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Boundedness of the Commutators of Fractional Maximal Operator on Variable Exponent Herz-Morrey Spaces

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Abstract: In this paper, using the Hölder inequality, by the properties of the fractional maximal operators and BMO functions, together with the definition of Herz-Morrey spaces with variable exponent, based on the boundedness of the commutators of the fractional maximal operator on $L^{p(\cdot)}$, the boundedness of the commutators of the fractional maximal operators on variable exponent Herz-Morrey spaces is proved. This result generalized the classical situation for non-variable exponent.
Keywords: variable exponent; fractional maximal operator; commutator; Herz-Morrey space.

INTRODUCTION

Let T be the singular integral operator. The commutator $[T,b]$ generated by T and a suitable function b is defined by

$$[T,b]=T(bf)-bT(f).$$

In 1965, Calderón[1] put forward the theory of commutator of singular integral operators. Milman and Schonbek[2] proved the boundedness of the commutators $[M,b]$ generated by BMO function and Hardy-Littlewood maximal functions on Lebesgue spaces. Zhang Pu and Wu Jianglong [3] introduced the commutator $[M_\beta,b]$ generated by BMO function and the fractional maximal function, and, they discussed some characterizations of b for which $[M_\beta,b]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Whereafter, they extended to the situation of variable exponent Lebesgue spaces[4].

Wang Lijuan and Shu Lisheng[5] obtained the boundedness of the commutators generated by BMO function and the fractional maximal function on Herz-Morrey spaces.

In this paper, we focus on the boundedness of commutators of fractional Hardy-Littlewood maximal operators on Herz-Morrey spaces associate to variable exponent.

PRELIMINARIES

We recall several usefull lemmas and definitions.

Definition 2.1 For $f\in L^1_{\text{loc}}(\mathbb{R}^n)$, the fractional Hardy-Littlewood maximal function M_β is defined by

$$M_\beta(f)(x)=\sup_{t>0}\frac{1}{|B(x,t)|^{1-\frac{\beta}{n}}}\int_{B(x,t)}|f(y)|\,\mathrm{d}y\ ,\ 0\le\beta<n,$$

where $B(x,t)=\{y\in\mathbb{R}^n:|x-y|<t\}$ denotes the ball with the center x and radius t.

The commutator generated by the fractional maximal function M_β and function b is defined by

$$M_{\beta,b}=bM_\beta(f)-M_\beta(bf).$$

As usual, let Ω be a bounded domain in $\mathbb{R}^n$. Suppose χ_s is the characteristic function of a measurable set S, $|S|$ is the Lebesgue measure of S, $B_k=(0,2^k)=\{x\in\Omega:|x|\le 2^k\}$, $A_k=B_k\setminus B_{k-1}$, $\chi_k=\chi_{A_k}$, $k\in\mathbb{Z}$. We always use the letter C to denote a absolute positive constant, which may change from one to another, and only depends on main parameters.

Definition 2.2 Let $p(\cdot)$ be a measurable function on Ω, $p(\cdot):\Omega\to[1,\infty)$. The variable exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, is defined by

$$L^{p(\cdot)}(\Omega)=\{f \text{ measurable}: \int_\Omega\left(\frac{|f(x)|}{\eta}\right)^{p(x)}dx<\infty, \eta>0\}.$$

The locally variable exponent Lebesgue space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega)=\{f \text{ measurable: for all compact subsets } K\subset\Omega,\ f\in L^{p(\cdot)}(K)\},$$

where

$$\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\{\eta>0:\int_\Omega\left(\frac{(f(x))}{\eta}\right)^{p(x)}dx\le 1\}.$$

Denote by $\mathscr{P}(\Omega)$ the set of all measurable function $p(x)$ on Ω such that

$$1<p_-<p(x)<p_+<\infty,$$

where

$$p_-=\operatorname{essinf}\{p(x):x\in\Omega\},\ p_+=\operatorname{ess\,sup}\{p(x):x\in\Omega\},$$

and by $\mathscr{B}(\Omega)$ the set of all $p(\cdot)\in\mathscr{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

Definition 2.3 Let $\alpha\in\mathbb{R}$, $0<q<\infty$, $p(\cdot)\in\mathscr{P}(\Omega)$, $0\le\lambda<\infty$. The variable exponent Herz-Morrey space is defined by

$$M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\Omega)=\{f\in L^{p(\cdot)}_{\mathrm{loc}}(\Omega\setminus\{0\}):\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}}<\infty\}.$$

The norm in $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\Omega)$ is defined as

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\Omega)}=\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha q}\|f\chi_k\|^q_{L^{p(\cdot)}(\Omega)}\Big)^{\frac{1}{q}}.$$

Definition 2.4 We say that a function $p:\mathbb{R}^n\to(0,\infty)$ is locally log-Hölder continuous, if there exists a constant $C>0$, such that

$$|p(x)-p(y)|\le\frac{C}{\log(e+1/|x-y|)},\ \text{for all } x,y\in\mathbb{R}^n.$$

If

$$|p(x)-p(0)|\le\frac{C}{\log(e+1/|x|)},\ \text{for all } x\in\mathbb{R}^n,$$

then we say that p is log-Hölder continuous at the origin(or has a log decay at the origin).

If, for $p_\infty=\lim_{|x|\to\infty}p(x)$ and $C>0$, there holds

$$|p(x)-p_\infty|\le\frac{C}{\log(e+|x|)}$$

for all $x\in\mathbb{R}^n$, then we say that p is log-Hölder continuous at infinity(or has a log decay at infinity).

We denote the class of all exponents $p\in\mathscr{P}(\Omega)$ which have a log decay at the origin and at infinity by $P_0^{\log}(\Omega)$ and $P_\infty^{\log}(\Omega)$, respectively.

Lemma 2.1(see[6]). Suppose $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. Then for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \le C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + \dfrac{1}{p^-} - \dfrac{1}{p^+}$.

Lemma 2.2(see[7]). Let $p(\cdot) \in \mathscr{B}(\Omega)$. Then there exists a constant $C > 0$, such that for all balls B in Ω and for all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_S\|_{L^{p(\cdot)}(\Omega)}} \le C \frac{|B|}{|S|}, \tag{2.1}$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_B\|_{L^{p(\cdot)}(\Omega)}} \le C \left(\frac{|S|}{|B|} \right)^{\delta}, \tag{2.2}$$

where the constant δ satisfies $0 < \delta < 1$.

If $p'(\cdot) \in \mathscr{B}(\Omega)$, by (2.1) and (2.2), we can take constant $0 < r < 1/(p_2')_+$ so that

$$\frac{\|\chi_S\|_{L^{p_2'(\cdot)}(\Omega)}}{\|\chi_B\|_{L^{p_2'(\cdot)}(\Omega)}} \le C \left(\frac{|S|}{|B|} \right)^{r},$$

for all balls B in Ω and all measurable subsets $S \subset B$.

Lemma 2.3(see[8]). Let $p(\cdot) \in \mathscr{B}(\Omega)$. Then there exists a constant $C > 0$, such that

$$C^{-1} \le \frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\Omega)} \|\chi_B\|_{L^{p'(\cdot)}(\Omega)} \le C,$$

for all balls B in Ω.

Lemma 2.4(see[9]). Let $b \in \mathrm{BMO}(\Omega)$. Then we have that for all balls $B \subset \Omega$ and all $j, i \in \mathbb{Z}$ with $j > i$,

$$C^{-1} \|b\|_{\mathrm{BMO}(\Omega)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\Omega)}} \|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\Omega)} \le C \|b\|_{\mathrm{BMO}(\Omega)},$$

$$\|(b - b_i)\chi_{B_j}\|_{L^{p(\cdot)}(\Omega)} \le C(j-i) \|b\|_{\mathrm{BMO}(\Omega)} \|\chi_{B_j}\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2.5(see[4]). Let $p(\cdot) \in \mathscr{P}(\Omega)$, $0 < \beta < \dfrac{n}{p^+}$, $\dfrac{q(\cdot)(n-\beta)}{n} \in \mathscr{B}(\Omega)$ and $\dfrac{1}{q(\cdot)} = \dfrac{1}{p(\cdot)} - \dfrac{\beta}{n}$. If $0 \le b(x) \in \mathrm{BMO}$, then $M_{\beta,b}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.6(see[10]). Let $p(\cdot) \in \mathscr{P}(\Omega)$. If $p(\cdot) \in P_0^{\log}(\Omega) \cap P_\infty^{\log}(\Omega)$, then $p(\cdot) \in \mathscr{B}(\Omega)$.

THE MAIN RESULT

The following theorem is the main result of this paper.

Theorem 3.1 Let $\alpha \in \mathbb{R}$, $0 < q_1 \le q_2 < \infty$, $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\Omega)$, satisfy the log-Hölder condition, and $\dfrac{1}{p_2(\cdot)} = \dfrac{1}{p_1(\cdot)} - \dfrac{\beta}{n}$. Set $0 < \lambda < \alpha$, $0 < \alpha < nr - \beta$, $0 < r < 1/(p_2')_+$. If $b \in \mathrm{BMO}$, then $M_{\beta,b}$ is bounded from $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\Omega)$ to $M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\Omega)$.

***Proof*:** Since $0 < q_1 \le q_2 < \infty$ and $b \in \mathrm{BMO}$ for all $f(x) \in M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\Omega)$, we write

$$f(x)=\sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x) \text{, where } j\in\mathbb{Z}.$$

Then, applying the Jensen inequality, we have

$$\| M_{\beta,b} f \|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\Omega)} \le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \Big(\sum_{j=-\infty}^{\infty} \| (M_{\beta,b} f_j)\cdot \chi_k \|_{L^{p_2(\cdot)}(\Omega)} \Big)^{q_1}$$

$$\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \Big(\sum_{j=-\infty}^{k-3} \| (M_{\beta,b} f_j)\cdot \chi_k \|_{L^{p_2(\cdot)}(\Omega)} \Big)^{q_1}$$

$$+C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \Big(\sum_{j=k-2}^{k+2} \| (M_{\beta,b} f_j)\cdot \chi_k \|_{L^{p_2(\cdot)}(\Omega)} \Big)^{q_1}$$

$$+C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \Big(\sum_{j=k+3}^{\infty} \| (M_{\beta,b} f_j)\cdot \chi_k \|_{L^{p_2(\cdot)}(\Omega)} \Big)^{q_1}$$

$$\triangleq D_1+D_2+D_3 .$$

We first estimate D_2. By the boundedness of $M_{\beta,b}$ from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$, we have

$$D_2 \le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k_0=-\infty}^{\infty} 2^{k\alpha q_1} \| f\chi_k \|^{q_1}_{L^{p_1(\cdot)}(\Omega)} \le C \| f \|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\Omega)} .$$

Now we turn to estimate D_1. Noting that $j\le k-3$, $x\in A_k$ and $y\in A_j$, by the Hölder inequality, we obtain

$$| M_{\beta,b} f_j(x)\chi_k(x) | \le C\cdot 2^{nk(\frac{\beta}{n}-1)} \int_{A_j} | b(x)-b(y) | \cdot | f_j(y) | \mathrm{d}y \cdot \chi_k(x)$$

$$\le C\cdot 2^{k(\beta-n)} \int_{A_j} | b(x)-b_{B_j} | \cdot | f_j(y) | \mathrm{d}y \cdot \chi_k(x)$$

$$+C\cdot 2^{k(\beta-n)} \int_{A_j} | b_{B_j}-b(y) | \cdot | f_j(y) | \mathrm{d}y \cdot \chi_k(x)$$

$$\le C\cdot 2^{k(\beta-n)} | b(x)-b_{B_j} | \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_j \|_{L^{p_1'(\cdot)}(\Omega)} \cdot \chi_k(x)$$

$$+C\cdot 2^{k(\beta-n)} \| (b_{B_j}-b(y))\cdot \chi_j \|_{L^{p_1'(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \chi_k(x) .$$

By Lemma 2.4, we have

$$\| M_{\beta,b} f_j(x)\chi_k(x) \|_{L^{p_2(\cdot)}(\Omega)}$$

$$\le C\cdot 2^{k(\beta-n)} \| (b(x)-b_{B_j})\cdot \chi_k(x) \|_{L^{p_2(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_j \|_{L^{p_1'(\cdot)}(\Omega)}$$

$$+C\cdot 2^{k(\beta-n)} \| (b_{B_j}-b(y))\cdot \chi_j \|_{L^{p_1'(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_k \|_{L^{p_2(\cdot)}(\Omega)}$$

$$\le C\cdot 2^{k(\beta-n)} (k-j) \| b \|_{\mathrm{BMO}(\Omega)} \cdot \| \chi_{B_k} \|_{L^{p_2(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)}$$

$$+C\cdot 2^{k(\beta-n)} \| b \|_{\mathrm{BMO}(\Omega)} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_{B_k} \|_{L^{p_2(\cdot)}(\Omega)}$$

$$\le C\cdot 2^{k(\beta-n)} (k-j) \| b \|_{\mathrm{BMO}(\Omega)} \cdot \| \chi_{B_k} \|_{L^{p_2(\cdot)}(\Omega)} \cdot \| f_j \|_{L^{p_1(\cdot)}(\Omega)} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)} .$$

Lemma 2.2 and Lemma 2.3 tell us

$$2^{k(\beta-n)} \| \chi_{B_k} \|_{L^{p_2(\cdot)}(\Omega)} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)} \le C\cdot 2^{k\beta} \| \chi_{B_k} \|^{-1}_{L^{p_2'(\cdot)}(\Omega)} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)}$$

$$\le C\cdot 2^{k\beta} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)} \| \chi_{B_j} \|^{-1}_{L^{p_2'(\cdot)}(\Omega)} \cdot \frac{\| \chi_{B_j} \|_{L^{p_2'(\cdot)}(\Omega)}}{\| \chi_{B_k} \|_{L^{p_2'(\cdot)}(\Omega)}}$$

$$\le C\cdot 2^{k\beta} \cdot \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\Omega)} \| \chi_{B_j} \|^{-1}_{L^{p_2'(\cdot)}(\Omega)} \cdot 2^{nr(j-k)} .$$

By the definition of M_β, we obtain

$$M_\beta(\chi_{B_j})(x) \ge M_\beta(\chi_{B_j})(x)\cdot \chi_{B_j}(x) = \frac{1}{|B_j|^{1-\frac{\beta}{n}}}\int_{B_j}|\chi_{B_j}|\,dy\cdot \chi_{B_j}(x)$$

$$= \frac{|B_j|}{|B_j|^{1-\frac{\beta}{n}}}\chi_{B_j} = |B_j|^{\frac{\beta}{n}}\chi_{B_j} = 2^{j\beta}\chi_{B_j}.$$

$$\|\chi_{B_j}\|^{-1}_{L^{p_2'(\cdot)}(\Omega)} \le C\cdot 2^{-nj}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\Omega)} \le C\cdot 2^{-nj}\cdot 2^{-j\beta}\|M_\beta(\chi_{B_j})\|_{L^{p_2(\cdot)}(\Omega)}$$
$$\le C\cdot 2^{-j\beta}\cdot 2^{-nj}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\Omega)} \le C\cdot 2^{-j\beta}\|\chi_{B_j}\|^{-1}_{L^{p_1'(\cdot)}(\Omega)}.$$

Hence,

$$D_1 \le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha q_1}\Big(\sum_{j=-\infty}^{k-3} 2^{k\beta}\cdot 2^{-j\beta}\cdot 2^{nr(j-k)}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}\|b\|_{\mathrm{BMO}(\Omega)}(k-j)\Big)^{q_1}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=-\infty}^{k-3}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}(k-j)\cdot 2^{(k-j)(\beta-nr+\alpha)}\cdot 2^{\alpha j}\Big)^{q_1}.$$

In the case of $1<q_1<\infty$, noting that $\beta-nr+\alpha<0$, by the Hölder inequality, we obtain

$$D_1 \le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=-\infty}^{k-3} 2^{\alpha j q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\Omega)}\cdot 2^{(k-j)(\beta-nr+\alpha)\cdot\frac{q_1}{2}}\Big)$$
$$\times\Big(\sum_{j=-\infty}^{k-3}(k-j)^{q_1'}\cdot 2^{(k-j)(\beta-nr+\alpha)\cdot\frac{q_1'}{2}}\Big)^{\frac{q_1}{q_1'}}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=-\infty}^{k-3} 2^{\alpha j q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\Omega)}\cdot 2^{(k-j)(\beta-nr+\alpha)\cdot\frac{q_1}{2}}\Big)$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k-3} 2^{\alpha j q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\Omega)}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\Omega)}.$$

In the case of $0<q_1<1$, we have

$$D_1 \le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\sum_{j=-\infty}^{k-3}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\Omega)}(k-j)^{q_1}\cdot 2^{(k-j)(\beta-nr+\alpha)q_1}\cdot 2^{\alpha j q_1}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\cdot\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k-3} 2^{\alpha j q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\Omega)}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\Omega)}.$$

Finally, we estimate D_3. Noting that $j\ge k+3$, by Lemma 2.5, we have

$$D_3 \le C\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha q_1}\Big(\sum_{j=k+3}^{\infty}\|b\|_{\mathrm{BMO}(\Omega)}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}\Big)^{q_1}$$
$$\le C\|b\|^{q_1}_{\mathrm{BMO}(\Omega)}\sup_{k_0\in Z} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=k+3}^{\infty} 2^{\alpha j}\cdot 2^{\alpha(k-j)}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}\Big)^{q_1}.$$

And note that $0<\lambda<\alpha$, choose $\delta>1$ such that $\lambda-\alpha/\delta<0$.

If $1<q_1<\infty$, by the Hölder inequality, there is

$$D_3 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \Big(\sum_{j=k+3}^{\infty} 2^{\alpha j q_1}\cdot 2^{\alpha(k-j)q_1/\delta}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}\Big) \times\Big(\sum_{j=k+3}^{\infty} 2^{\alpha(k-j)q_1'(\delta-1)/\delta}\Big)^{q_1/q_1'}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k+3}^{k_0-1} 2^{\alpha j q_1}\cdot 2^{\alpha(k-j)q_1/\delta}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}$$

$$+C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\alpha j q_1}\cdot 2^{\alpha(k-j)q_1/\delta}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}$$

$$\triangleq E_1+E_2 .$$

For $\alpha>0$,

$$E_1 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{\alpha j q_1}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1} \sum_{k=-\infty}^{j-3} 2^{\alpha(k-j)q_1/\delta}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{\alpha j q_1}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1}\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1} .$$

Noting that $\lambda-\alpha/\delta<0$,

$$E_2 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\alpha(k-j)q_1/\delta}\cdot 2^{j\lambda q_1}\|f_j\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \Big(\sum_{k=-\infty}^{k_0} 2^{\alpha k q_1/\delta}\Big)\Big(\sum_{j=k_0}^{\infty} 2^{(\lambda-\alpha/\delta)jq_1}\Big)\|f_j\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1}\cdot 2^{\alpha k_0 q_1/\delta}\cdot 2^{(\lambda-\alpha/\delta)k_0 q_1}\|f_j\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1}\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1} .$$

If $0<q_1<1$, similarly, we easily obtain that

$$D_3 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k+3}^{k_0-1} 2^{\alpha j q_1}\cdot 2^{\alpha(k-j)q_1}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}$$

$$+C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\alpha j q_1}\cdot 2^{\alpha(k-j)q_1}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1}$$

$$\triangleq E_3+E_4 .$$

$$E_3 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{\alpha j q_1}\|f_j\|_{L^{p_1(\cdot)}(\Omega)}^{q_1} \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1}\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1} .$$

$$E_4 \le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\alpha(k-j)q_1}\cdot 2^{\lambda j q_1}\|f_j\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1} \sup_{k_0\in Z} 2^{-k_0\lambda q_1}\cdot 2^{\alpha k_0 q_1}\cdot 2^{(\lambda-\alpha)k_0 q_1}\|f_j\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1}$$

$$\le C\|b\|_{\mathrm{BMO}(\Omega)}^{q_1}\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\Omega)}^{q_1} .$$

Thus, we complete the proof of Theorem 3.1.

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