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Abundant exact wave solutions to the standard KdV and the Burgers-Fisher equation

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Abstract **Original Research Article**

In this paper, the $(G'/G, 1/G)$-expansion method is applied to carry out the integration of standard KdV and the Burgers-Fisher equations. The effectiveness of the $(G'/G, 1/G)$-expansion method is exposed. Choosing the particular values of different parameters, soliton and other periodic type traveling waves can be derived. The method appears to be easier, economical, efficient and powerful by means of symbolic computation system to finding more new form exact solutions for various kinds of nonlinear evolution equations.

Keywords: The standard KdV equation, Burgers-Fisher equation, the $(G'/G, 1/G)$-expansion method, traveling wave solutions.

INTRODUCTION

In the mathematical language, where the rate of change involves the differential equations either ordinary or partial or systems of them evolves. Additionally, the true laws of nature never can be linear [21]. Thus the study of nonlinear evolution equation (NLEE) has made a significant progress in the past few decades. Exact solutions of these nonlinear phenomena assist to understand the qualitative and quantitative information about the underlying behavior of the mathematical expression in a direct way. Therefore, active research works have emerged worldwide in diverse branches of scientific fields to searching the exact solutions of nonlinear partial differential equations (PDEs). But, the largest parts of nonlinear models of real-life problems are still very difficult to solve either analytically or numerically. Many assumptions have to be made needlessly to make nonlinear models solvable particularly for traveling wave solutions. In the literature of traveling wave solutions many types of traveling waves are exposed such as soliton, kink, peakon, compacton, cuspon, cnoidal wave and many others. All of these types of wave pattern arise in the field of mathematics, physics and engineering which are very important. In this paper, the NLEEs the standard KdV and the Burger-Fisher equations are going to be studied to searching further new or more general form of exact solutions by the $(G'/G, 1/G)$-expansion method.

Since every nonlinear evolution equation has its own characteristics and the exact solution much more depends on its physical nature, no single method is suitable to handle all of these equations. Rather one has to rely on different techniques or necessary theories. So, researcher focuses on various methods for seeking exact solutions that are important for applications within and outside of mathematics, with the hope that these solutions will reflect the insight of PDE and clues of their origins. Moreover, the advent of modern computational technologies changes the scenario dramatically. Now, the nonlinear PDEs can be solved effectively by means of sophisticatedcomputers, with due attention to theaccuracy of the solutions.

There are many traditional and newly developed methods are used to carry out the integration of NLEEs. Some of them are the inverse scattering method, the Hirota method, the Backlund transform method, the exp-function method, the Jacobi elliptic function method, sine-cosine method, tanh-coth method and so on. Numerous asymptotic methods have also been suggested such as the foremost Adomain decomposition method which was introduced and developed by

George Adomain, in 1994, the variational iteration method, which was proposed by Ji-Huan He, in 1999. All of these common methods are available in most of the textbooks and contemporary scientific papers [27, 20].

The (G'/G)- expansion method was firstly proposed by Wang *et al*. in 2008 [25]. Since then, this (G'/G) extension method is one of the most efficient methods and applied to many NLEEs to construct new exact traveling wave solutions. Many researchers also worked on this (G'/G)-expansion method to improve and enhance this expansion method [9, 10]. Different modification and extensions have also been proposed by severals e.g. Zhang *et al*. [28], extended the method to deal with evolution equations with variable coefficients. A remarkable work was also done by Zhang [29], to some special nonlinear equations where the balance numbers are not integer. Akbar *et al*. modified as well as applied this method and derived abundant traveling wave solutions of different celebrated NLEEs [1-5].

In 2010, Li *et al*. firstly introduced the (G′/G, 1/G)-expansion method which can be thought of as an extension of the (G′/G)-expansion method [26]. They applied this extension method to the Zakharov equations and the traveling wave solutions are successfully obtained. The principle idea of this expansion method is that the exact traveling wave solutions of NLEEs can be expressed in the form a polynomial in two variables (G'/G) and $(1/G)$ where $G = G(\xi)$ satisfies a second order linear ordinary differential equation (LODE). Recently, Demiray*et al*. have also applied this method to the Boussinesq type equations for searching the exact solutions [7-8]. The $(G'/G, 1/G)$-method is also applied to various nonlinear PDEs such as Phi-Four equation, Boussinesq type equations, Gardner-KP equation [6-8], for analytical solutions. Very recently, Akbar *et al*. applied both the (G'/G) and $(G'/G, 1/G)$ expansion methods to the various nonlinear PDEs [13-16]. However, still this method is of little use to different nonlinear evolution equations due to an extra variable $(1/G)$, compared to (G'/G) expansion method.

In this work, we apply the (G′/G, 1/G)-expansion method to check the effectiveness in study of NLEEs and searching the new or more general form of exact solutions NLEEs the standard KdV and the Burger-Fisher equations.

The rest of the paper is organized as follows: In Section 2, we define the concept of $(G'/G, 1/G)$-expansion method briefly. In Sections 3, we have applied this extension method to the standard KdV and to the Burger-Fisher equations. Graphical representations of the obtaining results are presented in section 4. Results and discussions are pointed out in section 5. In section 6, the concluding remarks are given.

Outline of the $(G'/G, 1/G)$-expansion method

In this section, we briefly discuss the $(G'/G, 1/G)$-expansion method. In 2010, Li *et al*. firstly proposed this $(G'/G, 1/G)$-expansion method which is summarized as follows [26]:

Consider, the second order liner ordinary differential equation (LODE)

$$G''(\xi) + \lambda G(\xi) = \mu \tag{1}$$

where, G'' denotes the second order ordinary derivative, ξ is the independent variable and λ, μ are arbitrary constants. Now, consider two rational functions

$$\phi = G'/G, \ \psi = 1/G \tag{2}$$

where, G' denotes the first derivative and G is the solution of the Eq. (1). With a little effort from (1) and (2), we can obtain the following relation

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi \tag{3}$$

The general solution of the LODE (1) depends on the value of the parameter λ whether $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$.

Case1: when $\lambda < 0$, the general solution of the LODE (1) is,

$$G(\xi) = C_1 \sinh(\sqrt{-\lambda}\xi) + C_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda} \tag{4}$$

and we can obtain the following relation easily

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda) \tag{5}$$

where C_1 and C_2 are two arbitrary constants and $\sigma = C_1^2 - C_2^2$.

Case2: when $\lambda > 0$, the general solution of Eq. (1) has the form

$$G(\xi) = C_1 \sin(\sqrt{\lambda}\xi) + C_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda} \tag{6}$$

and again we can deduce the following relation

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}(\phi^2 - 2\mu\psi + \lambda) \tag{7}$$

where C_1 and C_2 are two arbitrary constants and $\sigma = C_1^2 + C_2^2$.

Case3: when $\lambda = 0$, then the general solution of the Eq. (1) is of the form

$$G(\xi) = \frac{\mu}{2}\xi^2 + C_1\xi + C_2 \tag{8}$$

and by some calculations, it can be deduced

$$\psi^2 = \frac{\lambda}{C_1^2 - 2\mu C_2}(\phi^2 - 2\mu\psi) \tag{9}$$

where C_1 and C_2 are two arbitrary constants.

Suppose that a nonlinear evolution equation in two independent variables x and t is given by,

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \tag{10}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u = u(x,t)$ and its various partial derivatives. The polynomial also consists of highest order derivatives and nonlinear terms. In the followings, we outline the main steps of the $(G'/G, 1/G)$-expansion method to solve Eq.(10).

Step 1: Use the traveling wave variable transformation

$$u(x,t) = u(\xi),\ \xi = x - Vt \tag{11}$$

where V is a constant to be determined latter? This transformation reduces the PDE (10) into an ODE for $u = u(\xi)$ in the form

$$P(u, -Vu', u', V^2u'', u'', -Vu'', \ldots) = 0 \tag{12}$$

where the primed denotes the ordinary derivatives of u with different order. Next, integrate the Eq. (12) as many times as possible and set the arbitrary constants of integration to be zero for simplicity.

Step 2: Suppose that the general solution of the ordinary differential equation (12) can be expressed by a polynomial in two rational functions ϕ and ψ as,

$$u(\xi) = \sum_{i=0}^{m} a_i\phi^i + \sum_{i=1}^{m} b_i\phi^{i-1}\psi \tag{13}$$

where $\phi = G'/G$, $\psi = 1/G$ and $G = G(\xi)$ satisfies the second order LODE (1), $a_i (i = 0,1,2,\ldots,m)$, $b_i (i = 1,2,\ldots,m), V, \lambda$ and μ are arbitrary constants which will be determined in latter. The degree m of the polynomial (13) can be determined using the principality by homogenous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(12).

Step 3: Now substituting the value of $u(\xi)$ of(13) into (12), using (3) and (5) (or using (3), (7) and (3), (9)) the left side of Eq. (12) transforms into a polynomial in ϕ and ψ. It is noted that applying the relations (5), (7) and (9), the degree of ψ will never be larger than 1. Equating each coefficient of the polynomial in ϕ,ψ to zero yields a system of algebraic equations in $a_i(i=0,1,2,...,m),\ b_i(i=1,2,...,m),\ V,\ \lambda,\ \mu,\ C_1$ and C_2.

Step 4: Solving the system of algebraic solutions arise in Step 3 with the mathematical graphing utility Maple or Mathematica and substituting the obtained values of $a_i(i=0,1,2,...,m),\ b_i(i=1,2,...,m),\ V,\lambda,\mu,C_1$ and C_2 into (13), we can finally obtain the traveling wave solutions expressed by the hyperbolic functions of Eq. (12) (or can be expressed by trigonometric function if $\lambda>0$ and rational function if $\lambda=0$).

Some application of the $(G'/G,1/G)$-expansion method

The standard KdV equation

In 1895, two Dutch physicists DiederickKorteweg and his student Gustav de Vries (KdV) [27] derived a nonlinear partial differential equation now which bears their name and the standard form is given by

$$u_t+6uu_x+u_{xxx}=0, \tag{14}$$

where $u(x,t)$ is a function of space variable x and time variable t. This equation models a variety of nonlinear phenomena such as plasma waves, shallow water waves and so on. Eq. (14) shows that the rate of change of the wave's height in time is governed by the sum of two terms, one is nonlinear terms that have the amplitude effect and the other is dispersive terms that have effect on waves of different wavelengths to travel with different velocities. The derivative u_t describes the time evolution of the wave propagation in one direction. The nonlinear term uu_x characterizes the steepening of the wave, whereas the linear term u_{xxx} accounts for the spreading or dispersion of the wave. This equation is the simplest nonlinear partial differential equation embodying two effects: nonlinearity represented by uu_x and linear dispersion represented by u_{xxx}. The fragile balance between the weak nonlinearity and the linear dispersion defines the formulation of soliton type travelling wave that consists of single humped wave.

Making use the traveling wave transform $u(x,t)=u(\xi),\ \xi=x-ct$, into eq. (14) and integrating, we obtain an ODE

$$-Vu+3u^2+u''=0\,. \tag{15}$$

Balancing the terms u^2 with u'' we obtain $m=2$. So the ansatz polynomial solution is in the form:

$$u(\xi)=a_0+a_1\phi+a_2\phi^2+b_1\psi+b_2\phi\psi, \tag{16}$$

where, a_0,a_1,a_2,b_1,b_2 are arbitrary constants, which are to be determined.

Case 1: When $\lambda<0$ (hyperbolic function solutions)

Substituting the value of $u(\xi)$, into (15) alongside with (3) and (5), and equating the coefficient of ϕ and ψ to zero yields to a system of algebraic equations (for brevity a few of them are given).

$$\phi^4:6\lambda^2\mu^2\sigma a_2^2+12\lambda^2\mu^2\sigma a_2+6\lambda^4\sigma^2a_2+3\lambda^4\sigma^2\sigma a_2^2-3\lambda^3\sigma b_2^2-3\lambda\mu^2b_2^2+3\mu a_2^2+6\mu^4a_2$$

$$\phi^3\psi:6\mu^4a_2b_2+6\lambda^4\sigma^2b_2+6\lambda^4\sigma^2a_2b_2+12\lambda^2\mu^2\sigma b_2+6\mu^4b_2+12\lambda^2\mu^2\sigma a_2b_2$$

$$\phi^3:2\mu^4a_1+6\lambda^4\sigma^2a_1a_2-6\lambda^3\sigma b_1b_2+6\lambda^3\mu\sigma b_2+4\lambda^2\mu^2\sigma a_1+12\lambda^2\mu^2\sigma a_1a_2+2\lambda^4\sigma^2a_1$$
$$+6\lambda\mu^3b_2+6\mu^4a_1a_2$$

$\cdots$

Solving this system of equations with the help of Maple, we get:

$$a_0 = -\lambda,\ a_1 = 0, a_2 = -1, b_1 = \mu, b_2 = \sqrt{\frac{-(\lambda^2\sigma + \mu^2)}{\lambda}}, V = \lambda\,. \tag{17}$$

Now, substituting these values in Eq. (16), we achieve

$$u(\xi) = \frac{-\lambda\left(C_1\cosh\left(\sqrt{-\lambda}\xi\right) + C_2\sinh\left(\sqrt{-\lambda}\xi\right)\right)^2}{\left(C_1\sinh\left(\sqrt{-\lambda}\xi\right) + C_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}\right)^2} + \frac{-\lambda\left(C_1\cosh\left(\sqrt{-\lambda}\xi\right) + C_2\sinh\left(\sqrt{-\lambda}\xi\right)\right)\sqrt{-\left(\lambda^2\sigma + \mu^2\right)}}{\sqrt{\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\xi\right) + C_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}\right)^2}$$
$$+ \frac{\mu}{C_1\sinh\left(\sqrt{-\lambda}\xi\right) + C_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}} - \lambda. \tag{18}$$

Setting up the particular values $C_1 = 0,\ C_2 = \sqrt{-\sigma}$ and $\mu = 0$, we obtain the traveling wave solution as:

$$u(\xi) = \lambda\tanh^2(\sqrt{-\lambda}\ \xi) + \lambda i\tanh(\sqrt{-\lambda}\ \xi)\sec h(\sqrt{-\lambda}\ \xi) - \lambda. \tag{19}$$

Now, substituting $\xi = x - Vt$, we finally obtain the solution of the standard KdV equation as

$$u(x,t) = \lambda\tanh^2(\sqrt{-\lambda}\ (x - Vt)) + \lambda i\tanh(\sqrt{-\lambda}\ (x - Vt))\sec h(\sqrt{-\lambda}\ (x - Vt)) - \lambda. \tag{20}$$

Similarly, if we set up $C_1 = \sqrt{\sigma},\ C_2 = 0$ and $\mu = 0$, we extract another wave solution as:

$$u(x,t) = \lambda\coth^2(\sqrt{-\lambda}\ (x - Vt)) + \lambda\coth(\sqrt{-\lambda}\ (x - Vt))\csc h(\sqrt{-\lambda}\ (x - Vt)) - \lambda. \tag{21}$$

Case 2: When $\lambda > 0$ (trigonometric function solutions)

Stepping similar to case1, substituting the value of $u(\xi)$, into (14) along with (3) and (7), and equating the coefficient of ϕ and ψ to be zero yields to a system of algebraic equations.

Equating each coefficient of equation to zero yields a system of algebraic equations (Herein, for conciseness and simplicity the algebraic equations are not given) and solving them by Maple the values of the arbitrary constants are obtained as follows:

$$a_0 = -\lambda,\ a_1 = 0, a_2 = -1, b_1 = \mu, b_2 = \sqrt{\frac{(\lambda^2\sigma - \mu^2)}{\lambda}}, V = \lambda\,. \tag{22}$$

While, we set $C_1 = \sqrt{\sigma},\ C_2 = 0$ and $\mu = 0$, then we have

$$u(x,t) = -\lambda\cot^2(\sqrt{\lambda}\ (x - Vt)) + \lambda\cot(\sqrt{\lambda}\ (x - Vt))\csc(\sqrt{\lambda}\ (x - Vt)) - \lambda\,. \tag{23}$$

In particular, if $C_1 = 0,\ C_2 = \sqrt{\sigma}$ and $\mu = 0$, we obtain the periodic wave solution

$$u(x,t) = -\lambda\tan^2(\sqrt{\lambda}\ (x - Vt)) - \lambda\tan(\sqrt{\lambda}\ (x - Vt))\sec(\sqrt{\lambda}\ (x - Vt)) - \lambda\,. \tag{24}$$

Case 3: When $\lambda = 0$ (rational function solutions)

Again following the similar to case 1 or case 2, substituting the value of $u(\xi)$, into (15) along with (3) and (9), we get the solution set as:

$$a_0 = -2\lambda, a_1 = 0, a_2 = -2, b_1 = 0, b_2 = 0, c = -4\lambda, \mu = 0\,. \tag{25}$$

This solution in turn provides the rational solution as:

$$u(x,t) = -\frac{2C_1^2}{(C_1(x - Vt) + C_2)^2} - 2\lambda\,, \tag{26}$$

where C_1 and C_2 are two arbitrary constants and $\xi = (x - Vt)$.

Burgers-Fisher equation

The Burgers-Fisher equation arises in field of applied mathematics and physics applications such as gas dynamics, traffic flow and financial mathematics and so on. This describes the interaction between the reaction mechanisms, convection effect, and diffusion transport. The Burgers-Fisher equation reads as [20]:

$$u_t - u_{xx} = uu_x + u(1-u)\,. \tag{27}$$

By substituting the wave variable $u(x,t) = u(\xi),\ \xi = x - Vt$, the Burgers-Fisher equation is converted to the ODE

$$Vu' + uu' + u'' + u(1-u) = 0\,. \tag{28}$$

Balancing the nonlinear term u^2 with the highest order derivative u'' gives again $m = 2$, that does not lead to any solution. However, balancing uu' with u'', gives $m = 1$. Therefore, the ansatz polynomial solution comes in the form:

$$u(\xi) = a_0 + a_1\phi + b_1\psi\,, \tag{29}$$

where, a_0, a_1, b_1 are arbitrary constants, which are to be determined.

Case 1: When $\lambda < 0$ (hyperbolic function solutions)

In the same way, as stated in *case 1* of standard KdV equation, substituting Eq. (29), along with Eqs. (3) and (5) into (28), the left hand side of (28) becomes a polynomial in ϕ and ψ. Again, setting each coefficient of this polynomial to zero, we find a set of algebraic equations for $V, a_0, a_1, b_1, \lambda$ and μ. For minimalism, the equations are not given herein. Solving these algebraic equations by Maple, the following values are obtained:

$$a_0 = \frac{1}{2},\ a_1 = 1, b_1 = \frac{1}{2}\sqrt{16\mu^2 + \sigma}, V = -\frac{5}{2}, \lambda = -\frac{1}{4}. \tag{30}$$

Considering, the particular values if $C_1 = 0,\ C_2 = \sqrt{-\sigma}$ and $\mu = 0$, we obtain the traveling wave solution as:

$$u(x,t) = \frac{1}{2}\tanh\left(\frac{1}{2}(x - Vt\right) + \frac{1}{2}i\,\mathrm{sec}h\left(\frac{1}{2}(x - Vt\right) + \frac{1}{2}. \tag{31}$$

Similarly, setting $C_1 = \sqrt{\sigma},\ C_2 = 0$ and $\mu = 0$, we extract another wave solution as:

$$u(x,t) = \frac{1}{2}\coth\left(\frac{1}{2}(x - Vt\right) + \frac{1}{2}\mathrm{csc}h\left(\frac{1}{2}(x - Vt\right) + \frac{1}{2}. \tag{32}$$

Case 2: When $\lambda > 0$ (trigonometric function solutions)

In a similar procedure, of *case 2* of standard KdV equation, substituting Eq. (29), along with Eqs. (3) and (7) into (28), the left hand side of (28) becomes a polynomial in ϕ and ψ. Solving the system of algebraic equations for $V, a, a_0, a_1, b_1, \lambda$ and μ we get (for shortness equations are not given):

$$a_0 = \frac{1}{2},\ a_1 = 1, b_1 = \frac{1}{2}\sqrt{16\mu^2 - \sigma}, c = -\frac{5}{2}, \lambda = -\frac{1}{4}. \tag{33}$$

It is interesting to note that for the case of $\lambda > 0$, the Burgers-Fisher equation provides the same solution as $\lambda < 0$, i.e., same as Eqs. (31) and (32).

Case 3: When $\lambda = 0$ (rational function solutions)

Following the previous steps, when $\lambda = 0$, the Burgers-Fisher equation provides solution as:

$$\left\{a_0 = a_0, a_1 = a_1, b_1 = 0, \mu = \frac{1}{2}\frac{C_1^2}{C_2}\right\}. \tag{34}$$

which in turn gives

$$u(x,t)=\frac{\left(\frac{1}{2}\frac{C_1^2\xi}{C_2}+C_1\right)}{\left(\frac{1}{4}\frac{C_1^2\xi^2}{C_2}+C_1\xi+C_2\right)}+a_0 \tag{35}$$

where, C_1 and C_2 are arbitrary constants and $\xi=x-Vt$.

Graphical representation of the obtaining results

In this section, we discuss the graphical explanation of the exact traveling wave solutions which are derived by the $(G'/G,1/G)$ -expansion method, particularly, to the exact solutions of the standard KdV equation. For briefness, we have only plotted the graphs of Eq. (20), (21) and (23) of the exact solution of the standard KdV equation in the figure (1), (2) and (3) respectively.

Taking, the absolute value of Eq. (20) and simplification, we get

$$|u(x,t)|=\operatorname{sec}h^2(x-Vt) \tag{36}$$

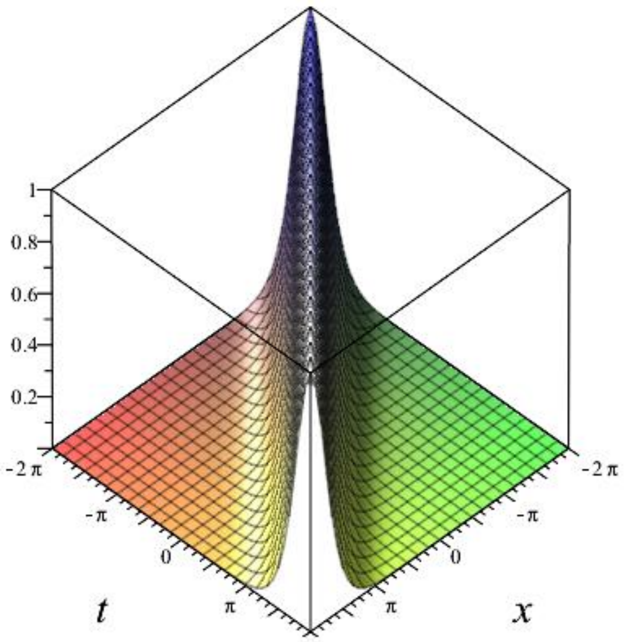

Fig-1: Shape of the solution Eq. (20) in the domain $-2\pi\le x,t\le 2\pi, V=1, \lambda=-1$

The shape of Fig. 1 is called the soliton solution which was named by Korteweg and de Vries during the investigation of water wave problem. The general form of soliton is either $\operatorname{sec}h^2$-bell shape or in the form of a kink in $\tanh$-mathematical form. Alternate form of soliton may appear in the mathematical form as $\operatorname{sec}h$ or $\arctan(e^{\alpha x})$. The Fig.1 looks like a single humped wave. Generally, this is a solitary wave of permanent form which is localized. Both wings of the graph decays and approaches a constant at infinity. Mathematically, the transition of the wave from the asymptotic state at $\xi=-\infty$ to the other asymptotic state $\xi=+\infty$ is localized in ξ, where $\xi=x-ct$, and c is the wave speed. Fig.1 is clearly characterized by infinite wings or infinite tails. The wave is spatially localized and hence $u'(\xi), u''(\xi)$ and $u'''(\xi)\to 0$ as $\xi\to\pm\infty, \xi=x-ct$. It is noticeable that the wave disturbance moving in the positive x-direction as the value of $V>0$. If we chose the negative value of V, the wave disturbance will moved towards the negative x-direction. From the standard KdV Eq.(14), the nonlinear term $(u_x)^2$ tends to steeping the wave whereas the dispersion term u_{xxx} spreads the wave out. The delicate balance between the weak nonlinearity of $(u_x)^2$ and the linear dispersion of u_{xxx} defines the formulation of soliton that consist of single humped wave.

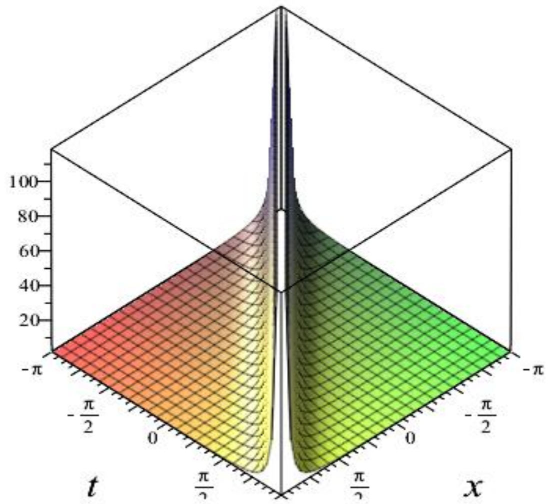

Fig-2: Graph of the solution Eq. (21) in the domain $-2\pi \le x, t \le 2\pi$

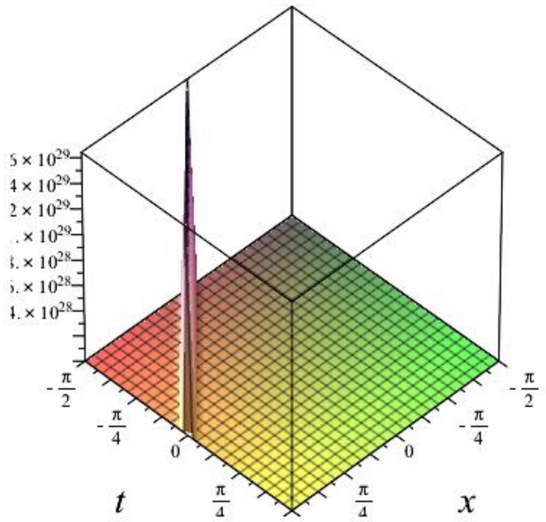

Fig-3: Graph of the solution Eq. (23) in the domain $-2\pi \le x, t \le 2\pi$

We have also depicted the traveling wave (21) of the standard KdV equation in the figure 2 The graph is plotted within the range of $-2\pi \le x, t \le 2\pi$ and the traveling wave velocity $V = 1.$ From the Fig.2, it is seen that the wave is highly peaked and vanishes from the one asymptotic state at $\xi = -\infty$ to the other asymptotic state $\xi = +\infty$ .The Fig.3 is depicted in the domain $-2\pi \le x, t \le 2\pi$. The Fig. 3 shows the periodic traveling wave nature of the standard KdV equation.

Results and Discussions

In this section, we discuss the advantages and disadvantages of $(G'/G, 1/G)$ -expansion method. The $(G'/G, 1/G)$ -expansion method is mainly an extension of (G'/G) method. The (G'/G) -method is based on the assumptions that the solution of the nonlinear evolution equations can be expressed by a polynomial in (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation (LODE) $G''(\xi) + \lambda G(\xi) + \mu G(\xi) = 0$ where

λ and μ are arbitrary constants. Alike to the (G'/G)-method, the principle of $(G'/G,1/G)$-expansion method is that the exact traveling wave solutions of NLEEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$ where $G = G(\xi)$ satisfies a second order linear ordinary differential Eq.(1). Making a homogeneous balance between the nonlinear term and highest order derivatives appear in a given NLEE we can determine the degree of the polynomial. Besides, the coefficients of the polynomial can be determined by solving a set of algebraic equations equating to zero resulting from the process of using the method. When $\mu = 0$ in Eq. (1) and $b_i = 0$ in Eq. (13), the $(G'/G,1/G)$ expansion method transforms into (G'/G)-expansion method. Moreover, if we take the special values of two parameters C_1 and C_2, soliton type of wave solution can be derived. Many techniques are available to searching the analytical solution of NLEEs and each of them have some advantages and disadvantages. Some methods provide solutions in a series form. In this case burning question arises to investigate the convergence of approximation series. For instance, Adomian decomposition method, variational iteration method [27], depends only on the initial conditions and converges to the exact solution of the problem. Some methods need linearization or to convert the inhomogenous boundary conditions to homogeneous, and so on [20]. In addition to, all numerical methods e.g. finite difference or finite element methods it is necessary to must have boundary and initial conditions. The main advantages of $(G'/G,1/G)$-expansion method over other methods are it can be applied directly without using linearization, perturbation or any other restrictive assumption.

CONCLUSION

In this paper, we tested the applicability of the $(G'/G,1/G)$-expansion method to the standard KdV and the Burgers-Fisher nonlinear evolution equations. There are several solutions are found by this ansatz method. We observe that when the parameters take certain special values, the soliton type and other trigonometric periodic traveling wave solutions are found. It is important to note that wave solutions to the standard KdV and the Burgers-Fisher that are obtained here were not found in the earlier studies. The above solutions might be fruitful to investigate the shallow water waves. The established results show that the $(G'/G,1/G)$-expansion method is effective and powerful to searching new or more general form of traveling wave solutions.

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