Therefore $h^{**}(y^{**}) = f^{r****r}(y^{**}, z^*)$ for every $y^{**} \in Y^{**}$. Now Theorem 2.2 implies that $f^{r****r}(Y^{**}, Z^*) \subseteq X^*$. Since $f^{r*}(z^*, Y) = X^*$ thus h is onto. Therefore h^{**} from Y^{**} into X^{***} is onto. Let $x^{***} \in X^{***}$ so there exists $y^{**} \in Y^{**}$ such that $x^{***} = h^{**}(y^{**}) = f^{r****r}(y^{**}, z^*) \in X^*$. Thus X is reflexive ■

REFERENCES

1. Arens A, The adjoint of a bilinear operation, Proc. Amer. Math. Soc, 2 (1951), 839-848.
2. Barootkoob S, Mohammadzadeh S and Vishki H.R, Topological centers of certain Banach module action, bulletin of the Iranian Mathematical Society, Vol. 35 No. 2 (2009), 25-36.
3. Dales H.G, Banach algebras and automatic continuity, London Math. Soc. Monographs 24 (Clarendon Press, Oxford,2000)
4. Dales H.G, Rodrigues-Palacios A, and Velasco M.V, The second transpose of a derivation, J. London Math. Soc. 64(2) (2001), 707-721.
5. Eshaghi Gordji M and Filali M, Arens regularity of module actions, Studia Math. 181 (3) (2007), 237-254.
6. Mohammadzadeh S and Vishki H.R, Arens regularity of module actions and the second adjoint of a derivation, Bull Austral. Mat. Soc. 77 (2008), 465-476.
7. Ulger A, Weakly compact bilinear forms and Arens regularity, Proc. Amer. Math. Soc. 101 (1978), 697-701.
8. Sheikhali A, Sheikhali A, Akhlaghi N, Arens regularity of Banach module actions and the strongly irregular property, J. Math. Computer Sci, 13 (2014), 41-46.