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Mathematical Modeling of Ghana's Greater Accra Region Population Growth

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Abstract: Population size and growth rate have great impact on the society and economy of every country. There is the need for leaders to be abreast with quality information on population so they can adequately provide the needed resources to meet the ever growing human population. The purpose of this study is to use mathematical models to predict the population growth of the Greater Accra Region (GAR) of Ghana. Two mathematical models: the Exponential and Logistic Growth Models were applied to model the population growth using GAR population data from 1960 to 2015. The Exponential Model predicted the population of GAR to be 18.1617 million in 2050, with a constant growth rate of 4.01% per year. On the other hand, the Logistic Model predicted GAR's population to be 12.2031 million in 2050, with a growth rate of 4.73 per year and a carrying capacity, which describes the maximum population that the environment can indefinitely sustain, to be 18.55 million. Based on our results, the Logistic Model fitted the GAR population better than the Exponential Model. The Logistic Model was fitted using the curve-fitting tool (cftool) in MATLAB. We hope the results will be very useful to the GAR authorities.
Keywords: Exponential Growth Model, Logistic Growth Model, Population growth rate, Carrying capacity. MAPE.

INTRODUCTION

Generally, the human population hardly declines in size in any part of the world. The size, growth and characteristics of a country's population greatly influence the sound policies and programs on education and literacy, employment and manpower, family planning, housing, maternal and child health, rural development, transportation and highway planning, urbanization and welfare, aimed at fostering the welfare of a country [1]. As Ghana's population trends have shown ever increasing growth rates, while government policy has been to keep growth rates in check, the need to understand and model population size and growth becomes more and more pressing. Due to ever increasing population growth naturally, it has become absolutely necessary to introduce the most common quantitative approach to population dynamics, taking note of the different theoretical foundations and assumptions to such population. Therefore, in this paper, we model the population growth of the Greater Accra Region of Ghana using the Exponential and Logistic Growth Models.

Ghana has 10 administrative regions with Accra being its capital city. Accra is located in the Greater Accra Region (GAR). The Greater Accra Region has the smallest area of Ghana's 10 administrative regions, occupying a total land surface of 3,245 square kilometers, which is less than all the other administrative regions [2]. This figure is just 1.4% of the total land area of Ghana. It is the second most populated region, after the Ashanti Region. Owing to in-migration and a high population growth rate, the region has the highest population density in the country. Therefore, it is of great importance to model the region's population to give the authorities much information to help them lay down good policies about the region. The Greater Accra region is the most urbanized region in Ghana.

There are numerous concerns about the consequences of human population on a country's development. It is of necessity to find an influential mathematical model to predict the present and future population growth. A mathematical model is defined as "a collection of equations based on quantitative description of a real world phenomenon; it is created in the hope that the predicted behavior will resemble the actual one" [3]. The widely used growth models are the Exponential Growth Model, proposed by Malthus in 1798 [5], and the Logistic Growth Model proposed by Verhulst in 1838 [7]. Both models originated from observations of biological reproduction process. However, they are now popularly used to model human population growth. According to the Exponential Model, the population grows exponentially without any upper bound or limit. Meanwhile, based on several factors that affect the population such as limited space, food supply and climatic conditions, it is unrealistic in practice to see a human population that can infinitely grow exponentially. The population will reach a point where the environment can no longer sustain its growth. In that case, the population would have reached its maximum. With this in mind, [7] incorporated the idea of "carrying capacity", which is described as the maximum population that an environment can sustain. He used this idea to modify the model by [5], which gave birth to the Logistic Growth Model. Several authors have applied these models to model population growth of places and countries. For example, Ghana's population growth has been modeled by [6]. In their work, they predicted Ghana's population to be 114.8207 million in 2050, with a growth rate of 3.15% per year using the Exponential Model. By the Logistic Growth Model, [6] predicted Ghana's population to be 341.2443 million in 2050, with a growth rate of 5.23% per year.

The rest of this paper is organized as follows: Section 2, presents the materials and methods used in this paper. In Section 3, the mathematical techniques for the prediction of population growth are elaborated and the explicit solution for each model is obtained. An implementation of the two models on the GAR population data, together with their results are presented in Section 4. In Section 5 we analyze and discuss the obtained results. Section 6 contains the conclusion and future research directions.

METHODS AND MATERIALS

In this paper, two mathematical models were employed. The Exponential and Logistic Growth. Models were used to analyze, predict and interpret a secondary data obtained from the Ghana National Population Council. The data consist of the population size of the GAR from year 1960 to 2015 and has 56 observations. The two models were applied to determine the population growth rate and were used to predict future population values. The two models were assessed based on the Mean Absolute Percentage Error (MAPE). The MATLAB software was used for the analysis.

Mathematical Techniques for predicting Population Growth

This section presents the two mathematical growth models used in modeling the GAR population growth. We looked at their mathematical equations and how their respective solutions are obtained.

The Exponential Growth Model

The first growth model we considered was the Exponential Growth Model, also known as the Malthus Model or the Natural Growth Model. This simple model of population growth was proposed by the British social scientist Reverend Thomas Robert Malthus 1798 [5]. The model is based on the assumption that the population grows at a rate proportional to the size of the population. This assumption seems only realistic to a population with ideal conditions (eg. unlimited environment and space, adequate nutrition, absence of predators, immunity from disease and absence of natural disasters). In simple terms, this simple model is appropriate in the initial stages of growth when there are no restrictions or constraints on the population. But in more realistic situations, there are limits to growth such as finite space or food supply. A more realistic model says that the population competes with itself.

Let the population at a given time $t = t_0$ be P_0 and the population at a future time $t = t_1$ be P. We were interested in finding a population function $P(t)$ for $t_0 \leq t \leq t_1$ with the initial condition $P(t_0) = P_0$. The Exponential Growth Model has the form:

$$\frac{dP}{dt} = rP(t), \quad t_0 \leq t \leq t_1, \quad P(t_0) = P_0, \tag{1}$$

where r is a constant called the *Malthus factor*, which represents the growth rate. If we assume that $P \neq 0$, then $P(t) > 0$ for all t. This shows that, for $r > 0$, then $\frac{dP}{dt} > 0$. This implies that the population is always growing and as $P(t)$ increases, $\frac{dP}{dt}$ increases. Equation (1) is a first order differential equation and we obtained its solution as follows:

Using separation of variables and integrating both sides, we have

$$\frac{dP}{P} = r\,dt$$
$$\int \frac{dP}{P} = r\int dt$$
$$ln\,P = rt + c$$
$$\Rightarrow P(t) = P_0\,e^{rt} \quad \text{or} \tag{2}$$
$$P(t) = P_0\,e^{r(t-t_0)}. \tag{3}$$

Equation (2) is the solution to Equation (1), which is the Exponential Growth Model. The t in Equation (2) represents the time difference whereas the one in Equation (3) represents the actual time.

The Logistic Growth Model

A slightly more realistic and largely used population growth model is the Logistic Growth Model proposed by Belgian scientist, Pierre Francois Verhulst in 1838. According to [7], population growth depends not only on population size, but also on the effect of a "carrying capacity" that would limit growth. In an environment that will support a limited population, it is assumed that the rate of growth of population decreases as the limiting population is approached. An appropriate model is the Logistic Model which has the form:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0, \tag{4}$$

where $K > 0$ is the carrying capacity; the size of population that the environment can long term sustain and r is the constant growth rate. Equation (4) is also a differential equation and has exact solution of the form:

$$P(t) = \frac{K}{1+A\,e^{-rt}}, \quad \text{where} A = \frac{K}{P_0} - 1. \tag{5}$$

[7] showed that the population growth not only depends on the population size but also on how far this size is from its upper limit (K). He studied in details the Exponential Model proposed by [5], and did indicate that it was by far too simplistic as it only included linear terms. [7] therefore modified the Exponential Model using the term:

$$\frac{\alpha - \beta P(t)}{\alpha} = 1 - \frac{\beta P(t)}{\alpha}, \tag{6}$$

where α and β are the vital components of the population. Equation (6) indicates how far the population is from its maximum value or limit. As the population value approaches $\frac{\alpha}{\beta}$, this term becomes very small and approximates to zero, providing the correct feedback to the population growth limit. Thus, the second term models the competition for available resources, which tends to limit the population growth [6]. Hence, the modified equation using the new term is:

$$\frac{dP}{dt} = \alpha P(t)\left[\frac{\alpha - \beta P(t)}{\alpha}\right], \quad t_0 < t < t_1, P(0) = P_0. \tag{7}$$

Equation (7) is known as the Logistic Law of population growth or Verhulst Logistic equation of population growth. We obtained the solution to Equation (7) as follows:

$$\frac{dP}{dt} = \alpha P - \beta P^2$$

Separating the variables, we have

$$\frac{dP}{P(\alpha - \beta P)} = dt$$
$$\frac{1}{\alpha}\left(\frac{1}{P} + \frac{\beta}{\alpha - \beta P}\right) dP = dt$$

Integrating both sides, we have

$$\int \frac{1}{\alpha}\left(\frac{1}{P} + \frac{\beta}{\alpha - \beta P}\right) dP = \int dt$$
$$\frac{1}{\alpha}[\ln(P) - \ln(\alpha - \beta P)] = t + c \tag{8}$$

Applying the initial condition $P(t) = P_0$ at $t = 0$, we find the constant c as

$$\frac{1}{\alpha}[\ln(P_0) - \ln(\alpha - \beta P_0)] = c$$

Substituting the value of c into Equation (8) gives

$$\frac{1}{\alpha}[\ln P - \ln(\alpha - \beta P)] = t + \frac{1}{\alpha}[\ln P_0 - \ln(\alpha - \beta P_0)]$$

Solving for P yields

$$P(t) = \frac{\frac{\alpha}{\beta}}{1+\left(\frac{\frac{\alpha}{\beta}}{P_0}-1\right)e^{-\alpha t}} \tag{9}$$

Now taking limit as $t \to \infty$ of Equation (9) gives

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left[\frac{\frac{\alpha}{\beta}}{1+\left(\frac{\frac{\alpha}{\beta}}{P_0}-1\right)e^{-\alpha t}}\right] = \frac{\alpha}{\beta} \tag{10}$$

This means that $\frac{\alpha}{\beta}$ is the maximum value that $P(t)$ can take. Thus, it is the carrying capacity. Now, [Verhulst, 1838] went ahead to find an expression for $\frac{\alpha}{\beta}$. Using Equation (9), putting $t = 1$ and $t = 2$, the values of $P(t)$ are P_1 and P_2 respectively, we obtain

$$\frac{\beta}{\alpha}(1 - e^{-\alpha}) = \frac{1}{P_1} - \frac{e^{-\alpha}}{P_0} \tag{11}$$

$$\frac{\beta}{\alpha}(1 - e^{-2\alpha}) = \frac{1}{P_2} - \frac{e^{-2\alpha}}{P_0} \tag{12}$$

Dividing Equation (12) by Equation (11), we get

$$1 + e^{-\alpha} = \frac{\left(\frac{1}{P_2}-\frac{e^{-2\alpha}}{P_0}\right)}{\left(\frac{1}{P_1}-\frac{e^{-\alpha}}{P_0}\right)}$$

$$e^{-\alpha} = \frac{\left(\frac{1}{P_2}-\frac{e^{-2\alpha}}{P_0}\right)}{\left(\frac{1}{P_1}-\frac{e^{-\alpha}}{P_0}\right)} - 1$$

Simplifying, we get

$$e^{-\alpha} = \frac{P_0(P_2-P_1)}{P_2(P_1-P_0)} \tag{13}$$

Substituting Equation (13) into Equation (11) and simplifying, we get

$$\frac{\beta}{\alpha} = \frac{P_1(P_0P_1-2P_0P_2+P_1P_2)}{{P_1}^2-P_0P_2} \tag{14}$$

Equation (14) is an expression for the carrying capacity. When we let $\frac{\beta}{\alpha} = K$ and $\alpha = r$, then Equation (7) looks exactly as Equation (4).

Figure 1 shows the difference between the Exponential and Logistic Growth Models. The Exponential Model is not bounded whereas the Logistic Model is bounded by the carrying capacity (K).

Mean Absolute Percentage Error (MAPE)

We assessed the performance of the two models using the MAPE. This is a commonly used statistic to assess the performance of forecasting models. It is usualy used as a tool to assess the Goodness of Fit of forecasting models. It is expressed in percentages and lower values indicate better models. [4] reports that MAPE values less than 10%, from 10% to 20%, from 21% to 50%, and above 50% represent a highly accurate, good, reasonable and inaccurate forecasting respectively. MAPE is given mathematically as

$$\text{MAPE} = \frac{1}{N}\sum_{i=1}^{N}\frac{|Y_i-\hat{Y}_i|}{Y_i} \times 100, \tag{15}$$

where Y and $\hat{Y}$ are the actual and predicted values respectively, and N is the number of observations.

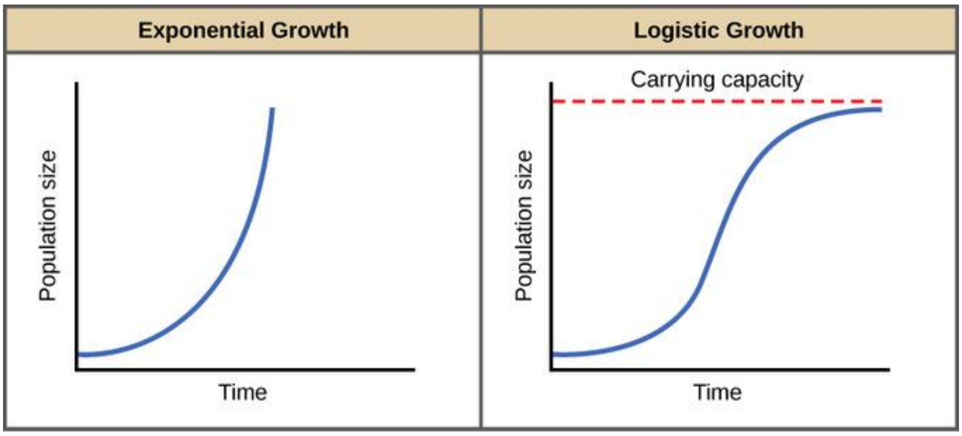

Fig-1: The difference between the Exponential and Logistic Growth Models

IMPLEMENTATION AND RESULTS

In this section, we present how the studied models were applied to the GAR population data and their respective results. We used the models to determine the population growth and predicted the population figures up to year 2050.

We first applied the Exponential Growth Model. Using the actual population (in millions) in Table 0 below, $t = 0$ correspond to year 1960, whose population is 0.4918 million. We need to find the growth rate r employing Equation (2). We thus have

$$P(t) = 0.4918 \times e^{rt}$$

Still using the actual population, for $t = 1$, $P_1 = 0.5119$. We use this information to determine the value of r.

$$0.5119 = 0.4918 \times e^{r}$$
$$\Rightarrow e^{r} = \frac{0.5119}{0.4918} = 1.04087$$
$$\Rightarrow r = 0.0401.$$

Thus, the general solution of the Exponential Growth Model using the GAR data is

$$P(t) = 0.4918 \times e^{\,0.0401\,(t-1960)}.$$

This resulting equation was used to predict the population up to year 2050. Figure 2 shows how this general solution fitted the data. We calculated a MAPE value for this Exponential Model to be 13.06%.

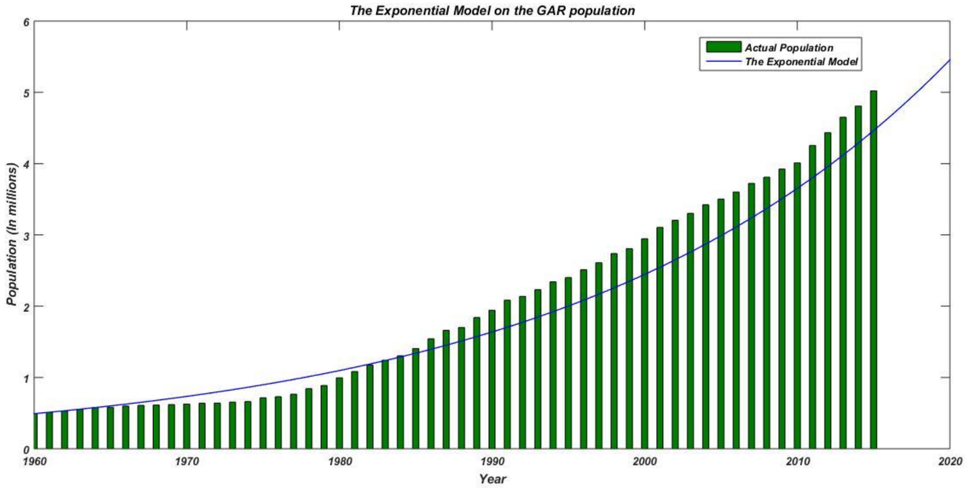

Fig-2: The Exponential Model on the GAR population

Next, we applied the Logistic Growth Model to the GAR data. One important feature about the Logistic Model is the carrying capacity K. Apart from the expression that [Verhulst, 1838] gave in Equation (14) , there are no vigorous mathematical expressions in literature to find K. The carrying capacity actually depends on lots of factors to find its real value in practice. Using the GAR actual population, $t = 2$ correspond to $P_2 = 0.5276$. Plugging in the values of P_1, P_2 and P_3 into Equation (14), we find K to be

$$K = \frac{0.5119(0.4918\times0.5119-2\times0.4918\times0.5276+0.5119\times0.5276)}{0.5119^2-0.4918\times0.5276} = 0.5748$$

Well, this value actually means that the maximum population that GAR can take is 0.5748 million. When we look at the actual population in Table 0, we observe that from year 1964 through to 2015, the population size is greater than this carrying capacity of 0.5748 million. Hence, we cannot use this value to make predictions. We henceforth shifted our attention to one of the mathematical software that can reasonably estimate the carrying capacity for this data.

The CFTOOL of the MATLAB software was used to fit the Logistic Growth Model to the GAR data. The resulting model is

$$P(t) = \frac{K}{1+\left(\frac{K}{0.4918}-1\right)e^{-r(t-1960)}},$$

where the population growth rate r was estimated to be 0.0473 with 95% confidence interval within $(0.0452{,}0.0494)$, while the carrying capacity K was found to be 18.55 with 95% confidence interval within $(10.6{,}26.49)$. This model has an R^2 value of 99.01%, which shows that the model best fits the data. Plugging in the values of r and K yields

$$P(t) = \frac{18.55}{1+(36.7186)e^{-0.0473(t-1960)}}.$$

We used this equation to predict the population of GAR up to year 1950. Table 0 shows the prediction results of the two models. Figure 3 shows how well the Logistic Model fitted the data.

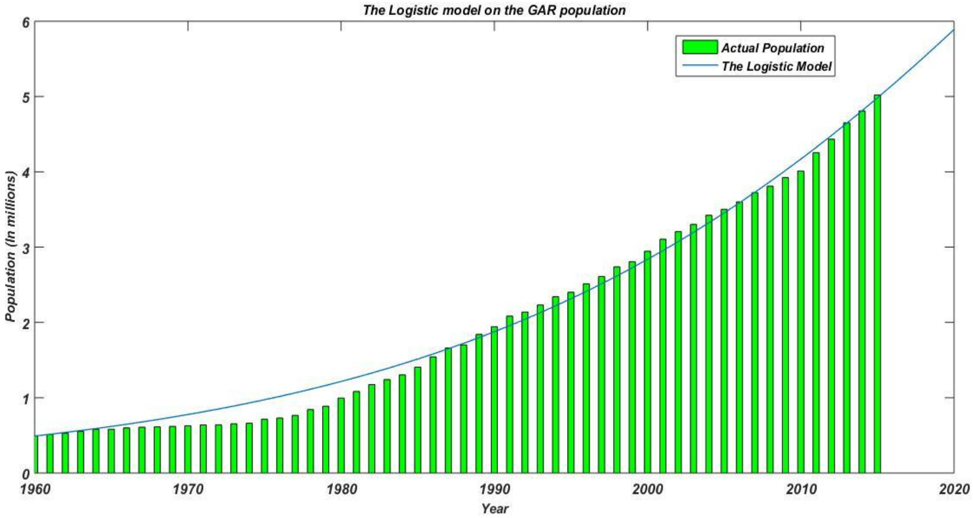

Fig-3: The Logistic Model on the GAR population

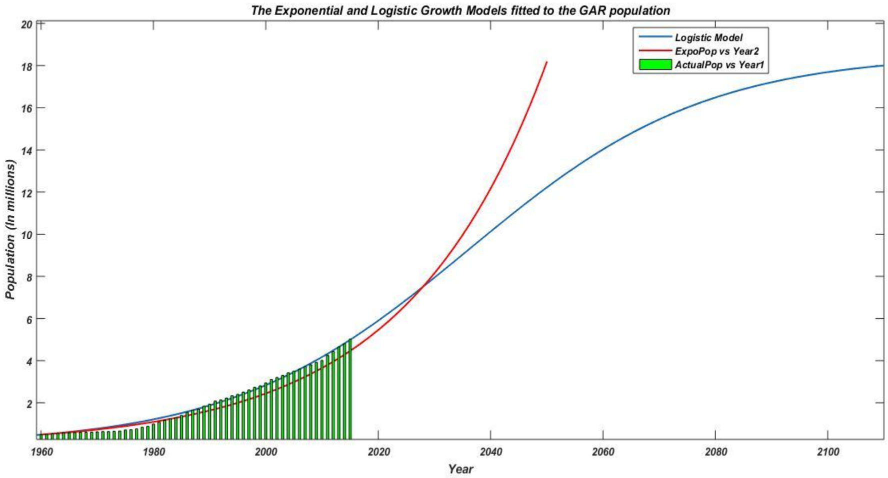

Fig-4: The Exponential and Logistic Models on the GAR population

Figure 4 shows how both models forecast the future GAR population size.

Table-1: Projection of GAR's Population using Exponential and Logistic Growth Models

Year	Actual Population (In millions)	Projected Population (In millions)	
		Exponential Model	Logistic Model
1960	0.4918	0.4918	0.4918
1961	0.5119	0.5119	0.5150
1962	0.5276	0.5329	0.5392
1963	0.5523	0.5547	0.5645
1964	0.5789	0.5774	0.5910
1965	0.5808	0.6010	0.6186
1966	0.5988	0.6256	0.6476
1967	0.6089	0.6512	0.6778
1968	0.6134	0.6778	0.7094
1969	0.6199	0.7055	0.7423
1970	0.6267	0.7344	0.7768
1971	0.6389	0.7645	0.8128
1972	0.6401	0.7957	0.8503
1973	0.6543	0.8283	0.8895

1974	0.6623	0.8622	0.9305
1975	0.7156	0.8975	0.9732
1976	0.7298	0.9342	1.0177
1977	0.7645	0.9724	1.0642
1978	0.8434	1.0122	1.1126
1979	0.8867	1.0536	1.1631
1980	0.9943	1.0967	1.2158
1981	1.0834	1.1416	1.2707
1982	1.1745	1.1883	1.3278
1983	1.2408	1.2369	1.3873
1984	1.3034	1.2875	1.4492
1985	1.4056	1.3402	1.5137
1986	1.5423	1.3950	1.5808
1987	1.6608	1.4521	1.6505
1988	1.7004	1.5115	1.7231
1989	1.8409	1.5734	1.7984
1990	1.9432	1.6377	1.8767
1991	2.0843	1.7047	1.9580
1992	2.1367	1.7745	2.0424
1993	2.2312	1.8471	2.1300
1994	2.3409	1.9227	2.2208
1995	2.4003	2.0013	2.3150
1996	2.5111	2.0832	2.4125
1997	2.6098	2.1685	2.5135
1998	2.7389	2.2572	2.6181
1999	2.8065	2.3495	2.7263
2000	2.9457	2.4457	2.8381
2001	3.1045	2.5457	2.9537
2002	3.2052	2.6499	3.0730
2003	3.3005	2.7583	3.1962
2004	3.4223	2.8712	3.3233
2005	3.5022	2.9886	3.4543
2006	3.6009	3.1109	3.5893
2007	3.7234	3.2382	3.7282
2008	3.8098	3.3707	3.8711
2009	3.9234	3.5086	4.0180
2010	4.0101	3.6522	4.1688
2011	4.2555	3.8016	4.3237
2012	4.4333	3.9571	4.4825
2013	4.6502	4.1190	4.6453
2014	4.8072	4.2876	4.8119
2015	5.0202	4.4630	4.9824
2016		4.6456	5.1566
2017		4.8357	5.3345
2018		5.0335	5.5161
2019		5.2395	5.7012
2020		5.4538	5.8896
2021		5.6770	6.0814
2022		5.9092	6.2763
2023		6.1510	6.4742
2024		6.4027	6.6749
2025		6.6646	6.8784
2026		6.9373	7.0843
2027		7.2212	7.2925
2028		7.5166	7.5029
2029		7.8242	7.7151
2030		8.1443	7.9291

2031		8.4775	8.1445
2032		8.8244	8.3612
2033		9.1854	8.5789
2034		9.5612	8.7974
2035		9.9524	9.0164
2036		10.3596	9.2357
2037		10.7835	9.4550
2038		11.2247	9.6741
2039		11.6840	9.8928
2040		12.1620	10.1108
2041		12.6596	10.3279
2042		13.1776	10.5438
2043		13.7167	10.7583
2044		14.2780	10.9712
2045		14.8621	11.1823
2046		15.4702	11.3913
2047		16.1032	11.5981
2048		16.7620	11.8024
2049		17.4479	12.0041
2050		18.1617	12.2031
MAPE		13.06%	10.19%

DISCUSSION

The predicted population size by both models increases from 1960 up to 1950 as can be observed in Table 1. The Exponential Growth Model predicted a constant population growth rate of $\mathbf{4.01}\%$ per year to the data. This shows that the population is fast growing and raises an alarm to the government of Ghana and the authorities of the GAR to put in measures to face the challenges that would come up as a result of the ever population increase. The Exponential Model predicts that the population grows without any bound, meanwhile this cannot possibly happen indefinitely. The population would at some point in time be constrained by limited resources. The Logistic Growth Model predicted the population growth at a constant rate of $\mathbf{4.73}\%$ per year (higher than that of the Exponential Model) and the maximum population ($\boldsymbol{K}$) that the environment can sustain indefinitely is $\mathbf{18.55}$ million. The Logistic Model relies heavily on the value of $\boldsymbol{K}$. It is very difficult to come up with the real value of $\boldsymbol{K}$ in a given environment or habitat. In reality, $\boldsymbol{K}$ is not fixed over time but rather, it changes always depending on many conditions. MAPE values of $\mathbf{13.06}\%$ and $\mathbf{10.19}\%$ were obtained by the Exponential and Logistic Models respectively. These MAPE values indicate that both models have good forecasts about the GAR data. However, since that of the Logistic Model is less, it shows that the Logistic Model does better in regard to this GAR data.

CONCLUSION

Population size and growth rates in any country directly have an impact on its policy making, culture, economy, education and the environment. We applied two mathematical growth models to the GAR data to model the population and its growth rate. A constant growth rate of 4.73% per year and a carrying capacity of 18.55 million were predicted by the Logistic Model. Based on the Exponential Model, a constant growth rate of 4.01% per year was predicted for the data. Further work should be directed to exploring more mathematical models to model human population and developing a better vigorous method of determining the carrying capacity. We hope the findings of this study can be useful to the government and authorities of the Greater Accra of Ghana to create economic policies accordingly. When the authorities are supplied with the right information about the population, concrete measures can be put in place to provide the needed resources to meet the challenges that come with population increase.

REFERENCES

1. United Nations Population Fund. The State of World Population 1998: The New Generations. United Nations Publications; 1998.
2. GSS. 2010 Population & Housing Census: District Analytical Report: Agotime Ziope District. 2010. Ghana statistical service.
3. Ledder G. Differential equations: a modeling approach. McGraw-Hill Higher Education. 2005.
4. Lewis CD. Industrial and business forecasting methods: A practical guide to exponential smoothing and curve fitting. Butterworth-Heinemann; 1982.
5. Malthus TR. An essay on the principle of population: or, A view of its past and present effects on human happiness. Reeves & Turner; 1888.

6. Ofori T, Ephraim L, Nyarko F. Mathematical model of ghana's population growth. International Journal of Modern Management Sciences, 2013. 2(2):57–66.
7. Verhulst PF. Notice sur la loi que la population suit dans son accroissement. Corresp. Math. Phys.. 1838;10:113-26.