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Existence of Time Periodic Solutions for the Modified Swift-Hohenberg Equation

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Abstract: In this paper, we consider the existence of time periodic solutions of the modified Swift-Hohenberg equation. We used the Galerkin method. Firstly, by Leray-Schauder fixed point theorem, we show the existence of approximate solutions of the modified Swift-Hohenberg equation, then we show the convergence of the approximate solutions, and we also get the uniqueness of the solution to the modified equation.
Keywords: Swift-Hohenberg equation, time periodic solutions, existence, uniqueness.

INTRODUCTION

In this paper we concerned the existence and uniqueness of time periodic solutions for the modified Swift-Hohenberg equation

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(x,t), \quad (x,t) \in \Omega \times R, \quad (1.1)$$

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega, \quad (1.2)$$

Where Ω is an open connected bounded domain in R^3 with smooth boundary $\partial\Omega$, *a* and b are arbitrary constant, g is an external forcing term.

The system is the usual Swift-Hohenberg equation if $b = 0$, $g \equiv 0$ in (1.1). Refer literature [1], we know the Swift-Hohenberg equation was introduced by Swift J. B. and Hohenberg P. C. in 1977 when they studied the convective hydrodynamics and viscous film flow.

In 2003, Peletier L. A. and Rottschafer in [8] researched the large time behaviour of solutions of the Swift-Hohenberg equation. In the same year, Zhou Hua and Tang Jian in [6] proved some properties and structures of solutions of the Swift-Hohenberg equation. In 2007, Wang Yanping in [5] proved the time-periodic solution for a generalized Swift-Hohenberg model equation; however, the modified Swift-Hohenberg equation does not satisfy its conditions. In 2009, Polat M. in [9] proved the global attractor for the modified Swift-Hohenberg equation. In 2014, Sun H. P. and Jong Y. P. in [8] researched pullback attractor for the non-autonomous modified Swift-Hohenberg equation. In 2017, Wang Z. and Du X. in [4] proved the pullback attractors for modified Swift-Hohenberg equation on unbounded domains with non-autonomous deterministic and stochastic forcing terms.

In present paper, the problems we have considered are as follows. Let the given external forces $g(x,t)$ be periodic in t with the period *T*, and then we try to prove the existence and uniqueness of periodic solutions *u* of the modified Swift-Hohenberg equation with the same period T,

$$u(x,t+T) = u(x,t) \quad (1.3)$$

under the critical smallness assumption,i.e.,

$$K \equiv \sup_{0 \le t \le T} \|g(x,t)\|_{L^N(\Omega)} \text{ is sufficiently small.}$$

Our main results are Theorem 5.1 and Theorem 5.2. To prove the time periodic solutions of the modified Swift-Hohenberg equation, we use the well-known Galerkin method which used to prove the existence of time periodic solutions and weak solutions for many systems, such as Navier-Stokes equations, Schrodinger -Boussinesq equation and quantum equation. So motivated by the ideas in [2,3,11], we can accomplished this paper.

Preliminaries

To describe our theorems accurately, we introduce some function space and notation. We denote $L^2(\Omega)$ -norm by $\|\cdot\|$, $L^p(\Omega)$ -norm by $\|\cdot\|_p$. $H^m(\Omega)$ is the Sobolev space. We define H_σ as the closure of C_0^∞ in $L_2(\Omega)$. Stokes operator A with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)\cap H_\sigma$. Let X be a Banach space. We denote by $C^k(T;X)$ the set of X-valued T-periodic functions on R^1 with continuous derivatives up to order k. then let us define the norm

$$\|g\|_{C^k(T;X)}=\sup_{0\le t\le T}\left\{\sum_{i=0}^{k}\|D_t^i g(t)\|_X\right\}$$

We denote by $L^p(T;X)$ $(1\le p<\infty)$ the set of T-periodic X-valued measurable functions g on R^1 such that

$$\|g\|_{L^p(T;X)}=\left(\int_0^T\|g\|_X^p dt\right)^{\frac{1}{p}}<+\infty \quad (1\le p<\infty),$$

$$\|g\|_{L^\infty(T;X)}=\sup_{0\le t\le T}\|g\|_X<+\infty$$

We denote by $W^{k,p}(T;X)$ the set of functions g which belong to $L^p(T;X)$ together with their derivatives up to order k, and in particular we write $H^k(T;X)=W^{k,2}(T;X)$ when X is a Hilbert space.

To prove our theorems, we shall use the following inequality, and we can refer literature [8] to get it.

Lemma 2.1 *(Gagliardo-Nirenberg Inequality) Let* Ω *be an open, bounded domain of the lipschitz class in* R^n. *Assume that* $1\le p\le\infty$, $1\le q\le\infty$, $1\le r$, $0\le\theta\le1$, *and let* $k-\frac{n}{p}\le\theta(m-\frac{n}{q})+(1-\theta)\frac{n}{r}$, *Then the following inequality hold*

$$\|D^k u\|_{L^p(\Omega)}\le c(\Omega)\|u\|_{W^{m,q}(\Omega)}^{\theta}\|u\|_{L^r(\Omega)}^{1-\theta}.$$

Approximate solutions

In this section, we will prove the existence of approximate solution of (1.1)-(1.3). Now let $w_k(k=1,2,\ldots)$ be the completely orthonormal system in H_σ consisting of the eigenfunctions of the Stokes operator A. Denote the form of the approximate solution u_n of the problem (1.1)-(1.3)

$$u_n=\sum_{k=1}^{m}a_{kn}(t)w_k$$

We consider the system of nonlinear differential equation

$$(u_{nt}+\Delta^2 u_n+2\Delta u_n+au_n+b|\nabla u_n|^2+u_n^3,w_k)=(g,w_k), \tag{3.1}$$

$$u_n(x,t+T)=u_n(x,t) \tag{3.2}$$

Let W_n' be the subspace of H_σ spanned by $w_1,w_2,\ldots,w_n$. It is well known that for any $v_n\in C^1(T,W_n')$, there exists a unique T-periodic solution $u_n\in C^1(T,W_n')$ of the linear equation

$$(u_{nt}+\Delta^2 u_n+2\Delta u_n+au_n,w_k)=(g-b|\nabla v_n|^2-v_n^3,w_k),$$

So we can see the mapping: $F:v_n\to u_n$ is continuous and compact in $C^1(T,W_n')$. Thus, we shall prove the existence of the solution of (3.1)-(3.2) by applying the Leray-schauder fixed point theorem, and it is only need to show the boundedness

$$\sup_{0\le t\le T}\|u_n(t)\|\le C$$

for all possible solutions of (3.1)-(3.2) replaced by $\delta(b|\nabla u_n|^2+u_n^3)$ $(0\le\delta\le1)$ instead of nonlinear terms $b|\nabla u_n|^2+u_n^3$. Where C is a constant independent of δ.

Multiplying (3.1) by $a_{kn}(t)$ and summing up over k, we see

$$(u_{nt}+\Delta^2 u_n+2\Delta u_n+au_n+\delta(b|\nabla u_n|^2+u_n^3),u_n)=(g,u_n), \tag{3.3}$$

using integration by Parts, we obtain

$$\frac{d}{dt}\|u_n\|^2+2\|\Delta u_n\|^2+2\delta\|u_n\|_4^4=4\|\nabla u_n\|^2-2a\|u_n\|^2-2b\delta\int_\Omega|\nabla u_n|^2u_n dx+2\int_\Omega gu_n dx. \tag{3.4}$$

Appling the Gagliardo-Nirenberg inequality with $k=1,n=3,p=r=m=q=2,\theta=\frac{1}{2}$ to the first term on the right hand sida of (3.4), we have

$$4\|\nabla u_n\|^2 \le c\|\Delta u_n\|\|u_n\| \le \frac{1}{4}\|\Delta u_n\|^2 + c\|u_n\|^2, \tag{3.5}$$

by using the Holder inequality, Gagliardo-Nirenberg inequality and Young inequality, we obtain

$$\begin{aligned} 2|b|\delta\int_\Omega |\nabla u_n|^2 u_n dx &\le 2|b|\delta\|\nabla u_n\|_4^2\|u_n\| \le 2|b|\delta\|\Delta u_n\|^{2\theta}\|u_n\|_4^{2(1-\theta)}\|u_n\| \\ &\le c\delta\|\Delta u_n\|^{2\theta}\|u_n\|_4^{3-2\theta} \le \frac{1}{4}\|\Delta u_n\|^2 + c\delta^{\frac{1}{1-\theta}}\|u_n\|_4^{\frac{3-2\theta}{1-\theta}}, \end{aligned} \tag{3.6}$$

Holder inequality, Young inequality and poincare inequality give that

$$2\int_\Omega g u_n dx \le \|g\|\|u_n\| \le \lambda^2\|u_n\|^2 + \frac{1}{\lambda^2}\|g\|^2 \le \|\Delta u_n\|^2 + \frac{1}{\lambda^2}\|g\|^2, \tag{3.7}$$

so from (3.4)-(3.7), using young inequality, seeing that $3 < \frac{3-2\theta}{1-\theta} < 4$, there exists $M > 0$ such that

$$\begin{aligned} \frac{d}{dt}\|u_n\|^2 + \frac{1}{2}\|\Delta u_n\|^2 + 2\delta\|u_n\|_4^4 &\le c\|u_n\|^2 + c\delta^{\frac{1}{1-\theta}}\|u_n\|_4^{\frac{3-2\theta}{1-\theta}} + \frac{1}{\lambda^2}\|g\|^2 \\ &\le M + \varepsilon(\delta)\|u_n\|_4^4 + \frac{1}{\lambda^2}\|g\|^2 \end{aligned}, \tag{3.8}$$

from (3.8), since $2\delta > \varepsilon(\delta)$, we have

$$\frac{d}{dt}\|u_n\|^2 + \frac{1}{2}\|\Delta u_n\|^2 + c\|u_n\|_4^4 \le M + \frac{1}{\lambda^2}K^2, \tag{3.9}$$

using the periodicity of u_n, integrating (3.9) over $[0,T]$ we get

$$\int_0^T (\frac{1}{2}\|\Delta u_n\|^2 + c\|u_n\|_4^4)dt \le MT + \frac{1}{\lambda}K^2 T, \tag{3.10}$$

by the first mean value theorems for definite integrals and (3.11), there exists $t^* \in [0,T]$ such that

$$\frac{1}{2}\|\Delta u_n(t^*)\|^2 \le \frac{1}{2}\|\Delta u_n(t^*)\|^2 + c\|u_n(t^*)\|_4^4 \le M + \frac{1}{\lambda}K^2, \tag{3.11}$$

using poincare inequality $\|A^\alpha u_n\| \le \lambda^{\alpha-\beta}\|A^\beta u_n\|$, we have

$$\|u_n(t^*)\|^2 \le \lambda^{-2}\|\Delta u_n(t^*)\|^2, \tag{3.12}$$

integrating (3.9) again over $[t^*, t+T]$ $(t \in [0,T])$, we obtain

$$\begin{aligned} \|u_n(t)\|^2 &\le \|u_n(t^*)\|^2 + \left|\int_0^T (\frac{1}{2}\|\Delta u_n\|^2 + c\|u_n\|_4^4)dt\right| + (t+T-t^*)(M + \frac{1}{\lambda^2}K^2) \\ &\le (2\lambda^{-2} + 2T)(M + \frac{1}{\lambda^2}K^2) = C \end{aligned}, \tag{3.13}$$

where C is independent of n and δ. So we proved the $u_n \in C^1(T, W_n')$ is the approximate solution of (3.1)-(3.2).

Estimates of derivatives of high order

In this section, we will show the convergence of the approximate solution.

Since the $w_k (k = 1,2,...)$ are the eigenfunctions of A, we can write

$$Aw_k = \lambda_k w_k, \quad A^s w_k = \lambda_k^s w_k, \tag{4.1}$$

where λ_k is the eigenvalue of A.

Lemma 4.1 *Let u_n be the solution of (3.1)-(3.2) given above. Set*

$$K_0 = \int_0^T \|g\|^2 dt$$

we have

$$\|\Delta u_n(t)\|^2 \le (\lambda^{-2}T^{-1} + 2)C(K, K_0),$$

where $C(K, K_0)$ denote constants depending on K, K_0 and independent of n.

Proof Considering (3.1) and (4.1), we see

$$(u_{nt} + \Delta^2 u_n + 2\Delta u_n + au_n + b|\nabla u_n|^2 + u_n^3, \Delta^2 u_n) = (g, \Delta^2 u_n),$$

using integration by Parts, we obtain

$$\frac{d}{dt}\|\Delta u_n\|^2 + 2\|\Delta^2 u_n\|^2 = -4\int_\Omega \Delta u_n \Delta^2 u_n dx - 2a\|\Delta u_n\|^2 - 2b\int_\Omega |\nabla u_n|^2 \Delta^2 u_n dx - 2\int_\Omega u_n^3 \Delta^2 u_n dx + 2\int_\Omega g\Delta^2 u_n dx, \tag{4.2}$$

Using Young inequality and Gagliardo-Nirenberg inequality, we can get

$$\left|\int_\Omega \Delta u_n \Delta^2 u_n dx\right| \le \|\Delta u_n\|\|\Delta^2 u_n\| \le \frac{5}{2}\|\Delta u_n\|^2 + \frac{1}{10}\|\Delta^2 u_n\|^2, \tag{4.3}$$

$$\begin{aligned}\left|b\int_\Omega |\nabla u_n|^2 \Delta^2 u_n dx\right| &\le |b|\|\nabla u_n\|_4^2\|\Delta^2 u_n\| \le \frac{5b^2}{2}\|\nabla u_n\|_4^4 + \frac{1}{10}\|\Delta^2 u_n\|^2 \\ &\le c\|\Delta u_n\|^2 + c\|u_n\|^6 + \frac{1}{10}\|\Delta^2 u_n\|^2\end{aligned}, \tag{4.4}$$

$$\left|\int_\Omega u_n^3 \Delta^2 u_n dx\right| \le \|u_n\|_6^3\|\Delta^2 u_n\| \le \frac{5}{2}\|u_n\|_6^6 + \frac{1}{10}\|\Delta^2 u_n\|^2 \le c\|\Delta u_n\|^2 + c\|u_n\|^{10} + \frac{1}{10}\|\Delta^2 u_n\|^2, \tag{4.5}$$

$$\left|\int_\Omega g\Delta^2 u_n dx\right| \le \|g\|\|\Delta^2 u_n\| \le \frac{5}{2}\|g\|^2 + \frac{1}{10}\|\Delta^2 u_n\|^2, \tag{4.6}$$

From (4.2)-(4.6), we have

$$\frac{d}{dt}\|\Delta u_n\|^2 + \|\Delta^2 u_n\|^2 \le c\|\Delta u_n\|^2 + c\|u_n\|^6 + c\|u_n\|^{10} + 5\|g\|^2, \tag{4.7}$$

using the periodicity of u_n, (3.10) and (3.13), integrating (4.7) over $[0,T]$ we get

$$\int_0^T \|\Delta^2 u_n\|^2 dt \le C(K, K_0), \tag{4.8}$$

so there exists $t^* \in [0,T]$ such that

$$\|\Delta^2 u_n(t^*)\|^2 \le \frac{1}{T}C(K, K_0), \tag{4.9}$$

integrating (3.10) again over $[t^*, t+T]$ $(t \in [0,T])$, using poincare inequality we obtain

$$\begin{aligned}\|\Delta u_n(t)\|^2 &\le \|\Delta u_n(t^*)\|^2 + \left|\int_0^T \|\Delta^2 u_n\|^2 dt\right| + C(K, K_0) \\ &\le (\lambda^{-2}T^{-1} + 2)C(K, K_0)\end{aligned}. \tag{4.10}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2 *Let u_n be the solution of (3.1)-(3.2) given above. Set*

$$K_1 = \int_0^T \|g_t\|^2 dt$$

we have

$$\|u_{nt}(t)\|^2 \le (\lambda^{-2}T^{-1} + 2)C(K, K_0, K_1),$$

$$\|\Delta u_{nt}\|^2 \le (\lambda^{-2}T^{-1} + 2)C(K, K_0, K_1),$$

$$\int_0^T \|\Delta^2 u_{nt}\|^2 dt \le C(K, K_0, K_1),$$

$$\int_0^T \|u_{ntt}\|^2 dt \le C(K, K_0, K_1),$$

where $C(K, K_0, K_1)$ denote constants depending on K, K_0, K_1 and independent of n.

Proof From (3.1) again, we see

$$(u_{nt} + \Delta^2 u_n + 2\Delta u_n + au_n + b|\nabla u_n|^2 + u_n^3, u_{nt}) = (g, u_{nt}), \tag{4.11}$$

using integration by Parts, Young inequality and Gagliardo-Nirenberg inequality, we obtain

$$\|u_{nt}\|^2 + \frac{d}{dt}\|\Delta u_n\|^2 \le c\|\Delta u_n\|^2 + c\|u_n\|^6 + c\|u_n\|^{10} + 5\|g\|^2, \tag{4.12}$$

from (3.13) and (4.10), we know

$$\|u_{nt}\|^2 + \frac{d}{dt}\|\Delta u_n\|^2 \le C(K, K_0), \tag{4.13}$$

using the periodicity of u_n, integrating (4.13) over $[0,T]$ we get

$$\int_0^T \|u_{nt}\|^2 \le C(K, K_0)T, \tag{4.14}$$

Multiplying (3.1) by $a_{kn}(t)$ and summing up over k, we see

$$(u_{nt} + \Delta^2 u_n + 2\Delta u_n + au_n + b|\nabla u_n|^2 + u_n^3, u_n) = (g, u_n), \tag{4.15}$$

Taking the derivative with respect to t of (4.15), using integration by Parts, we have

$$\frac{d}{dt}\|u_{nt}\|^2 + 2\|\Delta u_{nt}\|^2 = 4\|\nabla u_{nt}\|^2 - 2a\|u_{nt}\|^2 - 4b\int_\Omega |\nabla u_n||\nabla u_{nt}|u_{nt}dx - 6\int_\Omega u_n^2 u_{nt}\cdot u_{nt} + 2\int_\Omega g_t u_{nt}, \tag{4.16}$$

Using Young inequality and Gagliardo-Nirenberg inequality, we obtain

$$4\|\nabla u_{nt}\|^2 \le c\|\Delta u_{nt}\|\|u_{nt}\| \le \varepsilon\|\Delta u_{nt}\|^2 + c\|u_{nt}\|^2, \tag{4.17}$$

$$\begin{aligned} 4|b|\int_\Omega |\nabla u_n||\nabla u_{nt}|u_{nt}dx &\le c\|\nabla u_n\|_4\|\nabla u_{nt}\|_4\|u_{nt}\| \\ &\le \varepsilon\|\nabla u_{nt}\|_4^2 + c\|\nabla u_n\|_4^2\|u_{nt}\|^2 \\ &\le \varepsilon\|\Delta u_{nt}\|^2 + c\|\Delta u_n\|^2\|u_{nt}\|^2 \end{aligned}, \tag{4.18}$$

$$\begin{aligned} 6\left|\int_\Omega u_n^2 u_{nt}\cdot u_{nt}\right| &\le c\|u_n\|_4^2\|u_{nt}\|_4^2 \le c(\|\Delta u_n\|^2 + \|u_n\|^2)(\|\Delta u_{nt}\|^2 + \|u_{nt}\|^2) \\ &\le C(K, K_0)(\|\Delta u_{nt}\|^2 + \|u_{nt}\|^2) \end{aligned}, \tag{4.19}$$

from (4.16)-(4.19), let ε enough small, seeing that $C(K, K_0) < 1$, we get

$$\frac{d}{dt}\|u_{nt}\|^2 + \|\Delta u_{nt}\|^2 \le c\|u_{nt}\|^2 + c\|\Delta u_n\|^2\|u_{nt}\|^2 + c\|g_t\|^2, \tag{4.20}$$

using the periodicity of u_n, (4.10) and (4.14), integrating (4.20) over $[0,T]$ we get

$$\int_0^T \|\Delta u_{nt}\|^2 dt \le C(K, K_0, K_1), \tag{4.21}$$

so there exists $t^* \in [0,T]$ such that

$$\|\Delta u_{nt}(t^*)\|^2 \le \frac{1}{T}C(K, K_0, K_1), \tag{4.22}$$

integrating (4.20) again over $[t^*, t+T]\ (t \in [0,T])$, using poincare inequality we obtain

$$\begin{aligned} \|u_{nt}(t)\|^2 &\le \|u_{nt}(t^*)\|^2 + \left|\int_0^T \|\Delta u_{nt}\|^2 dt\right| + C(K, K_0, K_1) \\ &\le (\lambda^{-2}T^{-1} + 2)C(K, K_0, K_1) \end{aligned}. \tag{4.23}$$

By differentiating Eq. (3.1) and making the scalar product with $\Delta^2 u_{nt}$, using integration by Parts, we have

$$\begin{aligned} &\frac{d}{dt}\|\Delta u_{nt}\|^2 + 2\|\Delta^2 u_{nt}\|^2 \\ &= -4\int_\Omega \Delta u_{nt}\Delta^2 u_{nt}dx - 2a\|\Delta u_{nt}\|^2 - 4b\int_\Omega |\nabla u_n||\nabla u_{nt}|\Delta^2 u_{nt}dx - 6\int_\Omega u_n^2 u_{nt}\Delta^2 u_{nt}dx + 2\int_\Omega g_t\Delta^2 u_{nt}dx \end{aligned}, \tag{4.24}$$

applying Young inequality and Gagliardo-Nirenberg inequality, we can get

$$\frac{d}{dt}\|\Delta u_{nt}\|^2 + \|\Delta^2 u_{nt}\|^2 \le c(1 + \|\Delta u_n\|^2 + \|\Delta u_n\|^2\|u_n\|^2)\|\Delta u_{nt}\|^2 + c\|g_t\|^2.$$

Applying Lemma 4.1 and (4.21), use the similar way, we obtain

$$\int_0^T \|\Delta^2 u_{nt}\|^2 dt \le C(K, K_0, K_1),$$

$$\|\Delta u_{nt}\|^2 \le (\lambda^{-2}T^{-1} + 2)C(K, K_0, K_1).$$

Moreover, we also can get the following equation from (3.1),

$$(u_{ntt} + \Delta^2 u_{nt} + 2\Delta u_{nt} + au_{nt} + 2b|\nabla u_n||\nabla u_{nt}| + 3u_n^2 u_{nt}, u_{ntt}) = (g_t, u_{ntt}),$$

using integration by Parts,Young inequality, Gagliardo-Nirenberg inequality and the periodicity of u_n, we can get

$$\int_0^T \|u_{ntt}\|^2 dt \le C(K, K_0, K_1).$$

This completes the proof of Lemma 4.2. □

T-Periodic solutions

Theorem 5.1 *Let* $g \in L^{\infty}(T, H^{2}(\Omega))(T>0)$ *.then there exists a constant* $C_0 = C_0(N) > 0$ *, if*

$$K \equiv \sup_{0\le t\le T} \|g\|_{L^N(\Omega)} \le C_0$$

the problem (1.1)-(1.3) has a T-periodic solution u *, it satisfies*

$$u \in H^{2}(T; H_{\sigma}) \cap H^{1}(T; D(A^{2})) \cap L^{\infty}(T; D(A)) .$$

Proof In section 4, we get the u_n and u_{nt} estimate in $H^{2}(\Omega)$, use of compactness theorem, we know there exists a subsequence u_n lending to u in such a way

$$u_n \to u \ \text{ weakly}^{*} \text{ in } L^{\infty}(0,T; D(A)) ,$$

$$u_n \to u \ \text{ strongly in } L^{\infty}(0,T; D(A^{\frac{1}{2}})) ,$$

$$u_{nt} \to u_t \ \text{ weakly}^{*} \text{ in } L^{\infty}(0,T; D(A)) ,$$

$$u_{nt} \to u_t \ \text{ strongly in } L^{\infty}(0,T; D(A^{\frac{1}{2}})) .$$

By the above estimate we know that the nonliner terms are well defined. If $n \to \infty$, uniformly in t, we have

$$(b|\nabla u_n|^2 - b|\nabla u|^2) + (u_n^3 - u^3) \to b(\nabla u_n - \nabla u)(\nabla u_n + \nabla u) + (u_n - u)(u_n^2 + u_n u + u^2) \to 0 .$$

Consequently ,we see that

$$(u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3, w_k) = (g, w_k) ,$$

so we get

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g .$$

Thus, the proof of Theorem 5.1 is complete. □

Theorem 5.2 *The solution of (1.1)-(1.3) given in Theorem 5.1 is unique.*

Proof Let u_1 and u_2 be two T-periodic solutions of problem (1.1)-(1.3), define $u = u_1 - u_2$. Then it follows

$$\frac{du}{dt} + \Delta^2 u + 2\Delta u + au + b\nabla u(\nabla u_1 + \nabla u_2) + u(u_1^2 + u_1 u_2 + u_2^2) = 0 , \qquad (5.1)$$

Taking the inner product of (5.1) with *u*, using integration by parts, Young inequality, Gagliardo-Nirenberg inequality and Lemma 4.1, we have

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \le C(K, K_0)\|\Delta u\|^2 .$$

Since $C(K, K_0) < 1$, poincare inequality can give that

$$\frac{d}{dt}\|u\|^2 \le (C(K, K_0) - 1)\lambda^2\|u\|^2 = -L\|u\|^2 ,$$

where $L \equiv (1 - C(K, K_0))\lambda^2 > 0$, so it follows

$$\|u\|^2 \le \|u\|^2(0)\exp(-Lt) , \text{ for any } t \in (0, +\infty) .$$

Since u is T-periodic in t, for any positive integer N, for any $t \ge 0$,we have

$$\|u\|^2(t) = \|u\|^2(t + NT) .$$

Hence,it follows

$$\|u\|^2 \le \|u\|^2(0)\exp(-LNt) ,$$

which implies $\|u\|^2 = 0$. The proof of theorem 5.2 is complete. □

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