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Orbit of an Artificial Satellite in a Great-Circle Plane Relative to Spherical Rotating Earth

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Abstract **Original Research Article**

This paper mainly deals with the trajectory of a near-Earth satellite constrained to move in a great-circle plane by means of a propulsion system contained in it. The vector equations of motion of the satellite with respect to a rectangular frame rigidly fixed to the surface of the Earth are formed. Its trajectory relative to the rotating Earth is analyzed taking into consideration that it is acted on by two forces: gravitational force and coriolis force; the aerodynamic forces being very small are neglected. The component of the coriolis force perpendicular to the great-circle plane is balanced by the side thrust generated in the satellite at all time instants. Two numerical examples related to an elliptic orbit and a circular orbit respectively is cited. The centrifugal force being very small compared to the coriolis force is also neglected. Finally, an approximate formula has been derived to determine the drift of a satellite from its initial plane, ie, great-circle plane, if it is allowed to move freely for a short interval of time compared to its orbital period, ie, the satellite propulsion is shut off during that interval of time and the magnitude of deflection caused by the coriolis force has been numerically found in two cases.
Keywords: satellite, great-circle, Earth.

INTRODUCTION

This paper deals with trajectory of an artificial satellite that travels in the neighborhood of the Earth and is characterized by relatively small interval of time

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ie, small with respect to the period of revolution of the Earth around the Sun and the moon around the Earth. The trajectory of the satellite with reference to non-rotating frame with the origin rigidly fixed to the centre of the Earth is an unperturbed circular or elliptic orbit depending on the initial conditions of projection. The time period of such a satellite is 90 to100 minutes, far less than the time of the Earth's one revolution around the Sun. So linear motion of the Earth is negligible compared to that of the satellite.

For analysis of the trajectory relative to a frame rigidly attached to the rotating Earth, we have considered along with gravitational force, another force known as coriolis force which arises due to interaction of the Earth's rotation about the polar axis with the high velocity of the satellite. Motion of the satellite over the spherical rotating Earth is, however, discussed with assumptions that inertia terms associated with the transport (centrifugal) acceleration and effects of lift and drag are negligible as compared to the weight of the satellite. Because of high velocity of the satellite the coriolis force is much greater than the centrifugal force which is negligible.

Nomenclature

X, h =the distance on the Earth's surface of the satellite (o), its altitude respectively from the point E on the Earth's surface.

$\theta = \frac{X}{r_0}$= its true anamoly ,ie, angular travel from QE,Q the centre of the Earth

m=the mass of the satellite, r_0 = the radius of the Earth.

γ = the path inclination of the satellite to the horizon

h_1=the height of satellite at the time of launching. $r_1 = r_0 + h_1$

g_0=acceleration due to gravity on the Earth's surface.

V_1= initial velocity of projection of the satellite at an angle α to the vertical, relative to the Earth, $\mu = g_0 r_0^2$ = constant of gravitation.V= its velocity at time t.

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r=$r_0 + h$ =the radius vector joining the instantaneous position (o) of the satellite to the centre (Q) of the Earth.
(p_e, q_e, r_e)=the components of the angular velocity $\overrightarrow{\omega}$ of the Earth on the Earth axes.
$\overrightarrow{\omega} = ip_e$+j$q_e + kr_e$, (i,j,k) =unit vectors along the Earth axes in order $E_{X_e}, E_{Y_e}, E_{Z_e}$.
$(\hat{t}, \hat{n}, k)$= unit vectors directed along the wind axes in order $O_{x_w}, O_{y_w}, O_{z_w}$.
k' =unit vector along the polar axis, giving the direction of rotation of the Earth.
ω = The Earth's physical angular velocity about the polar axis; $\overrightarrow{\omega}$ =$k'\omega$,
$\emptyset = the$ angle of inclination of the polar axis with respect to the perpendicular to the great-circle plane.
P=the magnitude of the instantaneous balancing force directed perpendicular to the great-circle plane. $\hat{r}$ =the unit vector along QO. $\hat{\theta}$=unit vector perpendicular to QO, being positive in the direction of θ increasing. $\overrightarrow{g}$=g$\hat{r}$

System of axes

So as to deal with motion of the satellite in a great-circle plane let us define several co- ordinate systems and establish relationships which describe the position and orientation of one system with respect to another system.

Earth Axes System: The Earth axes system is a Cartesian reference frame rigidly attached to the surface of the Earth. The origin E is a point on the surface; the y_e axis is perpendicular to the x_e axis and is lying in the great-circle plane, while z_e axis is perpendicular to x_e y_e plane and is directed in such a way that trihedral Ex_e y_e z_e is a right-handed system. X_e axis is vertical and Is positive downwards.

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Wind Axes System:It is defined as follows: The x_w axis is the tangent to the flight path and is positive in the forward direction; the y_w axis is perpendicular to the x_w axis and is lying in the great-circle plane. Finally z_w axis is perpendicular to the $x_w y_w$ plane in such a way that trihedral O $x_w y_w z_w$ constitutes a right-handed system. Body axes meant to study orientation of thrust has no role to play in this design because of the tangent to the flight path and the thrust lying in the same vertical plane [1].

Equations of motion

The equations[2,3] governing motion of the satellite are given by

$$m\vec{g}=m(\frac{d\vec{v}}{dt}+2\vec{\omega} \times \vec{v}) \qquad (1)$$

where $\vec{g}$ is the acceleration due to gravity, t the time, $\vec{\omega}$ the angular velocity of the Earth with respect to the fixed stars and m the mass of the satellite: (vide Figure 1to 6)

$$\vec{v} = \frac{d\overline{EO}}{dt} \qquad (2)$$

represents the velocity of the satellite relative to the Earth, $\overrightarrow{EO}$ being the vector joining the point E on the surface of the Earth to the satellite O. m($2\vec{\omega} \times \vec{v}$) is the coriolis force, which vanishes when the flight is parallel to the polar axis and is maximum when it is perpendicular to this axis.In the Northern hemisphere the coriolis force tends to deflect a projectile from the initial plane of projection to the right of the direction of travel. In the Southern hemisphere the coriolis deflection being zero at the Equator, reverses its direction. The coriolis force is to be taken into consideration in case of vehicles travelling at satellite speed or vehicles approaching escape velocity. The magnitude of the coriolis acceleration with v as the satellite velocity, when maximum, is $2\omega v = 1.5 \times 10^{-4}$v

km/sec^2.g_0=.0098km/sec^2 on the Earth's surface If v=6km/sec,this coriolis force per unit mass is 9×10^{-4}km/sec^2 [2].

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The scalar equations associated with vectorial relationships (1) and (2) can be derived for the general case of paths in three- dimensions and in this case, are reduced to those relevant to flight in great- circle plane by employing necessary side thrust by means of a propulsion system carried by the satellite. If $\vec{P}$ be the instantaneous side thrust required to constrain the satellite to travel in the great-circle plane, equation (1) assumes the form[1]

$$\vec{P} + m\vec{g}= m (\frac{d\vec{v}}{dt}+2\vec{\omega} \times \vec{v}) \qquad (3)$$

where $i = -\hat{t}\sin(\gamma-\theta) - \hat{n}\cos(\gamma-\theta)$ (4)

$j = \hat{t}\cos(\gamma-\theta) - \hat{n}\sin(\gamma-\theta)$

which are obtained in conformity with Figure 1.

$r_e = \omega\cos\emptyset$ (5)

which denotes the component of the Earth's angular velocity ω along the z axis of the Earth axes system. Using equations (3),(4) and (5) the kinematic and dynamical relationships describing the satellite motion with its velocity V in great-circle plane can be obtained as

$\dot{r} = \dot{h} = V\sin\gamma$ (6)

$r\dot{\theta} = \frac{r\dot{X}}{r_0} = V\cos\gamma$ (7)

$\frac{\vec{P}}{m} + \vec{g} = \dot{\vec{v}} + 2(\vec{\omega}\times\vec{v})$

Or, $\frac{\vec{P}}{m} - \frac{\mu}{r^2}(\hat{t}\sin\gamma + \hat{n}\cos\gamma) = \dot{V}\hat{t} + V(\dot{\gamma} - \dot{\theta})\hat{n} + 2(i\,P_e + jq_e + kr_e)\times\hat{t}V$

$-\frac{\mu}{r^2}\sin\gamma = \dot{v}$ (8)

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$-\frac{\mu}{r^2}\cos\gamma = V[(\dot{\gamma} - \dot{\theta}) + 2\omega cos\emptyset]$ (9)

$\vec{P} = kP$ so that using (3), (4) and(8), we get

$P = 2mV[q_e\sin(\gamma - \frac{X}{r_0}) + p_e\cos(\gamma - \frac{X}{r_0})]$ (10)

Where the dot sign denotes the derivative with respect to time t. The side force P equal to the coriolis force, is a function of the satellite velocity and the components of the Earth's angular velocity along the Earth axes.

Solutions to the equations

The situation clearly indicates that the coriolis force perpendicular to the great circle plane has to be balanced by a side force to maintain the satellite in that plane. In fact this variable side force is necessary to convert three-dimensional motion into two-dimensional motion of the

satellite in a great − circle plane. Multiplying equations(8) and (9) by cosγ and sinγ respectively and the substracting one gets

$\dot{V}\cos\gamma - V(\dot{\gamma} - \dot{\theta} + 2\omega\cos\emptyset)\sin\gamma = 0,$

Using (6) and (7),

$\frac{d(r\dot{\theta})}{dt} + \dot{r}\dot{\theta} = 2\omega\dot{r}\cos\emptyset$

$\frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = 2\omega\dot{r}\cos\emptyset$

Or, $r^2\dot{\theta} = \omega r^2\cos\emptyset + A$

In order to find the constant A, we assume that initially when $r = r_1 = r_0 + h_1$,

$r^2\dot{\theta} = V_1\sin\alpha.$

$$\dot{\theta} = \omega\cos\emptyset + \frac{K}{r^2} \qquad (11)$$

where $K = r_1 V_1 \sin\alpha - \omega {r_1}^2 \cos\emptyset$ (11.1)

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Now combining equations(6) and (8) is obtained

$\frac{-\mu}{r^2}\frac{dr}{dt} = V\frac{dV}{dt}$, integrating which, $\frac{\mu}{r} = \frac{V^2}{2} + B$

Initially when $r = r_1, V = V_1$ so that the value of the constant B can be evaluated as $B = \frac{\mu}{r_1} - \frac{V_1^2}{2}$ so that

$$V^2 = V_1^2 + 2\mu\left(\frac{1}{r} - \frac{1}{r_1}\right) \qquad (12)$$

Using equations (6), (7),(11)and (12), equation (12) reduces to the form

$$\left(\omega\cos\emptyset + \frac{K}{r^2}\right)^2\left(\frac{dr}{d\theta}\right)^2 = V_1^2 + 2\mu\left(\frac{1}{r} - \frac{1}{r_1}\right) - r^2\left(\omega\cos\emptyset + \frac{K}{r^2}\right)^2$$

Since motion of the satellite is considered in the neighborhood of the Earth as such $r - r_1 \ll r_0,$ we can replace the small term $\omega^2 r^2$ by its initial value $\omega^2 {r_1}^2$or by $\omega^2\left(\frac{r_{max}+r_{min}}{2}\right)^2$ for approximation without sacrifice of the accuracy:

$$d\theta = \frac{\left(\omega\cos\emptyset + \frac{K}{r^2}\right)dr}{\sqrt{V_1^2 + 2\mu\left(\frac{1}{r} - \frac{1}{r_1}\right) - 2\omega K\cos\emptyset - \frac{K^2}{r^2} - \omega^2 {r_1}^2 \cos^2\emptyset}}$$

$$= \frac{-Kd\left(\frac{1}{r}\right)}{\sqrt{-C - \frac{K^2}{r^2} + \frac{2\mu}{r}}} + \frac{\omega\cos\emptyset\, dr}{\sqrt{-C - \frac{K^2}{r^2} + \frac{2\mu}{r}}} \qquad (13)$$

where $-C = V_1^2 - 2\mu\left(\frac{1}{r_1}\right) - 2\omega K\cos\emptyset - \omega^2 {r_1}^2 \cos^2\emptyset$ (13.1)

$\sqrt{\frac{2\mu}{r_1}}$ =Escape velocity

$$V_1^2 < 2\mu\left(\frac{1}{r_1}\right) + 2\omega K\cos\emptyset + \omega^2 {r_1}^2 \cos^2\emptyset$$

Integrating θ from 0 to θ and r from r_1 to r we get

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$$\theta = \int_{r_1}^{r} \frac{-d\left(\frac{K}{r} - \frac{\mu}{K}\right)}{\sqrt{\frac{\mu^2}{K^2} - C - \left(\frac{K}{r} - \frac{\mu}{K}\right)^2}} + \omega\cos\emptyset \int_{r_1}^{r} \frac{rdr}{\sqrt{-Cr^2 + 2\mu r - K^2}}$$

$$= \cos^{-1}\frac{\frac{K}{r} - \frac{\mu}{K}}{\sqrt{\frac{\mu^2}{K^2} - C}} - \cos^{-1}\frac{\frac{K}{r_1} - \frac{\mu}{K}}{\sqrt{\frac{\mu^2}{K^2} - C}} + \omega I\cos\emptyset$$

where $I = \int_{r_1}^{r} \frac{rdr}{\sqrt{-Cr^2 + 2\mu r - K^2}} = \frac{1}{\sqrt{C}} \int_{r_1}^{r} \frac{\left\{\left(r - \frac{\mu}{C}\right) + \frac{\mu}{C}\right\}dr}{\sqrt{\left(\frac{\mu^2}{C^2} - \frac{K^2}{C}\right) - \left(r - \frac{\mu}{C}\right)^2}}$ $\left(\frac{\mu^2}{C^2} - \frac{K^2}{C} > 0 \quad \text{or } \mu^2 > K^2 C\right)$

$$=\frac{1}{\sqrt{C}}\left[\left\{\sqrt{\left(\frac{\mu^2}{C^2}-\frac{K^2}{C}\right)-\left(r_1-\frac{\mu}{C}\right)^2}-\sqrt{\left(\frac{\mu^2}{C^2}-\frac{K^2}{C}\right)-\left(r-\frac{\mu}{C}\right)^2}\right\}+\frac{\mu}{C}\left\{\cos^{-1}\frac{r_1-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}-\cos^{-1}\frac{r-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}\right\}\right]$$

$$=\frac{1}{C}\left[\sqrt{2\mu r_1-K^2-Cr_1^2}-\sqrt{2\mu r-K^2-Cr^2}\right]+\frac{\mu}{C^{\frac{3}{2}}}\left\{\cos^{-1}\frac{r_1-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}-\cos^{-1}\frac{r-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}\right\}$$

Again, utilizing equations (6), (7),(11) and (12)along with approximate relationship $\omega^2r^2=\omega^2r_1^2$ one can write

$$\dot{r}^2+2\omega K\cos\emptyset+\frac{K^2}{r^2}=V_1^2+2\mu\left(\frac{1}{r}-\frac{1}{r_1}\right)-\omega^2r_1^2\cos^2\emptyset$$

Or, $\dot{r}^2=\frac{2\mu r-K^2-Cr^2}{r^2}$, hence $I=t=\int_{r_1}^{r}\frac{rdr}{\sqrt{-Cr^2+2\mu r-K^2}}$ (At t=0,r=r_1)

$$t=\frac{1}{C}\left[\sqrt{2\mu r_1-K^2-Cr_1^2}-\sqrt{2\mu r-K^2-Cr^2}\right]+\frac{\mu}{C^{\frac{3}{2}}}\left\{\cos^{-1}\frac{r_1-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}-\cos^{-1}\frac{r-\frac{\mu}{C}}{\sqrt{\frac{\mu^2}{C^2}-\frac{K^2}{C}}}\right\} \quad (14)$$

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Now θ can be expressed as

$$\theta=\cos^{-1}\frac{\frac{K}{r}-\frac{\mu}{K}}{\sqrt{\frac{\mu^2}{K^2}-C}}-\epsilon+\omega t\cos\emptyset$$

where $\cos^{-1}\frac{\frac{K}{r_1}-\frac{\mu}{K}}{\sqrt{\frac{\mu^2}{K^2}-C}}=\epsilon$

$$\frac{\frac{K}{r}-\frac{\mu}{K}}{\sqrt{\frac{\mu^2}{K^2}-C}}=\cos(\theta+\epsilon-\omega t\cos\emptyset)$$

Or $$r=\frac{\frac{K^2}{\mu}}{1+\sqrt{1-\frac{CK^2}{\mu^2}}.\cos\left(\frac{X}{r_0}+\epsilon-\omega t\cos\emptyset\right)} \quad (15)$$

Using equations (14) and (15) we can find out the instantaneous anomaly θ and altitude h of the satellite as functions of time t.The trajectory of the satellite in the great-circle plane can be determined by eliminating t between equations(14) and (15).But employing equations(7),(11) and (12) we get

$$\cos\gamma=\frac{\omega r\cos\emptyset+\frac{K}{r}}{\sqrt{V_1^2+2\mu\left(\frac{1}{r}-\frac{1}{r_1}\right)}} \quad (16)$$

Now combining equations (10) and (12) we determine the magnitude of the instantaneous side force given by

$$P=2m\sqrt{V_1^2+2\mu\left(\frac{1}{r}-\frac{1}{r_1}\right)}\left\{q_e\sin\left(\gamma-\frac{X}{r_0}\right)+p_e\cos\left(\gamma-\frac{X}{r_0}\right)\right\} \quad (17)$$

Hence the instantaneous side force or thrust can be found out as a function of time by means of equations (14) to (17). Thus equation (17) can be employed a

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Posteriori in order to predict the time history of the side force required to maintain the satellite motion in the great-circle plane. Thrust generated by the propulsion system carried by the satellite can cater to such force.

Discussion

If we consider coriolis force as a perturbation,the basic orbit of the satellite is an ellipse represented by the equation

$$r=\frac{\frac{K'^2}{\mu}}{1+\sqrt{1-\frac{C'K^2}{\mu^2}}.\cos(\frac{X}{r_0}-\epsilon')} \qquad (18)$$

where $C'=2\frac{\mu}{r_1}-V_1^2,\ K'=r_1V_1\sin\alpha,\ \epsilon'=\cos^{-1}(\frac{\frac{K'}{r_1'}-\frac{\mu}{K'}}{\sqrt{\frac{\mu^2}{K'^2}-C'}}$ (18.1)

It can be mentioned that the same equation is obtained when we consider motion of the satellite with respect to the fixed stars or with respect to the non-rotating Earth where $\omega=0.$ Hence equation (15)stands for a perturbed orbit which is obtained on consideration variation of the basic elliptic orbit defined by three elements:

(1) Its eccentricity (2) the direction of the major axis (3) length of the major axis. It is usually assumed in Astronomical work that the perturbed orbit remains an ellipse but the elements continuously vary, ie, the real orbit is split up into infinitesimal lengths of a myriad of different ellipses. Hence the equation of the perturbed orbit which is an ellipse can be in the present contest rewritten as

$$r=\frac{\frac{K^2}{\mu}}{1+\sqrt{1-\frac{CK^2}{\mu^2}}.\cos(\frac{X}{r_0}-\beta)} \qquad (19)$$

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where $\beta(=\omega t\cos\emptyset-\epsilon)$ governs the rotation of the major axis such that $\frac{d\beta}{dt}=\omega\cos\emptyset.$ Therefore the major axis rotates with an angular velocity $\omega\cos\emptyset$ in the orbital plane. Figures 1 to 6 depict different configurations at different stages.

NUMERICAL EXAMPLE 1

Let us take the case of a satellite constrained to orbit in a great-circle plane as discussed in the foregoing analysis where

V_1 = 8Km/sec, $=85^0 33'$, $\emptyset=60^0$,r_1 =6800Km, $\omega=7.5\times10^{-5}$ radians/second, r_0 =6378Km, g_0 =.0098Km/sec^2 .

Then K= $=r_1V_1\sin\alpha-\omega {r_1}^2 cos\emptyset=52638.8\text{Km}^2$ /sec , $\frac{d\beta}{dt}=\omega\cos\emptyset=3.75\times10^{-5}$radians/sec, $\mu=g_0r_0^2=398654.2$ $\frac{\text{Km}^3}{sec^2}$, $C=-V_1^2+2\mu\left(\frac{1}{r_1}\right)+2\omega K cos\emptyset+\omega^2{r_1}^2\cos^2\emptyset$ =57.26 Km^2 $/sec^2$, $\frac{CK^2}{\mu^2}=.9982$, $\sqrt{1-\frac{CK^2}{\mu^2}}$ =.0424 , $\frac{K^2}{\mu}=$ 6950Km. From equation (15) the minimum and maximum values of r :

$$r_{min}=\frac{6950}{1.0424}=6667Km$$

$$r_{max}=\frac{6950}{.9576}=7258\text{Km}$$

Hence the minimum and maximum altitudes of the satellite are 289 and 880 kilometers respectively. With such data the orbit of the satellite becomes an ellipse whose major axis rotates at the rate of 3.75radians /sec.

Example 2

In this example we deal with a nearly circular orbit where

V_1=7.9965 Km/sec, $\alpha=90^0$

The rest of the data remain the same as in the example1.

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K=52601.6, C=57.31, $\frac{CK^2}{\mu^2} = 1(\text{approx})$, $\frac{K^2}{\mu}$=6782Km.

Hence the orbit of the satellite is roughly circular with radius equal to 6800 Km.

Three–dimensional motion of the satellite of short duration

In the last section of the paper we study three-dimensional motion of the satellite relative to the Earth for a short interval of time as compared to its period of revolution around the Earth. Such a maneuver can be accomplished by shutting off the propulsion engine of the satellite so that it can travel freely leading to its three-dimensional motion as it gradually deviates from the original great- circle plane of motion. Since the free movement of the satellite is of short duration, the orientation of the plane of its motion is so small that the component P of the coriolis force can be assumed to act along Z_e axis (perpendicular to the great-circle plane) during that interval of time [3].

The instantaneous position of the satellite can approximately be defined by the cylindrical co-ordinates (r,θ, z) where z is its distance from the great − circle plane, r the radius vector joining the centre of the Earth to the projection of the satellite on the great-circle plane and θ its angular travel from x_e axis(QE)so that (r, θ)can be roughly determined as functions of time t from equations (14)and (15).So as to find the coriolis deviation of the satellite from its initial plane of motion, equation (10) gives

$$\frac{d^2z}{dt^2}=2V[q_e\sin(\gamma - \frac{X}{r_0}) + p_e \cos(\gamma - \frac{X}{r_0})] =2V[(q_e cos\theta + p_e sin\theta)\sin\gamma - (q_e sin\theta - p_e cos\theta)\cos\gamma]$$ (Using equations (6) and (7))

$$\frac{d^2z}{dt^2} = 2[\dot{r}\cos(\theta - \epsilon_0) - r\dot{\theta}\sin(\theta - \epsilon_0)]\sqrt{p_e^2 + q_e^2} \qquad (20)$$

where $\tan\epsilon_0 = \frac{p_e}{q_e}$ and $\sqrt{p_e^2 + q_e^2} = \omega\sin\emptyset$ (21)

If initially at t=0; z=$\dot{z}$=0, $\theta = \theta_2$ and r= r_2 ,then

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$$\dot{z} =\{2\omega r\cos(\theta - \epsilon_0) - 2\omega r_2\cos(\theta_2 - \epsilon_0)\}\sin\emptyset \qquad (22)$$

Using equation (11) and neglecting the small term $\omega^2\cos\emptyset$, one can write

$$\frac{dz}{d\theta} = \frac{2r^3\omega\cos(\theta-\epsilon_0)\sin\emptyset}{K} - \frac{2r^2\omega r_2\cos(\theta-\epsilon_0)\sin\emptyset}{K} \qquad (23)$$

Using equation (15) where a=$\frac{K^2}{\mu}$ and

b=$\sqrt{1 - \frac{CK^2}{\mu^2}}$ and neglecting the small term $\omega t_2\cos\emptyset$, we get

$$\frac{dz}{d\theta} = \frac{2a^3\omega\cos(\theta-\epsilon_0)\sin\emptyset}{K\{1+bcos(\theta+\in)\}^3} - \frac{2a^2\omega r_2\cos(\theta_2-\epsilon_0)\sin\emptyset}{K\{1+bcos(\theta+\in)\}^2}$$

Recalling the value of r_2 from (15)wherein $r = r_2$ and $t = t_2$, substituting θ+∈= λ and integrating we get

$$z=\int\left[\frac{2a^3\omega\cos\{\{\lambda-(\in+\epsilon_0)\}\sin\emptyset}{K\{1+bcos\lambda\}^3} - \frac{2a^2\omega r_2\cos(\in+\epsilon_0)\sin\emptyset}{K\{1+bcos\lambda\}^2}\right]d\lambda$$

$$=\frac{2a^3\omega\sin\emptyset}{K}\left[\int\frac{\cos\{\{\lambda-(\in+\epsilon_0)\}d\lambda}{\{1+bcos\lambda\}^3} - \frac{\cos(\theta_2-\epsilon_0)}{1+b\cos(\theta_2+\in)}\int\frac{d\lambda}{\{1+bcos\lambda\}^2}\right]$$

$$=\frac{2a^3\omega\sin\emptyset}{K}\left[\sin(\in+\epsilon_0)\int\frac{\sin\lambda d\lambda}{\{1+bcos\lambda\}^3} + \cos(\in+\epsilon_0)\int\frac{\cos\lambda d\lambda}{\{1+bcos\lambda\}^3}\right.$$

$$\left.-\frac{\cos(\theta_2-\epsilon_0)}{1+b\cos(\theta_2+\in)}\int\frac{d\lambda}{\{1+bcos\lambda\}^2}\right]$$

$=\frac{2a^3\omega\sin\emptyset}{K}\left[\frac{\sin(\epsilon+\epsilon_0)}{2b\{1+b\cos\lambda\}^2}-\frac{\cos(\epsilon+\epsilon_0)}{b}\{\int\frac{d\lambda}{\{1+b\cos\lambda\}^2}-\int\frac{d\lambda}{\{1+b\cos\lambda\}^3}\}-\frac{\cos(\theta_2-\epsilon_0)}{\{1+b\cos(\theta_2+\epsilon)\}}\int\frac{d\lambda}{\{1+b\cos\lambda\}^2}\right]$ +(constant)

(24)

where $I_2=\int\frac{d\lambda}{\{1+b\cos\lambda\}^2}=\frac{-b\sin\lambda}{(1-b^2)(1+b\cos\lambda)}+\frac{I_1}{(1-b^2)}$

$I_3=\int\frac{\cos\lambda d\lambda}{\{1+b\cos\lambda\}^3}=\frac{-b\sin\lambda}{2(1-b^2)\{1+b\cos\lambda\}^2}+\frac{3I_2}{2(1-b^2)}-\frac{I_1}{2(1-b^2)}$

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$I_1=\int\frac{d\lambda}{1+b\cos\lambda}=\frac{2}{\sqrt{1-b^2}}tan^{-1}\,(tan\frac{\lambda}{2})\sqrt{\frac{1-b}{1+b}}$

Value of the constant can be evaluated by use of the initial conditions. Thus relationship (24) is an approximate expression for the drift of the satellite from the great-circle plane during a short interval of time of shutting off the propulsion engine.

Numerical example 3

In order to cite a simple numerical example relating to equation (24), let us take the case of the satellite having the conditions of projection:

$\epsilon+\epsilon_0=\theta_2-\epsilon_0=90^0$, and the same orbital elements given in the example1.Now the magnitude of the linear deflection of the satellite , perpendicular to its initial plane of motion in the light of the foregoing theory can be calculated as

$z_1=\frac{2a^3\omega\sin\emptyset}{Kb}\left[\frac{1}{(1+b\cos\lambda_1)^2}-\frac{1}{(1+b\cos\lambda_2)^2}\right]$ (where $\lambda_1=180^0$ and $\lambda_2=188^0$)
$=-42.4$Km.

The minus sign indicates that the deflection takes place towards the Southern Hemisphere. The time taken by the satellite for this travel can be determined with the help of equations (14) and (15).

Numerical example 4

In the last numerical example we compute the deviation of the satellite after five minutes of its free motion while it was originally executing circular orbit in a great-circle plane as referred to in example 2.If $\epsilon_0=90^0$ and initially at t=0, z=0,$\dot{z}$=0, the solution to equation (20) takes up the form

$z=-2V_1\omega\sin\emptyset[\frac{\cos\Omega t}{\Omega^2}]_0^t=\frac{4V_1\omega}{\Omega^2}sin\phi sin^2(\frac{\Omega t}{2})$ (25)

where $\Omega=\dot{\theta}=\frac{V_1}{r_1}=$ constant for circular orbit. Hence for small values of t,

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$z=V_1\omega t^2\sin\emptyset$, $\sin(\frac{\Omega t}{2})=\frac{\Omega t}{2}$ (26)

Using the data mentioned in example2, deflection at t= 5 minutes is z=46.8 Km

Propell ant consumption

In order to maintain the satellite orbit in a great-circle plane, thrust is applied leading to consumption of propellant carried by it. The thrust equation is given by

$P=-V_E\frac{dm}{dt}$ (27)

where m is the instantaneous mass of the satellite and V_E its exhaust velocity, ie, velocity of the gaseous elements emitted through the nozzles.

Combining equation (27) with equation (6),(7) and(10) is obtained

$V_E \frac{dm}{dt} = -2Vm\omega[\dot{r}\cos(\theta - \epsilon_0) - r\dot{\theta}\sin(\theta - \epsilon_0)]\sin\dot{\emptyset}$ (27.1)

Introducing the initial conditions: t=0, $\theta = 0, m = m_0$ and r= $r_1 = r_0 + h_1$, solution to equation (27.1) becomes

$\log\frac{m}{m_0} = \frac{-2\omega a'}{V_E}[\frac{\cos(\theta-\epsilon_0)}{1+b'\cos(\theta+\epsilon')} - \frac{\cos\epsilon_0}{1+b'\cos(\theta+\epsilon')}]\sin\emptyset$ (28)

where $a' = \frac{K'^2}{\mu}$ and $b' = 1 - \frac{C'K'^2}{\mu^2}$ (29)

The values of C', K', ϵ' are mentioned along with equation (18) and the satellite travels in the same hemisphere. For each initial mass m_0, altitude h_1, launch velocity V_1, angular travel θ, exhaust velocity V_E of the satellite, latitude χ of the reference point E depending on ϵ_0 and angle $\emptyset$ of inclination of the orbital plane to the Equator there exists an angle α of projection of the satellite such that propellant consumption is minimum. By use of (28). it can be found that

$\frac{d}{d\alpha}\left(\log\frac{m}{m_0}\right) = 0$ when $\sin 2\alpha = 0$, ie, $\alpha = 90^0$ or 0

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$\frac{d^2}{d\alpha^2}\left(\log\frac{m}{m_0}\right) < 0$ *when* $\alpha = 90^0$ *(30)*

Hence m is maximum,ie, propellant consumption is minimum if the satellite is launched horizontally while other aspects of the orbital program are pre-assigned.

Let us determine the propellant consumption during a complete orbital period, ie, the time taken by the satellite to complete one revolution around the Earth with respect to the line QE rigidly fixed to the Earth. Propellant consumption of the satellite during its travel in the Northern Hemisphere can be determined by integrating equation (27.1) with respect to θ *from* $\theta = \pi + \epsilon_0$ *to* $\theta = \epsilon_0$. Since the coriolis force perpendicular to the great circle-plane reverses its direction as soon as the satellite moves into the Southern Hemisphere. From the Northern Hemisphere or vice-versa, the thrust required to balance this coriolis force is to be applied in the reversed direction. By means of equation (29) it can be readily shown that the propellant consumption during the travel in the Northern Hemisphere is equal to that in the Southern Hemisphere and as such the total propellant consumption can be obtained as

$\mathrm{Log}(1 - \rho_{2\pi}) = \frac{-8a'\omega\sin\emptyset}{V_E\{1-b'^2\cos^2(\epsilon_0+\epsilon')\}}$ (31)

where $\rho_{2\pi}$ is the propellant consumption to initial mass ratio. It can easily be verified that the propellant consumption is minimum for initial angle of projection $\alpha = 90^0$, the other elements in equation (31) being pre-chosen. In the following analysis we have expressed the components p_e and q_e of the Earth's angular velocity in terms of the latitude of the place E of observation and have proved that the angular travel of the satellite at the time of crossing the Equatorial plane is ϵ_0.

Let us assume a right-handed frame of axes $QX'Y'Z$ fixed in space such that QX' is the line of intersection of the great-circle plane and the Equatorial plane and is perpendicular to QY' lying in the former plane while finally QZ is

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Perpendicular to QX' and QY'. See Figure 5.If (i', j', k) be the unit vectors along QX',

QY', QZ respectively then the unit vector along QE is given by

$-i = i'\cos\psi + j'\sin\psi$ (32)

where ψ is the angle beteen the lines QE and QX',

$k' = k\cos\phi + j'\sin\phi$ (33)

The dot product of (32) and (33) yields

$-i.k' = \cos\chi = \sin\phi\sin\psi$ (34)

Then p_e and q_e *are* given by

$p_e = -\omega\sin\phi\sin\psi$ *and* $q_e = -\omega\sin\phi\cos\psi$ (35)

$\frac{p_e}{q_e} = \tan\psi = \tan\epsilon_0$, ie, $\psi = \epsilon_0$

Hence the angular displacement of the satellite, when it cuts the line QX', ie, it crosses the Equatorial plane, is equal to ϵ_0. Combing (34) and (35), we get

$p_e = -\omega\sin\chi$ and $q_e = -\omega\sqrt{\sin^2\phi - \cos^2\chi}$ (36)

Thrust analysis

Recollecting equation (10) and (35) we observe that the coriolis force perpendicular to the great-circle plane disappears when

$\tan(\gamma - \theta) = -\frac{p_e}{q_e} = -\tan\epsilon_0$

$\gamma - \theta = 180^0.n - \epsilon_0$ $(n = 0,1,2,3 \ldots \ldots \ldots \ldots \ldots)$

ie, the thrust reduces to zero at that instant. As referred to in Figure 6 the tangent to the satellite path makes an angle $90^0 + ((\gamma - \theta))$ *with QE. Hence the* instant at which the thrust vanishes, it

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Makes an $90^0 + (180^0.n - \epsilon_0)$ where $(n = 0,1,2 \ldots..)$ with the line QE, ie, the satellite travels at right angles to the nodal line QX' (Figure 6). Now let us find the time at which the instantaneous thrust is maximum for a given exhaust velocity and other given initial conditions.

$$\frac{dP}{dt} = 2(V\frac{dm}{dt} + m\frac{dV}{dt})\{q_e \sin(\gamma - \theta) + p_e \cos(\gamma - \theta)\}$$

$$+2mV\{q_e \sin(\gamma - \theta) - p_e \cos(\gamma - \theta)\}\frac{d(\gamma-\theta)}{dt} = 0$$

Employing equations (8), (9) and (27) and neglecting the term containing ω^2,

$$\frac{2V^2}{V_E}\{q_e \sin(\gamma - \theta) + p_e \cos(\gamma - \theta)\}^2 = -\frac{\mu\omega}{r^2}\cos(\theta - \epsilon_0)\sin\emptyset \qquad (37)$$

where $\theta - \epsilon_0 > 90^0$

Hence the time at the end of which the instantaneous thrust is maximum during travel of the satellite in the same Hemisphere can be determined by solving simultaneous equations (12), (14),(15) and (37):

$$P_{max} = m_M\sqrt{-\frac{(2/\mu\omega)V_E}{r_M^2}\cos(\theta_M - \epsilon_0)\sin\emptyset}$$

where $(\theta_M - \epsilon_0) > 90^0$ and the thrust is maximum when m=m_M, r=r_M and $\theta = \theta_M$ Which can be evaluated as functions of time t by solving the simultaneous equations (12),(14),(15), (28) and (37).

Conclusion

In the meridian plane where $\emptyset = 90^0$, coriolis affect leading to deflection of the satellite is maximum. But there is no such effect in the Equatorial plane$(\emptyset = 0)$and no side force is required to hold the satellite in that plane. The present analysis can only approximately be applied for practical purpose as the shape of

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The Earth deviates from a perfect sphere, as a result of which, the gravitational field is not of that value assumed in this paper and moreover the effects of aerodynamic forces even at high altitudes (200-600Kms) cannot be totally neglected. The deviation of the gravitational field from the Inverse-square law and the aerodynamic forces, however small may be, may cause perturbations to the satellite orbit. Hence the balancing force required to constrain the satellite to move in a great-circle plane may not be exactly given by equation (17). However, discussion about a satellite falling into the atmosphere is out of scope in this paper.

As the satellite goes on orbiting the Earth, it happens that it is gradually dragged more and more into the Earth's atmosphere and is ultimately burnt to decay following the tremendous amount of heat generated owing to the friction between the atmosphere and surface of the satellite. Finally it can be noted that instantaneous orbital plane of a free satellite, defined by the plane inclined at a constant angle to the Equator which instantaneously contains the satellite, rotates about the polar axis.

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