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Another Proof of the Hahn-Banach Theorem

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Abstract: Hahn-Banach Theorem is a famous extension theorem of linear functional analysis. Generally, the proof of the Hahn-Banach Theorem is completed through Zorn Lemma, which doesn't matter with the convex analysis. But it is in fact equivalent to Separating Hyperplane Theorem which can also be considered as the geometric form of Hahn-Banach Theorem. In this paper, we try to use the Separating Hyperplane Theorem to give another proof on the Hahn-Banach Theorem.
Keywords: Hahn-Banach Theorem, Separating Hyperplane Theorem, Convex Analysis.

INTRODUCRION

Hahn-Banach Theorem

The extension method was first came up with F. Riesz, but he just gave the L^p case .Then, in [1], Helly further proves the method is viable based on Riesz theorem .Hahn [2] and Banach [3] extend it into complete normed linear space by Helly's key inequality. Separating Hyperplane Theorem is a fundamental theorem of convex analysis [4].But it is usually just considered as the geometric form of Hahn-Banach Theorem.The equivalence of the two theorems is often ignored.In the following,we assume X is a normed linear space.

Theorem 1.1. (Hahn-Banach Theorem) Let $p: X \to R$ be a function satisfying

$$p(\lambda x) = \lambda p(x) \qquad \forall x \in X \text{ and } \forall \lambda > 0, \tag{1}$$

$$p(x+y) \le p(x) + p(y) \qquad \forall x, y \in X \quad . \tag{2}$$

Let $G \subset X$ be a linear subspace and let $g: G \to R$ be a linear functional such that

$$g(x) \le p(x), \qquad \forall x \in G \tag{3}$$

Then there exists a linear function f defined on all of X that extends g , i.e., $g(x) = f(x), \forall x \in G$, and such that

$$f(x) \le p(x). \qquad \forall x \in X \tag{4}$$

Theorem 1.2. (Separating Hyperplane Theorem) Let $A \subset X$ and $B \subset X$ be two nonempty convex subsets such that $A \cap B = \emptyset$ .Assume that one of them is open. Then there exists a closed hyperplane that separates A 和 B .

1.2. A Important Lemma

In order to give a complete proof of the Hahn-Banach theorem , we should know some preliminary knowledge used in this paper.

Lemma 1.1.([5]) Let $A \subset X$ be convex, we claim that

(i) $\overline{A}$ and Int A are convex.

(ii) $\overline{A} = \overline{IntA}$ whenever $IntA \neq \emptyset$.

Proof. (i) Let $x, y \in \overline{A}$,then there exist $x_n \in A$ and $y_n \in A$ such that

$$x = \lim x_n \text{ and } y = \lim y_n ,$$

It follows that

$$tx + (1-t)y = \lim [tx_n + (1-t)y_n], \qquad \forall t \in [0,1].$$

Apparently,

$$tx_n + (1-t)y_n \in A .$$

And therefore,

$$tx + (1-t)y \in \overline{A}, \qquad \forall t \in [0,1].$$

Set $x, y \in \operatorname{Int} A$,so there is some $r > 0$ that satisfies $B(x,r) \subset A$ and $B(y,r) \subset A$.

Since A is a convex subset, it is clear that

$$tB(x,r) + (1-t)B(y,r) \subset A, \qquad \forall t \in [0,1].$$

In fact, it is easy to see that

$$tB(x,r) + (1-t)B(y,r) = B(tx + (1-t)y, r). \qquad \forall t \in [0,1].$$

The proof is completed.

(ii) First, it is obvious that $\overline{A} \supset \overline{IntA}$.

In order to prove that $\overline{A} = \overline{IntA}$,we fix any $y_0 \in \operatorname{Int} A$ and give $x \in A$.

Then, we have

$$x = \lim_{n \to \infty} \left[(1 - \frac{1}{n})x + \frac{1}{n} y_0 \right]$$

On the other hand, there exists some $r > 0$ such that $B(y_0, r) \subset A$ .It follows that

$$(1 - t)x + tB(y_0, r) \subset A, \qquad \forall t \in [0,1].$$

In particular, choosing $t = \frac{1}{n}$,we can obtain

$$B((1 - \frac{1}{n})x + \frac{1}{n} y_0, \frac{1}{n} r) \subset A$$

and thus

$$(1 - \frac{1}{n})x + \frac{1}{n} y_0 \in \operatorname{Int} A .$$

Therefore,

$$x \in \overline{IntA}$$

This proves that $A \subset \overline{IntA}$ and hence $\overline{A} \subset \overline{IntA}$.

USING SEPARATING HYPERPLANE THEOREM TO PROVE hahn-banach THEOREM

Proof. Set A =epi $p = \{(x,\alpha) | p(x) \le \alpha, x \in X, \alpha \in R\}, B = \{(x,\alpha) | g(x) = \alpha, x \in G, \alpha \in R\}$.

First, we can claim that A is a convex subset in $X \times R$.

Indeed, for every $\lambda \in [0,1]$,choosing any $(x_1, \alpha_1) \in A$ and $(x_2, \alpha_2) \in A$,we claim by (1)and (2) that

$$p(\lambda x_1 + (1-\lambda)x_2) \le p(\lambda x_1) + p((1-\lambda)x_2) = \lambda p(x_1) + (1-\lambda)p(x_2) .$$

By the definition of A ,we have

$$\lambda p(x_1) + (1-\lambda)p(x_2) \le \lambda\alpha_1 + \lambda\alpha_2 .$$

and thus

$$(\lambda x_1 + (1-\lambda)x_2), \ \lambda\alpha_1 + (1-\lambda)\alpha_2) \in A .$$

By Lemma1.1,we claim that Int A and $\overline{A}$ are also convex. Moreover, Int A is nonempty and open . On the other hand,since g is a linear functional, it is clear that B is a nonempty subspace, hence which is also convex.

Moreover, Int $A \cap B = \varnothing$.

We apply Theorem1.2(Separating Hyperplane Theorem)in the space $X \times R$ with Int A and B .Thus there exists a closed hyperplane H that separates Int A and B ,but H also separates $\overline{IntA}$ and B .

Moreover, we know from Lemma1.1 that $\overline{A} = \overline{IntA}$ .Hence there exist $h \in X^*$, $k \in R$ and $\beta \in R$ such that $H = \{(x,\alpha) \in X \times R | \Phi(x,\alpha) = \beta\}$ separates A and B ,where

$$\Phi((x,\alpha)) = \langle h, x\rangle + k\alpha, \qquad \forall (x,\alpha) \in X \times R.$$

It follows that

$$\langle h, x\rangle + k\alpha \geq \beta, \qquad \forall (x,\alpha) \in A,$$

(5)

$$\langle h, x\rangle + k\alpha \leq \beta, \qquad \forall (x,\alpha) \in B.$$

(6)

By the definition of B , (6) equals to

$$\langle h, x\rangle + k\langle g, x\rangle \leq \beta, \qquad \forall x \in G.$$

(7)

Since α can be any real number, so we can set $\alpha \to +\infty$ in (5), which implies $k \geq 0$ .Furthermore, we claim that

$$k > 0.$$

(8)

In fact, since G is linear subspace, for any $t \in R$ and $x \in G$, we have $tx \in G$ .Therefore, the left part of (7) equals to zero.

So we claim that

$$\langle h, x\rangle + k\langle g, x\rangle = 0, \qquad \forall x \in G,$$

(9)

and by (5),we have

$$\langle h, x\rangle + k\alpha \geq 0, \quad \forall (x,\alpha) \in A.$$

(10)

Assuming $k = 0$, we know from (10) that

$$h(x) = \langle h, x\rangle \geq 0, \forall x \in X$$

(11)

But h is a linear functional so that $h(tx) = th(x), \forall t \in R$.

Therefore, by (11) ,we can get

$$h(tx) = th(x) \geq 0, \forall t \in R, \forall x \in X$$

(12)

Hence, $h(x) = 0, \forall x \in X$.

Since otherwise, if there exists some $x_0 \in X$ such that $h(x_0) \neq 0$, we have the following facts:

(i) If $h(x_0) > 0$,we can take $t < 0$, then $th(x_0) < 0$.

(ii) If $h(x_0) < 0$,we can take $t > 0$, then $th(x_0) < 0$.

If $k = 0$,then $\Phi \equiv 0$,which leads to a contradiction and we completes the proof of (8).

Let $f = -\frac{h}{k}$.

From (9) we obtain

$$\langle -kf, x\rangle + k\langle g, x\rangle = 0, \quad \forall x \in G, \tag{13}$$

so that

$$\langle f, x\rangle = \langle g, x\rangle, \qquad \forall x \in G. \tag{14}$$

By (10) we obtain

$$\langle -kf, x\rangle + k\alpha \geq 0, \qquad \forall (x, \alpha) \in A \tag{15}$$

and

$$\langle f, x\rangle \leq \alpha, \qquad \forall (x, \alpha) \in A. \tag{16}$$

Since $(x, p(x)) \in A$,we can choose $\alpha = p(x)$ and the proof of (4) is finished.

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