Solution and Green’s Function of the Sturm-Liouville Problem with Fuzzy Forcing Function
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Abstract: In this paper, Green’s function of Sturm-Liouville problem with the fuzzy forcing function is examined and the solution of the problem using the Green’s function is investigated. Solution is shown on an example.

Keywords: Fuzzy boundary value problems, Hukuhara differentiability, Green’s function.

INTRODUCTION
Fuzzy set theory is powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models [3]. Fuzzy differential equation is a very important topic from theoretical point of view [7, 8, 12, 14] and its applications, for example, in population models [5], civil engineering [13] and medicine [1, 2].

There are different views on the concept of differentiable fuzzy-valued function. Historically, the first approach is to consider differentiability in the sense of Hukuhara.

In this paper, a investigation is made on the solution and Green’s function of Sturm-Liouville problem with the fuzzy forcing function by using Hukuha differentiability.

Preliminaries
Definition 1. [11] A fuzzy number is a function \( u: \mathbb{R} \rightarrow [0,1] \) satisfying the properties, \( u \) is normal, \( u \) is convex fuzzy set, \( u \) is upper semi-continuous on \( \mathbb{R} \), \( \text{cl}\{x \in \mathbb{R} | u(x) > 0\} \) is compact where \( \text{cl} \) denotes the closure of a subset.

Let \( \mathbb{R}_p \) denote the space of fuzzy numbers.

Definition 2. [10] Let \( u \in \mathbb{R}_p \). The \( \alpha \)-level set of \( u \), denoted \( [u]^{\alpha} \), \( 0 < \alpha \leq 1 \), is \( [u]^{\alpha} = \{ x \in \mathbb{R} | u(x) \geq \alpha \} \). If \( \alpha = 0 \), the support of \( u \) is defined \( [u]^0 = \text{cl}\{x \in \mathbb{R} | u(x) > 0\} \). The notation, \( [u]^{\alpha} = [\underline{u}_\alpha, \overline{u}_\alpha] \) denotes explicitly the \( \alpha \)-level set of \( u \).

Remark 1. [9] The sufficient and necessary conditions for \( [u]^{\alpha}, \overline{u}_\alpha \) to define the parametric form of a fuzzy number as follows:

i) \( \underline{u}_\alpha \) is bounded monotonic increasing (nondecreasing) left-continuous function on \( (0,1] \) and right-continuous for \( \alpha = 0 \),

ii) \( \overline{u}_\alpha \) is bounded monotonic decreasing (nonincreasing) left-continuous function on \( (0,1] \) and right-continuous for \( \alpha = 0 \),

iii) \( \underline{u}_\alpha \leq \overline{u}_\alpha, 0 \leq \alpha \leq 1 \).

Definition 3. [11] If \( A \) is a symmetric triangular number with support \( [a, \overline{a}] \), the \( \alpha \)-level set of \( A \) is

\[ [A]^{\alpha} = \left[ a + \frac{\overline{a} - a}{2} \alpha, \overline{a} - \frac{\overline{a} - a}{2} \alpha \right]. \]

Definition 4. [10] For \( u, v \in \mathbb{R}_p \) and \( \lambda \in \mathbb{R} \), the sum \( u + v \) and the product \( \lambda u \) are defined by \( [u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, \ [\lambda u]^{\alpha} = \lambda[u]^{\alpha}, \forall \alpha \in [0,1] \), where \( [u]^{\alpha} + [v]^{\alpha} \) means the usual addition of two intervals (subsets) of \( \mathbb{R} \) and \( \lambda[u]^{\alpha} \) means the usual product between a scalar and a subset of \( \mathbb{R} \).

The metric structure is given by the Hausdorff distance
by
\[ D(u, v) = \sup_{u \in [0, 1]} \max \left\{ \|u_u - v_v\|, \|u_v - v_u\| \right\} \] \[ [9]. \]

Definition 5. \[ [11] \] Let \( u, v \in \mathbb{R}_F \). If there exist \( w \in \mathbb{R}_F \) such that \( u = v + w \), then \( w \) is called the H-difference of \( u \) and \( v \) and it is denoted \( u - v \).

Definition 6. \[ [9] \] Let \( I = (a, b) \), for \( a, b \in \mathbb{R} \), and \( F: I \rightarrow \mathbb{R}_F \) be a fuzzy function. We say \( F \) is Hukuhara differentiable at \( t \in I \) if there exists an element \( \frac{F(t + h) - F(t)}{h} \) and \( \frac{F(t) - F(t - h)}{h} \) exist and equal \( F'(t) \). Here the limits are taken in the metric space \((\mathbb{R}_F, D)\).

Theorem 1. \[ [4] \] Let \( f: I \rightarrow \mathbb{R}_F \) be a function and denote
\[ f(t) = [f_a(t), f_b(t)] \]
where \( \lambda > 0 \), \([f(t)]^a = [f_a(t), f_a(t)]\) triangular fuzzy function, \([\beta]^a = [\beta_a^+, \beta_a^-]\), \([\gamma]^a = [\gamma_a^+, \gamma_a^-]\) are symmetric triangular fuzzy numbers, \( \zeta_i \geq 0, i = 1, 2, 3, 4, \zeta_1^2 + \zeta_2^2 = 0, \zeta_3^2 + \zeta_4^2 = 0, t \) is a continuous function and positive on \((0, \ell)\).

Firstly, consider the boundary value problem
\[ (1) \]
u'' + q(t)u + \lambda u = \int f(t) t \in [0, \ell],
\[ (2) \]
\[ \zeta_1 u(0) + \zeta_2 u'(0) = [\beta]^a, \]
\[ (3) \]
\[ \zeta_3 u(\ell) + \zeta_4 u'(\ell) = [\gamma]^a, \]
Let \([\phi(t, \lambda)]^a = [\phi_a(t, \lambda), \phi_b(t, \lambda)]\) be the solution satisfying the conditions
\[ (7) \]
u(0) = \zeta_2, \quad u'(0) = -\zeta_1,
and \([\chi(t, \lambda)]^a = [\chi_a(t, \lambda), \chi_b(t, \lambda)]\) be the solution satisfying the conditions
\[ (8) \]
\[ u(\ell) = \zeta_4, \quad u'(\ell) = -\zeta_3. \]
of fuzzy differential equations (4). The eigenvalues of the fuzzy boundary value problem (4)-(6) if and only if are consist of the zeros of Wronskian functions
\[ (9) \]
\[ W_a(\lambda) = W(\phi_a(t, \lambda), \chi_a(t, \lambda)) = \phi_a(t, \lambda)\chi_a(t, \lambda) - \phi_a(t, \lambda)\chi_a(t, \lambda), \]
\[ (10) \]
\[ W_a(\lambda) = W(\phi_b(t, \lambda), \chi_b(t, \lambda)) = \phi_b(t, \lambda)\chi_b(t, \lambda) - \phi_b(t, \lambda)\chi_b(t, \lambda), \]
Then, if \( \lambda \) is not the eigenvalue of the fuzzy boundary value problem (4)-(6), since \( W_a(\lambda) \neq 0 \) and \( W_a(\lambda) \neq 0 \), the lower and upper solutions \( \phi_a(t, \lambda), \chi_a(t, \lambda) \) and \( \phi_b(t, \lambda), \chi_b(t, \lambda) \) will be linear independent. According to this, the general solution of the fuzzy differential equation (4) is
\[ (11) \]
\[ [u(t, \lambda)]^a = [\phi_a(t, \lambda), \chi_a(t, \lambda)], \]
\[ (12) \]
\[ \phi_a(t, \lambda) = a_a(\lambda)\phi_a(t, \lambda) + b_a(\lambda)\chi_a(t, \lambda), \]
\[ (13) \]
\[ \chi_a(t, \lambda) = a_a(\lambda)\phi_a(t, \lambda) + b_a(\lambda)\chi_a(t, \lambda), \]
Using the method variation of parameters, we can search for the general solution of the fuzzy differential equation (1) as
\[ (14) \]
\[ [u(t, \lambda)]^a = [u_a(t, \lambda), u_b(t, \lambda)]. \]
Derivating the Hukuhara of (14)-(16) according to \( x \), choosing the functions \( a_a(t, \lambda), b_a(t, \lambda), \overline{a}_a(t, \lambda) \) and \( \overline{b}_a(t, \lambda) \) that

\[
\begin{align*}
\frac{d}{dt}a_a(t, \lambda)\varphi_a(t, \lambda) + \frac{d}{dt}b_a(t, \lambda)\chi_a(t, \lambda) &= 0, \quad (17) \\
\frac{d}{dt}\overline{a}_a(t, \lambda)\overline{\varphi}_a(t, \lambda) + \frac{d}{dt}\overline{b}_a(t, \lambda)\overline{\chi}_a(t, \lambda) &= 0, \quad (18)
\end{align*}
\]

later, derivating of \([u'(t, \lambda)]^a\) once again and substituting in the fuzzy differential equation (1), making the necessary operations, the equations

\[
\begin{align*}
\frac{d}{dt}a_a(t, \lambda)\varphi_a(t, \lambda) + \frac{d}{dt}b_a(t, \lambda)\chi_a(t, \lambda) &= f_a(t), \quad (19) \\
\frac{d}{dt}\overline{a}_a(t, \lambda)\overline{\varphi}_a(t, \lambda) + \frac{d}{dt}\overline{b}_a(t, \lambda)\overline{\chi}_a(t, \lambda) &= \overline{f}_a(t), \quad (20)
\end{align*}
\]

are obtained. Then, solving the equations (17), (19) and (18), (20) by looking at the system as a linear equation system according to the variables, yields

\[
\begin{align*}
a'_a(t, \lambda) &= -\frac{1}{W_a(\lambda)} \chi_a(t, \lambda)f_a(t), \\
b'_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \varphi_a(t, \lambda)f_a(t), \\
\overline{a}'_a(t, \lambda) &= -\frac{1}{W_a(\lambda)} \overline{\chi}_a(t, \lambda)\overline{f}_a(t), \\
\overline{b}'_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \overline{\varphi}_a(t, \lambda)\overline{f}_a(t).
\end{align*}
\]

From here, we have

\[
\begin{align*}
a_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \int_0^t \chi_a(x, \lambda)f_a(x)dx + a_a(\lambda), \\
b_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \int_0^t \varphi_a(x, \lambda)f_a(x)dx + b_a(\lambda), \\
\overline{a}_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \int_0^t \overline{\chi}_a(x, \lambda)\overline{f}_a(x)dx + \overline{a}_a(\lambda), \\
\overline{b}_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \int_0^t \overline{\varphi}_a(x, \lambda)\overline{f}_a(x)dx + \overline{b}_a(\lambda).
\end{align*}
\]

Substituting these equations in (15) and (16), the general solution of the fuzzy differential equation (1) is obtained as

\[
[u(t, \lambda)]^a = [a_a(t, \lambda), \overline{a}_a(t, \lambda)],
\]

\[
\begin{align*}
\varphi_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \left\{ \varphi_a(t, \lambda) \int_0^{\xi_1} \chi_a(x, \lambda)f_a(x)dx + \chi_a(t, \lambda) \int_{\xi_1}^t \varphi_a(x, \lambda)f_a(x)dx \right\} \\
&\quad + a_a(\lambda)\varphi_a(t, \lambda) + b_a(\lambda)\chi_a(t, \lambda). \\
\overline{\varphi}_a(t, \lambda) &= \frac{1}{W_a(\lambda)} \left\{ \overline{\varphi}_a(t, \lambda) \int_0^{\xi_1} \overline{\chi}_a(x, \lambda)\overline{f}_a(x)dx + \overline{\chi}_a(t, \lambda) \int_{\xi_1}^t \overline{\varphi}_a(x, \lambda)\overline{f}_a(x)dx \right\} \\
&\quad + \overline{a}_a(\lambda)\overline{\varphi}_a(t, \lambda) + \overline{b}_a(\lambda)\overline{\chi}_a(t, \lambda).
\end{align*}
\]

Using the boundary condition (2) and the equations

\[
\xi_1 \varphi_a(0, \lambda) + \xi_2 \varphi'_a(0, \lambda) = 0, \quad \xi_1 \overline{\varphi}_a(0, \lambda) + \xi_2 \overline{\varphi}'_a(0, \lambda) = 0,
\]

we have

\[
\begin{align*}
b_a(\lambda) \left\{ \xi_1 \chi_a(0, \lambda) + \xi_2 \chi'_a(0, \lambda) \right\} &= \beta_a, \\
\overline{b}_a(\lambda) \left\{ \xi_1 \overline{\chi}_a(0, \lambda) + \xi_2 \overline{\chi}'_a(0, \lambda) \right\} &= \overline{\beta}_a.
\end{align*}
\]
From (7), (9) and (10), we obtained
\[ \zeta_1 \chi_a(0, \lambda) + \zeta_2 \chi'_a(0, \lambda) = \overline{W}_a(\lambda), \quad \zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda) = \overline{\overline{W}}_a(\lambda). \]

Since \( \lambda \) is not eigenvalue, \( W_a(\lambda) \neq 0, \overline{W}_a(\lambda) \neq 0 \). Then, yields
\[ b_\alpha(\lambda) = \frac{\beta_\alpha}{\zeta_1 \chi_a(0, \lambda) + \zeta_2 \chi'_a(0, \lambda)}, \quad \overline{b}_\alpha(\lambda) = \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda)}. \]

Similarly,
\[ a_\alpha(\lambda) = \frac{\gamma_\alpha}{\zeta_3 \varphi_a(\ell, \lambda) + \zeta_4 \varphi'_a(\ell, \lambda)}, \quad \overline{a}_\alpha(\lambda) = \frac{\overline{\gamma}_\alpha}{\zeta_3 \overline{\varphi}_a(\ell, \lambda) + \zeta_4 \overline{\varphi}'_a(\ell, \lambda)}. \]

are obtained. Then, from (21)-(23) yields
\[ [u(t, \lambda)]^a = [u_a(\lambda), \overline{u}_a(\lambda)]. \]

\[ u_a(\lambda) = \frac{1}{W_a(\lambda)} \left\{ \varphi_a(\lambda, \lambda) \int_0^t \chi_a(x, \lambda) f_a(x) \, dx + \chi_a(\lambda, \lambda) \int_0^t \varphi_a(x, \lambda) f_a(x) \, dx \right\} + \left( \frac{\gamma_\alpha}{\zeta_3 \varphi_a(\ell, \lambda) + \zeta_4 \varphi'_a(\ell, \lambda)} \right) \varphi_a(\lambda, \lambda) + \left( \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda)} \right) \chi_a(\lambda, \lambda). \]

\[ \overline{u}_a(\lambda) = \frac{1}{\overline{W}_a(\lambda)} \left\{ \overline{\varphi}_a(\lambda, \lambda) \int_0^t \overline{\chi}_a(x, \lambda) \overline{f}_a(x) \, dx + \overline{\chi}_a(\lambda, \lambda) \int_0^t \overline{\varphi}_a(x, \lambda) \overline{f}_a(x) \, dx \right\} + \left( \frac{\overline{\gamma}_\alpha}{\zeta_3 \overline{\varphi}_a(\ell, \lambda) + \zeta_4 \overline{\varphi}'_a(\ell, \lambda)} \right) \overline{\varphi}_a(\lambda, \lambda) + \left( \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda)} \right) \overline{\chi}_a(\lambda, \lambda). \]

If we take
\[ G_a(t, x, \lambda) = \begin{cases} \frac{\chi_a(t, \lambda) \varphi_a(x, \lambda)}{W_a(\lambda)}, & 0 \leq x \leq t, \\ \frac{\varphi_a(\lambda, \lambda) \chi_a(x, \lambda)}{W_a(\lambda)}, & t \leq x \leq \ell. \end{cases} \]
\[ \overline{G}_a(t, x, \lambda) = \begin{cases} \frac{\overline{\chi}_a(t, \lambda) \overline{\varphi}_a(x, \lambda)}{\overline{W}_a(\lambda)}, & 0 \leq x \leq t, \\ \frac{\overline{\varphi}_a(\lambda, \lambda) \overline{\chi}_a(x, \lambda)}{\overline{W}_a(\lambda)}, & t \leq x \leq \ell. \end{cases} \]

in (25) and (26), it is written as
\[ [u(t, \lambda)]^a = [u_a(\lambda), \overline{u}_a(\lambda)]. \]

\[ u_a(\lambda) = \int_0^t G_a(t, x, \lambda) f_a(x) \, dx + \left( \frac{\gamma_\alpha}{\zeta_3 \varphi_a(\ell, \lambda) + \zeta_4 \varphi'_a(\ell, \lambda)} \right) \varphi_a(\lambda, \lambda) + \left( \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda)} \right) \chi_a(\lambda, \lambda). \]
\[ \overline{u}_a(\lambda) = \int_0^t \overline{G}_a(t, x, \lambda) \overline{f}_a(x) \, dx + \left( \frac{\overline{\gamma}_\alpha}{\zeta_3 \overline{\varphi}_a(\ell, \lambda) + \zeta_4 \overline{\varphi}'_a(\ell, \lambda)} \right) \overline{\varphi}_a(\lambda, \lambda) + \left( \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_a(0, \lambda) + \zeta_2 \overline{\chi}'_a(0, \lambda)} \right) \overline{\chi}_a(\lambda, \lambda). \]

where, \([G(t, x, \lambda)]^a = [G_a(t, x, \lambda), \overline{G}_a(t, x, \lambda)]\) is the Green’s function of the problem.

If
\[ \frac{\partial u_a(\lambda)}{\partial \alpha} \geq 0, \quad \frac{\partial \overline{u}_a(\lambda)}{\partial \alpha} \leq 0 \quad \text{and} \quad u_a(\lambda) \leq \overline{u}_a(\lambda), \]

for all \( t \in [0, \ell] \) and \( \alpha \in [0,1] \), the solution (29)-(31) of the fuzzy boundary value problem (1)-(3) is a valid \( \alpha \)-level set. Consequently, the solution of the fuzzy boundary value problem (1)-(3) is (29)-(31) for \( \lambda > 0 \) satisfying inequalities (32).
Example Consider the fuzzy boundary value problem
\[(33)\]
\[u'' + \lambda u = [t]^\alpha, \quad u(0) = [1]^\alpha, \quad u(1) = [0]^\alpha.\]
where \([t]^\alpha = \{t - 1 + \alpha, t + 1 - \alpha\}, [1]^\alpha = [\alpha, 2 - \alpha], [0]^\alpha = [-1 + \alpha, 1 - \alpha].\]
Let be \(k = k^2, k > 0\), let be
\[
[\varphi(t, \lambda)]^\alpha = \left[\varphi_a(t, \lambda), \varphi_b(t, \lambda)\right]
\]
the solution satisfying the condition \(u(0) = 0\) and
\[
[\chi(t, \lambda)]^\alpha = \left[\chi_a(t, \lambda), \chi_b(t, \lambda)\right]
\]
the solution satisfying the condition \(u(1) = 0\) of the fuzzy differential equation \(u'' + \lambda u = 0\). According to this,
\[
\varphi_a(1, \lambda) = a\sin(k), \quad \varphi_b(1, \lambda) = (2 - a)\sin(k),
\]
\[
\chi_a(0, \lambda) = a\sin(k), \quad \chi_b(0, \lambda) = (2 - a)\sin(k),
\]
and
\[
W_a(\lambda) = -ka^2\sin(k), \quad W_b(\lambda) = -(2 - a)^2\sin(k).
\]
Then,
\[
\underline{u}_a(t, \lambda) = \frac{1}{-ka^2\sin(k)} \left\{a^2\sin(k) \int_0^1 (x - 1 + a)(\sin(k)\cos(kx) - \cos(k)\sin(kx))dx + a^2(\sin(k)\cos(k) - \cos(k)\sin(k)) \int_0^1 (x - 1 + a)\sin(kx)dx \right\}
\]
\[
+ (\frac{-1 + a}{a\sin(k)}) a(\sin(k)\cos(k) - \cos(k)\sin(k))
\]
Where if we take,
\[
\underline{G}_a(t, x, \lambda) = \begin{cases}
\alpha^2(\sin(k)\cos(k\lambda - \cos(k)\sin(k\lambda)), & 0 \leq x \leq t \\
\alpha^2 sin(k)(\sin(k)\cos(k\lambda - \cos(k)\sin(k\lambda)), & t \leq x \leq 1 \end{cases}
\]
\[
\underline{u}_a(t, \lambda) = \int_0^1 \underline{G}_a(t, x, \lambda)(x^2 - 1 + a)dx + \left(\frac{-1 + a}{a\sin(k)}\right) a(\sin(k)\cos(k) - \cos(k)\sin(k))
\]
is obtained. Therefore,
\[
\underline{u}_a(t, \lambda) = \frac{a}{k^2\sin(k)} \left\{\frac{\sin(k)}{\sin(\lambda)} + \frac{t - 1 + \alpha}{k^2} + \frac{-1 + \alpha}{k^2}(\cot(k)\sin(k) - \cos(k)) \right\}
\]
\[
+ \alpha\left(\frac{\sin(k)}{\sin(\lambda)} \right) a(\sin(k)\cos(k) - \cos(k)\sin(k))
\]
From this, yields
\[(34)\]
\[
\underline{u}_a(t, \lambda) = \left(\alpha + \frac{1 - a}{k^2}\right) \sin(k) + \left(\frac{-a}{k^2} - 1 + \frac{a^2}{k^2}\right) a(\sin(k)\cos(k) - \cos(k)\sin(k))
\]
Similarly,
\[(35)\]
\[
\overline{u}_a(t, \lambda) = \left(2 - \alpha + \frac{a - 1}{k^2}\right) \cos(k) + \left(\frac{a^2}{k^2} - 1 + \frac{a}{k^2}\right) a\sin(k)\cos(k) - \cos(k)\sin(k) + \frac{t + 1 + a}{k^2}
\]
is obtained. Then the solution of the fuzzy boundary value problem \((33)\) is
\[(36)\]
\[
[u(t, \lambda)]^\alpha = \left[\underline{u}_a(t, \lambda), \overline{u}_a(t, \lambda)\right].
\]
(37) \[ \left( 1 - \frac{1}{k^2} \right) \cos(kt) + \left( \frac{1}{k^2} \frac{\cos(k) - 1}{\sin(k)} \right) \sin(kt) + \frac{1}{k^2} \geq 0 \]

\[ u(t, \lambda) \] is a valid \( \alpha \)-level set. From this, the solution of the fuzzy boundary value problem (33) is (34)-(36) for the values \( \lambda = k^2 \) satisfying the inequality (37).

REFERENCES