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Solution and Green's Function of the Sturm-Liouville Problem with Fuzzy Forcing Function

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Abstract: In this paper, Green's function of Sturm-Liouville problem with the fuzzy forcing function is examined and the solution of the problem using the Green's function is investigated. Solution is shown on an example.
Keywords: Fuzzy boundary value problems, Hukuhara differentiability, Green's function.

INTRODUCTION

Fuzzy set theory is powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models [3]. Fuzzy differential equation is a very important topic from theoretical point of view [7, 8, 12, 14] and its applications, for example, in population models [5], civil engineering [13] and medicine [1,2].

There are different views on the concept of differentiable fuzzy-valued function. Historically, the first approach is to consider differentiability in the sense of Hukuhara.

In this paper, a investigation is made on the solution and Green's function of Sturm-Liouville problem with the fuzzy forcing function by using Hukuhara differentiability.

Preliminaries

Definition 1. [11] A fuzzy number is a function $u: \mathbb{R} \longrightarrow [0,1]$ satisfying the properties, u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$, $cl\{x \epsilon \mathbb{R} | u(x) > 0\}$ is compact where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2. [10] Let $u \epsilon \mathbb{R}_F$. The α-level set of u, denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \epsilon \mathbb{R} | u(x) \geq \alpha\}$. If $\alpha = 0$, the support of u is defined $[u]^0 = cl\{x \epsilon \mathbb{R} | u(x) > 0\}$. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u.

Remark 1. [9] The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

i) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

ii) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

iii) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 3. [11] If A is a symmetric triangular number with support $[\underline{a}, \overline{a}]$, the α-level set of A is

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right].$$

Definition 4. [10] For $u, v \epsilon \mathbb{R}_F$ and $\lambda \epsilon \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \epsilon [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D:\mathbb{R}_F\times\mathbb{R}_F\longrightarrow\mathbb{R}_+\cup\{0\},$$

by

$$D(u,v)=\sup_{\alpha\in[0,1]}\max\left\{\left|\underline{u}_\alpha-\underline{v}_\alpha\right|,\left|\overline{u}_\alpha-\overline{v}_\alpha\right|\right\}$$ [9].

Definition 5. [11] Let $u,v\epsilon\mathbb{R}_F$. If there exist $w\epsilon\mathbb{R}_F$ such that $u=v+w$, then w is called the H-difference of u and v and it is denoted $u\underset{H}{-}v$.

Definition 6. [9] Let I=(a,b), for $a,b\epsilon\mathbb{R}$, and $F:I\longrightarrow\mathbb{R}_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0\in I$ if there exists an element $F'(t_0)\epsilon\mathbb{R}_F$ such that the limits

$$\lim_{h\to0^+}\frac{F(t_0+h)-F(t_0)}{h}\ \text{and}\ \lim_{h\to0^+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist and equal $F'(t_0)$. Here the limits are taken in the metric space$(\mathbb{R}_F,D)$.

Theorem 1. [4] Let $f:I\longrightarrow\mathbb{R}_F$ be a function and denote $[f(t)]^\alpha=[\underline{f}_\alpha(t),\overline{f}_\alpha(t)]$, for each $\alpha\epsilon[0,1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[f'(t)]^\alpha=[\underline{f}'_\alpha(t),\overline{f}'_\alpha(t)]$.

FINDINGS AND DISCUSSION

Consider the fuzzy boundary value problem

(1) $$u''+q(t)u+\lambda u=[f(t)]^\alpha, t\epsilon(0,\ell),$$

(2) $$\zeta_1u(0)+\zeta_2u'(0)=[\beta]^\alpha,$$

(3) $$\zeta_3u(\ell)+\zeta_4u'(\ell)=[\gamma]^\alpha,$$

where $\lambda>0$, $[f(t)]^\alpha=\left[\underline{f_\alpha}(t),\overline{f}_\alpha(t)\right]$ triangular fuzzy function, $[\beta]^\alpha=\left[\underline{\beta_\alpha},\overline{\beta}_\alpha\right]$, $[\gamma]^\alpha=\left[\underline{\gamma_\alpha},\overline{\gamma}_\alpha\right]$ are symmetric triangular fuzzy numbers, $\zeta_i\geq0, i=1,2,3,4$, $\zeta_1^2+\zeta_2^2\neq0$, $\zeta_3^2+\zeta_4^2\neq0$, $q(t)$ is a continuous function and positive on $(0,\ell)$.

Firstly, consider the boundary value problem

(4) $$u''+q(t)u+\lambda u=0, t\epsilon(0,\ell),$$

(5) $$\zeta_1u(0)+\zeta_2u'(0)=0,$$

(6) $$\zeta_3u(\ell)+\zeta_4u'(\ell)=0.$$

Let $[\varphi(t,\lambda)]^\alpha=\left[\underline{\varphi_\alpha}(t,\lambda),\overline{\varphi}_\alpha(t,\lambda)\right]$ be the solution satisfying the conditions

(7) $$u(0)=\zeta_2,\ u'(0)=-\zeta_1,$$

and $[\chi(t,\lambda)]^\alpha=\left[\underline{\chi_\alpha}(t,\lambda),\overline{\chi}_\alpha(t,\lambda)\right]$ be the solution satisfying the conditions

(8) $$u(\ell)=\zeta_4,\ u'(\ell)=-\zeta_3$$

of fuzzy differential equations (4). The eigenvalues of the fuzzy boundary value problem (4)-(6) if and only if are consist of the zeros of Wronskian functions

(9) $$\underline{W_\alpha}(\lambda)=W\left(\underline{\varphi_\alpha},\underline{\chi_\alpha}\right)(t,\lambda)=\underline{\varphi_\alpha}(t,\lambda)\underline{\chi'_\alpha}(t,\lambda)-\underline{\chi_\alpha}(t,\lambda)\underline{\varphi'_\alpha}(t,\lambda),$$

(10) $$\overline{W}_\alpha(\lambda)=W\left(\overline{\varphi}_\alpha,\overline{\chi}_\alpha\right)(t,\lambda)=\overline{\varphi}_\alpha(t,\lambda)\overline{\chi}'_\alpha(t,\lambda)-\overline{\chi}_\alpha(t,\lambda)\overline{\varphi}'_\alpha(t,\lambda)$$ [6].

Then, if λ is not the eigenvalue of the fuzzy boundary value problem (4)-(6), since $\underline{W_\alpha}(\lambda)\neq0$ and $\overline{W}_\alpha(\lambda)\neq0$, the lower and upper solutions $\underline{\varphi_\alpha}(t,\lambda)$, $\underline{\chi_\alpha}(t,\lambda)$ and $\overline{\varphi}_\alpha(t,\lambda)$, $\overline{\chi}_\alpha(t,\lambda)$ will be linear independent. According to this, the general solution of the fuzzy differential equation (4) is

(11) $$[u(t,\lambda)]^\alpha=\left[\underline{u_\alpha}(t,\lambda),\overline{u}_\alpha(t,\lambda)\right],$$

(12) $$\underline{u_\alpha}(t,\lambda)=\underline{a_\alpha}(\lambda)\underline{\varphi_\alpha}(t,\lambda)+\underline{b_\alpha}(\lambda)\underline{\chi_\alpha}(t,\lambda),$$

(13) $$\overline{u}_\alpha(t,\lambda)=\overline{a}_\alpha(\lambda)\overline{\varphi}_\alpha(t,\lambda)+\overline{b}_\alpha(\lambda)\overline{\chi}_\alpha(t,\lambda).$$

Using the method variation of parameters, we can search for the general solution of the fuzzy differential equation (1) as

(14) $$[u(t,\lambda)]^\alpha=\left[\underline{u_\alpha}(t,\lambda),\overline{u}_\alpha(t,\lambda)\right],$$

(15) $$\underline{u}_\alpha(t,\lambda) = \underline{a}_\alpha(t,\lambda)\underline{\varphi}_\alpha(t,\lambda) + \underline{b}_\alpha(t,\lambda)\underline{\chi}_\alpha(t,\lambda),$$

(16) $$\overline{u}_\alpha(t,\lambda) = \overline{a}_\alpha(t,\lambda)\overline{\varphi}_\alpha(t,\lambda) + \overline{b}_\alpha(t,\lambda)\overline{\chi}_\alpha(t,\lambda).$$

Derivating the Hukuhara of (14)-(16) according to x, choosing the functions $\underline{a}_\alpha(t,\lambda)$, $\underline{b}_\alpha(t,\lambda)$, $\overline{a}_\alpha(t,\lambda)$ and $\overline{b}_\alpha(t,\lambda)$ that

(17) $$\underline{a}'_\alpha(t,\lambda)\underline{\varphi}_\alpha(t,\lambda) + \underline{b}'_\alpha(t,\lambda)\underline{\chi}_\alpha(t,\lambda) = 0,$$

(18) $$\overline{a}'_\alpha(t,\lambda)\overline{\varphi}_\alpha(t,\lambda) + \overline{b}'_\alpha(t,\lambda)\overline{\chi}_\alpha(t,\lambda) = 0,$$

later, derivating of $[u'(t,\lambda)]^\alpha$ once again and substituing in the fuzzy differential equation (1), making the necessary operations, the equations

(19) $$\underline{a}'_\alpha(t,\lambda)\underline{\varphi}'_\alpha(t,\lambda) + \underline{b}'_\alpha(t,\lambda)\underline{\chi}'_\alpha(t,\lambda) = \underline{f}_\alpha(t),$$

(20) $$\overline{a}'_\alpha(t,\lambda)\overline{\varphi}'_\alpha(t,\lambda) + \overline{b}'_\alpha(t,\lambda)\overline{\chi}'_\alpha(t,\lambda) = \overline{f}_\alpha(t)\ ,$$

are obtained. Then, solving the equations (17), (19) and (18), (20) by looking at the system as a linear equation system according to the variables, yields

$$\underline{a}'_\alpha(t,\lambda) = -\frac{1}{\underline{W}_\alpha(\lambda)}\underline{\chi}_\alpha(t,\lambda)\underline{f}_\alpha(t),$$
$$\underline{b}'_\alpha(t,\lambda) = \frac{1}{\underline{W}_\alpha(\lambda)}\underline{\varphi}_\alpha(t,\lambda)\underline{f}_\alpha(t),$$
$$\overline{a}'_\alpha(t,\lambda) = -\frac{1}{\overline{W}_\alpha(\lambda)}\overline{\chi}_\alpha(t,\lambda)\overline{f}_\alpha(t),$$
$$\overline{b}'_\alpha(t,\lambda) = \frac{1}{\overline{W}_\alpha(\lambda)}\overline{\varphi}_\alpha(t,\lambda)\overline{f}_\alpha(t).$$

From here, we have

$$\underline{a}_\alpha(t,\lambda) = \frac{1}{\underline{W}_\alpha(\lambda)}\int_t^\ell \underline{\chi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx + \underline{a}_\alpha(\lambda),$$
$$\underline{b}_\alpha(t,\lambda) = \frac{1}{\underline{W}_\alpha(\lambda)}\int_0^t \underline{\varphi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx + \underline{b}_\alpha(\lambda),$$
$$\overline{a}_\alpha(t,\lambda) = \frac{1}{\overline{W}_\alpha(\lambda)}\int_t^\ell \overline{\chi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx + \overline{a}_\alpha(\lambda),$$
$$\overline{b}_\alpha(t,\lambda) = \frac{1}{\overline{W}_\alpha(\lambda)}\int_0^t \overline{\varphi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx + \overline{b}_\alpha(\lambda).$$

Substituing these equations in (15) and (16), the general solution of the fuzzy differential equation (1) is obtained as

(21) $$[u(t,\lambda)]^\alpha = [\underline{u}_\alpha(t,\lambda), \overline{u}_\alpha(t,\lambda)],$$

(22) $$\underline{u}_\alpha(t,\lambda) = \frac{1}{\underline{W}_\alpha(\lambda)}\left\{\underline{\varphi}_\alpha(t,\lambda)\int_t^\ell \underline{\chi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx + \underline{\chi}_\alpha(t,\lambda)\int_0^t \underline{\varphi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx\right\}$$
$$+\underline{a}_\alpha(\lambda)\underline{\varphi}_\alpha(t,\lambda) + \underline{b}_\alpha(\lambda)\underline{\chi}_\alpha(t,\lambda),$$

(23) $$\overline{u}_\alpha(t,\lambda) = \frac{1}{\overline{W}_\alpha(\lambda)}\left\{\overline{\varphi}_\alpha(t,\lambda)\int_t^\ell \overline{\chi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx + \overline{\chi}_\alpha(t,\lambda)\int_0^t \overline{\varphi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx\right\}$$
$$+\overline{a}_\alpha(\lambda)\overline{\varphi}_\alpha(t,\lambda) + \overline{b}_\alpha(\lambda)\overline{\chi}_\alpha(t,\lambda).$$

Using the boundary condition (2) and the equations

$$\zeta_1\underline{\varphi}_\alpha(0,\lambda) + \zeta_2\underline{\varphi}'_\alpha(0,\lambda) = 0,\ \ \zeta_1\overline{\varphi}_\alpha(0,\lambda) + \zeta_2\overline{\varphi}'_\alpha(0,\lambda) = 0,$$

we have

$$\underline{b}_\alpha(\lambda)\left\{\zeta_1\underline{\chi}_\alpha(0,\lambda) + \zeta_2\underline{\chi}'_\alpha(0,\lambda)\right\} = \underline{\beta}_\alpha,$$
$$\overline{b}_\alpha(\lambda)\{\zeta_1\overline{\chi}_\alpha(0,\lambda) + \zeta_2\overline{\chi}'_\alpha(0,\lambda)\} = \overline{\beta}_\alpha.$$

From (7), (9) and (10), we obtained

$$\zeta_1 \underline{\chi}_\alpha(0,\lambda) + \zeta_2 \underline{\chi}'_\alpha(0,\lambda) = \underline{W}_\alpha(\lambda), \quad \zeta_1 \overline{\chi}_\alpha(0,\lambda) + \zeta_2 \overline{\chi}'_\alpha(0,\lambda) = \overline{W}_\alpha(\lambda).$$

Since λ is not eigenvalue, $\underline{W}_\alpha(\lambda) \neq 0$, $\overline{W}_\alpha(\lambda) \neq 0$. Then, yields

$$\underline{b}_\alpha(\lambda) = \frac{\underline{\beta}_\alpha}{\zeta_1 \underline{\chi}_\alpha(0,\lambda) + \zeta_2 \underline{\chi}'_\alpha(0,\lambda)}, \qquad \overline{b}_\alpha(\lambda) = \frac{\overline{\beta}_\alpha}{\zeta_1 \overline{\chi}_\alpha(0,\lambda) + \zeta_2 \overline{\chi}'_\alpha(0,\lambda)}.$$

Similarly,

$$\underline{a}_\alpha(\lambda) = \frac{\underline{\gamma}_\alpha}{\zeta_3 \underline{\varphi}_\alpha(\ell,\lambda) + \zeta_4 \underline{\varphi}'_\alpha(\ell,\lambda)}, \qquad \overline{a}_\alpha(\lambda) = \frac{\overline{\gamma}_\alpha}{\zeta_3 \overline{\varphi}_\alpha(\ell,\lambda) + \zeta_4 \overline{\varphi}'_\alpha(\ell,\lambda)}.$$

are obtained. Then, from (21)-(23) yields

(24) $$[u(t,\lambda)]^\alpha = [\underline{u}_\alpha(t,\lambda), \overline{u}_\alpha(t,\lambda)],$$

(25) $$\underline{u}_\alpha(t,\lambda) = \frac{1}{\underline{W}_\alpha(\lambda)}\left\{\underline{\varphi}_\alpha(t,\lambda)\int_t^\ell \underline{\chi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx + \underline{\chi}_\alpha(t,\lambda)\int_0^t \underline{\varphi}_\alpha(x,\lambda)\underline{f}_\alpha(x)dx\right\}$$
$$+\left(\frac{\underline{\gamma}_\alpha}{\zeta_3\underline{\varphi}_\alpha(\ell,\lambda)+\zeta_4\underline{\varphi}'_\alpha(\ell,\lambda)}\right)\underline{\varphi}_\alpha(t,\lambda) + \left(\frac{\underline{\beta}_\alpha}{\zeta_1\underline{\chi}_\alpha(0,\lambda)+\zeta_2\underline{\chi}'_\alpha(0,\lambda)}\right)\underline{\chi}_\alpha(t,\lambda),$$

(26) $$\overline{u}_\alpha(t,\lambda) = \frac{1}{\overline{W}_\alpha(\lambda)}\left\{\overline{\varphi}_\alpha(t,\lambda)\int_t^\ell \overline{\chi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx + \overline{\chi}_\alpha(t,\lambda)\int_0^t \overline{\varphi}_\alpha(x,\lambda)\overline{f}_\alpha(x)dx\right\}$$
$$+\left(\frac{\overline{\gamma}_\alpha}{\zeta_3\overline{\varphi}_\alpha(\ell,\lambda)+\zeta_4\overline{\varphi}'_\alpha(\ell,\lambda)}\right)\overline{\varphi}_\alpha(t,\lambda) + \left(\frac{\overline{\beta}_\alpha}{\zeta_1\overline{\chi}_\alpha(0,\lambda)+\zeta_2\overline{\chi}'_\alpha(0,\lambda)}\right)\overline{\chi}_\alpha(t,\lambda).$$

If we take

(27) $$\underline{G}_\alpha(t,x,\lambda) = \begin{cases} \frac{\underline{\chi}_\alpha(t,\lambda)\underline{\varphi}_\alpha(x,\lambda)}{\underline{W}_\alpha(\lambda)}, 0 \le x \le t \\ \frac{\underline{\varphi}_\alpha(t,\lambda)\underline{\chi}_\alpha(x,\lambda)}{\underline{W}_\alpha(\lambda)}, t \le x \le \ell \end{cases},$$

(28) $$\overline{G}_\alpha(t,x,\lambda) = \begin{cases} \frac{\overline{\chi}_\alpha(t,\lambda)\overline{\varphi}_\alpha(x,\lambda)}{\overline{W}_\alpha(\lambda)}, 0 \le x \le t \\ \frac{\overline{\varphi}_\alpha(t,\lambda)\overline{\chi}_\alpha(x,\lambda)}{\overline{W}_\alpha(\lambda)}, t \le x \le \ell \end{cases}$$

in (25) and (26), it is written as

(29) $$[u(t,\lambda)]^\alpha = [\underline{u}_\alpha(t,\lambda), \overline{u}_\alpha(t,\lambda)],$$

(30) $$\underline{u}_\alpha(t,\lambda) = \int_0^\ell \underline{G}_\alpha(t,x,\lambda)\underline{f}_\alpha(x)dx + \left(\frac{\underline{\gamma}_\alpha}{\zeta_3\underline{\varphi}_\alpha(\ell,\lambda)+\zeta_4\underline{\varphi}'_\alpha(\ell,\lambda)}\right)\underline{\varphi}_\alpha(t,\lambda)$$
$$+\left(\frac{\underline{\beta}_\alpha}{\zeta_1\underline{\chi}_\alpha(0,\lambda)+\zeta_2\underline{\chi}'_\alpha(0,\lambda)}\right)\underline{\chi}_\alpha(t,\lambda),$$

(31) $$\overline{u}_\alpha(t,\lambda) = \int_0^\ell \overline{G}_\alpha(t,x,\lambda)dx\,\overline{f}_\alpha(x)dx + \left(\frac{\overline{\gamma}_\alpha}{\zeta_3\overline{\varphi}_\alpha(\ell,\lambda)+\zeta_4\overline{\varphi}'_\alpha(\ell,\lambda)}\right)\overline{\varphi}_\alpha(t,\lambda)$$
$$+\left(\frac{\overline{\beta}_\alpha}{\zeta_1\overline{\chi}_\alpha(0,\lambda)+\zeta_2\overline{\chi}'_\alpha(0,\lambda)}\right)\overline{\chi}_\alpha(t,\lambda),$$

where, $[G(t,x,\lambda)]^\alpha = [\underline{G}_\alpha(t,x,\lambda), \overline{G}_\alpha(t,x,\lambda)]$ is the Green's function of the problem.
If

(32) $$\frac{\partial \underline{u}_\alpha(t,\lambda)}{\partial\alpha} \ge 0, \ \frac{\partial \overline{u}_\alpha(t,\lambda)}{\partial\alpha} \le 0 \text{ and } \underline{u}_\alpha(t,\lambda) \le \overline{u}_\alpha(t,\lambda)$$

for all $t \in [0,\ell]$ and $\alpha \in [0,1)$, the solution (29)-(31) of the fuzzy boundary value problem (1)-(3) is a valid α-level set. Consequently, the solution of the fuzzy boundary value problem (1)-(3) is (29)-(31) for $\lambda > 0$ satisfying inequalities (32).

Example Consider the fuzzy boundary value problem

(33) $$u'' + \lambda u = [t]^{\alpha},\ u(0) = [1]^{\alpha},\ u(1) = [0]^{\alpha},$$

where $[t]^{\alpha} = [t-1+\alpha, t+1-\alpha]$, $[1]^{\alpha} = [\alpha, 2-\alpha]$, $[0]^{\alpha} = [-1+\alpha, 1-\alpha]$.

Let be $\lambda = k^2, k > 0$, let be

$$[\varphi(t,\lambda)]^{\alpha} = \left[\underline{\varphi}_{\alpha}(t,\lambda), \overline{\varphi}_{\alpha}(t,\lambda)\right] = [\alpha, 2-\alpha] sin(kt)$$

the solution satisfying the condition $u(0) = 0$ and

$$[\chi(t,\lambda)]^{\alpha} = \left[\underline{\chi}_{\alpha}(t,\lambda), \overline{\chi}_{\alpha}(t,\lambda)\right] = [\alpha, 2-\alpha]\big(sin(k)cos(kt) - cos(k)sin(kt)\big)$$

the solution satisfying the condition $u(1) = 0$ of the fuzzy differential equation $u'' + \lambda u = 0$. According to this,

$$\underline{\varphi}_{\alpha}(1,\lambda) = \alpha sin(k),\quad \overline{\varphi}_{\alpha}(1,\lambda) = (2-\alpha)sin(k),$$
$$\underline{\chi}_{\alpha}(0,\lambda) = \alpha sin(k),\quad \overline{\chi}_{\alpha}(0,\lambda) = (2-\alpha)sin(k),$$

and

$$\underline{W}_{\alpha}(\lambda) = -k\alpha^2 sin(k),\quad \overline{W}_{\alpha}(\lambda) = -k(2-\alpha)^2 sin(k).$$

Then,

$$\underline{u}_{\alpha}(t,\lambda) = \frac{1}{-k\alpha^2 sin(k)}\Big\{\alpha^2 sin(kt)\int_t^1 (x-1+\alpha)\big(sin(k)cos(kx) - cos(k)sin(kx)\big)dx$$
$$+\alpha^2\big(sin(k)cos(kt) - cos(k)sin(kt)\big)\int_0^t (x-1+\alpha)sin(kx)dx\Big\}$$
$$+\left(\frac{-1+\alpha}{\alpha sin(k)}\right)\big(\alpha sin(kt)\big) + \left(\frac{\alpha}{\alpha sin(k)}\right)\alpha\big(sin(k)cos(kt) - cos(k)sin(kt)\big).$$

Where if we take,

$$\underline{G}_{\alpha}(t,x,\lambda) = \begin{cases} \frac{\alpha^2(sin(k)cos(kt) - cos(k)sin(kt))sin(kx)}{-k\alpha^2 sin(k)}, & 0 \le x \le t \\ \frac{\alpha^2 sin(kt)(sin(k)cos(kx) - cos(k)sin(kx))}{-k\alpha^2 sin(k)}, & t \le x \le 1 \end{cases},$$

$$\underline{u}_{\alpha}(t,\lambda) = \int_0^1 \underline{G}_{\alpha}(t,x,\lambda)(x^2-1+\alpha)dx + \left(\frac{-1+\alpha}{\alpha sin(k)}\right)\alpha sin(kt)$$
$$+\left(\frac{\alpha}{\alpha sin(k)}\right)\alpha\big(sin(k)cos(kt) - cos(k)sin(kt)\big)$$

is obtained. Therefore,

$$\underline{u}_{\alpha}(t,\lambda) = -\frac{\alpha}{k^2}\frac{sin(kt)}{sin(k)} + \frac{t-1+\alpha}{k^2} + \frac{-1+\alpha}{k^2}\big(cot(k)sin(kt) - cos(kt)\big)$$
$$+\frac{(-1+\alpha)sin(kt)}{sin(k)} + \alpha\left(cos(kt) - \frac{cos(k)sin(kt)}{sin(k)}\right).$$

From this, yields

(34) $$\underline{u}_{\alpha}(t,\lambda) = \left(\alpha + \frac{1-\alpha}{k^2}\right)cos(kt) + \left(\frac{-\frac{\alpha}{k^2} + \alpha - 1 + \left(\frac{\alpha-1}{k^2} - \alpha\right)cos(k)}{sin(k)}\right)sin(kt) + \frac{t-1+\alpha}{k^2}$$

Similarly,

(35) $$\overline{u}_{\alpha}(t,\lambda) = \left((2-\alpha) + \frac{\alpha-1}{k^2}\right)cos(kt) + \left(\frac{-\frac{(2-\alpha)}{k^2} + 1 - \alpha + \left(\frac{1-\alpha}{k^2} - (2-\alpha)\right)cos(k)}{sin(k)}\right)sin(kt) + \frac{t+1-\alpha}{k^2}$$

is obtained. Then the solution of the fuzzy boundary value problem (33) is

(36) $$[u(t,\lambda)]^{\alpha} = \left[\underline{u}_{\alpha}(t,\lambda), \overline{u}_{\alpha}(t,\lambda)\right].$$

If

(37) $$\left(1-\frac{1}{k^2}\right)cos(kt)+\left(\frac{\left(1-\frac{1}{k^2}\right)(cos(k)-1)}{sin(k)}\right)sin(kt)+\frac{1}{k^2}\geq 0$$

$[u(t,\lambda)]^{\alpha}$ is a valid α-level set. From this, the solution of the fuzzy boundary value problem (33) is (34)-(36) for the values $\lambda=k^2$ satisfying the inequality (37).

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