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Weighted Inequalities of the Vector-Valued Intrinsic Square Functions on Herz-Morrey Spaces with Variable Exponent

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Abstract **Review Article**

Based on the properties of function spaces with variable exponents, applying the real variable theory of harmonic analysis, as well as the Hölder and Jensen inequalities to estimate the main formulas, by the properties of intrinsic square functions, we proved the weighted inequalities of the vector-valued intrinsic square functions on Herz-Morrey spaces with variable exponents.
Keywords: Variable exponent; intrinsic square function; Herz-Morrey space; weighted.

INTRODUCTION

In the 1930s, Orlicz [1] introduced a class of variable exponential Lebesgue spaces $L^{p(\cdot)}$ that generalize the classical Lebesgue spaces. In 1991, the generalized Hölder inequality in $\mathbb{R}^n$ was given by Kováčik and Rákosnik[2], and the basic properties of variable exponential Lebesgue spaces $L^{p(\cdot)}$ were described in detail. This made a breakthrough in the study of variable exponential Lebesgue spaces $L^{p(\cdot)}$, and attracted extensive attention of scholars (see [3-12]).

As we all know, the classical A_p weighted of Muckenhoupt[13] is widely used in harmonic analysis and partial differential equations. Using Muckenhoupt's classical A_p weighted, Cruz-Uribe, Fiorenza and Neugebauer defined a new class of $A_{p(\cdot)}$ type condition on weighted in [14]. The equivalence conditions for weighted and the boundedness of Hardy-Littlewood maximal operators on variable exponential Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ were given. Influenced by the development of variable exponential function spaces, in 2009, Izuki [15] introduced Herz spaces and Herz-Morrey spaces with variable exponents, and extended the classical Herz spaces and Herz-Morrey spaces to the case of variable exponents. The boundedness of a class of vector-valued sublinear operators on these two classes of spaces was also proved. The intrinsic square function was introduced by Wilson [16] in 2007. This function was a kind of Littlewood-Paley operator, which is an improvement of classical Littlewood-Paley area function. At the same time, the author showed the definition of vector-valued intrinsic square function, which included Littlewood-Paley g-function and its deformed Littlewood-Paley g_λ^*-function. Next, Wang Hua studied the boundedness of intrinsic square functions in weighted Morrey spaces and weighted Herz spaces in [17-19]. The boundedness of β-order intrinsic square function from weighted Herz space to weighted weak Herz space was also obtained. Kwok-Pun Ho [20] proved the boundedness of vector-valued singular integral operators and Fefferman-Stein vector-valued maximal operators on Morrey spaces with variable exponents. Recently, Izuki and Noi using the properties of $A_{p(\cdot)}$ type weighted and variable exponential space in [21], gave the boundedness of the β-order intrinsic square function on homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$. The results for the intrinsic square function can also be found in [22]. Here, we are interested in the boundedness of order vector-valued intrinsic square functions on Herz-Morrey spaces with variable exponents, and obtain satisfactory weighted inequalities.

In the whole article, for measurable sets $S \subset \mathbb{R}^n$, we use $|S|$ to denote the Lebesgue measure of S and χ_s to denote the characteristic function of S. We also let $B = \{y \in \mathbb{R}^n : |x-y| < r\}$ to denote the sphere which centered at x radius of r. The constant C may take different values for different conditions but have nothing to do with the main parameters.

PRELIMINARIES

Firstly, we show the definition of variable exponential Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 1.1 [5] Let $p(\cdot): \mathbb{R}^n \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponents is as follows:

There exists $\lambda > 0$ and a set of measurable functions f such that $\int_{\mathbb{R}^n} (\frac{|f(x)|}{\lambda})^{p(x)} dx < \infty$. And $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} (\frac{|f(x)|}{\lambda})^{p(x)} dx \le 1\}$.

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space with the norm $\|\bullet\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Obviously, if $p(x) = p_0$ (constant), then $L^{p(\cdot)}(\mathbb{R}^n)$ is classical Lebesgue space $L^{p_0}(\mathbb{R}^n)$.

For $p(\cdot): \mathbb{R}^n \to (0,\infty)$, let $p_- = \text{essinf}\{p(x) : x \in \mathbb{R}^n\}$, $p_+ = \text{esssup}\{p(x) : x \in \mathbb{R}^n\}$.
Suppose $\mathcal{P}(\mathbb{R}^n)$ is all of $p(\cdot)$ that satisfying $1 < p_- \le p(x) < p_+ < \infty$ and $\mathcal{P}^0(\mathbb{R}^n)$ is all of $p(\cdot)$ that satisfying $0 < p_- \le p(x) < p_+ < \infty$.

$\mathcal{B}(\mathbb{R}^n)$ is all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfying Hardy-Littlewood maximal operator M bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. We use $p'(\cdot)$ to denote that the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Let $p(\cdot) \in \mathcal{P}$. (1) If there is a constant C_0 such that $|p(x) - p(0)| \le \frac{C_0}{\log(e + 1/|x|)}$, $x \in \mathbb{R}^n$.
Then we say that $p(\cdot)$ is log-Hölder continuous at the origin, and denote by $p(\cdot) \in LH_0$.
(2) If there exists p_∞ and a constant $C_\infty > 0$, such that $|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}$, $x \in \mathbb{R}^n$,
where $p_\infty = \lim_{x\to\infty} p(x)$. Then we say $p(\cdot)$ is log-Hölder continuous at the infinity, and denote by $p(\cdot) \in LH_\infty$. If $p(\cdot) \in LH_0 \cap LH_\infty$, we denote by $p(\cdot) \in LH$. Obviously, if $p(\cdot) \in LH$, then $p'(\cdot) \in LH$.

Secondly, we give the definition of $A_{p(\cdot)}$ type weighted [14]: Let ω be a locally integrable non-negative function defined in $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

If $\sup_B \| \omega^{\frac{1}{p(\cdot)}} \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| \omega^{-\frac{1}{p(\cdot)}} \chi_B \|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty$, then ω is called $A_{p(\cdot)}$ type weighted function. If $p(\cdot)$ is a constant and $p \in (1,\infty)$, the above definition is equivalent to the A_p weighted of Muckenhoupt.

Definition 1.2[14] Let $\omega \in A_{p(\cdot)}$. Then the weighted variable Lebesgue space $L^{p(\cdot)}(\omega)$ is defined as $L^{p(\cdot)}(\omega) = L^{p(\cdot)}(\mathbb{R}^n, \omega^{\frac{1}{p(\cdot)}})$, whose norm is $\| f\|_{L^{p(\cdot)}(\omega)} = \| f \omega^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}}$.

Let $B_k = B(0, 2^k)$, $C_k = B_k \setminus B_{k-1}$, $\chi_k = \chi_{C_k}$, $k \in \mathbb{Z}$. Based on $A_{p(\cdot)}$ type weighted, we give the definition of weighted variable exponential Herz-Morrey space. Suppose $\Omega \subset \mathbb{R}^n$ is a measurable set,

$p(\cdot): \Omega \to [1, \infty)$ is a measurable function and ω is a non-negative local integrable function defined on Ω. $L^{p(\cdot)}_{loc}(\Omega, \omega^{\frac{1}{p(\cdot)}})$ is a set that contains all the functions f satisfying the following condition: for any measurable set $E \subset \Omega$, there exists a constant $\lambda > 0$, such that $\int_E |\frac{f(x)}{\lambda}|^{p(x)} \omega(x)dx < \infty.$

Definition 1.3[22] Let $\alpha \in \mathbb{R}$, $0 \le \lambda < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, ω be a weighted function. Then the weighted variable exponential Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)$ is defined as

$$M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega) = \{ f \in L^{q(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}, \omega^{\frac{1}{q(\cdot)}}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)} < \infty \},$$

where $\| f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} (\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f \chi_k\|^p_{L^{q(\cdot)}(\omega)})^{\frac{1}{p}}$. Obviously, $M\dot{K}^{\alpha,0}_{p,q(\cdot)}(\omega) = \dot{K}^{\alpha,p}_{q(\cdot)}(\omega)$.

The concept of intrinsic square function is given as follows (see [16], etc.).

For $x \in \mathbb{R}^n$, we define the set $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$, where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. Let $0 < \beta \le 1$ be a constant, and the set $\mathcal{C}_\beta$ be all functions φ on $\mathbb{R}^n$ satisfying the following three conditions.

(1) $\operatorname{supp}\varphi \subset \{|x| \le 1\}$;

(2) $\int_{\mathbb{R}^n} \varphi(x)dx = 0$;

(3) $|\varphi(x_1) - \varphi(x_2)| \le |x_1 - x_2|^\beta$, for $x_1, x_2 \in \mathbb{R}^n$.

For any $(y,t) \in \mathbb{R}^{n+1}_+$, let $\varphi_t(y) = t^{-n}\varphi(\frac{y}{t})$. Then for any $f \in L^1_{loc}(\mathbb{R}^n)$, we define maximal function $A_\beta f(y,t) = \sup_{\varphi \in \mathcal{C}_\beta} |f * \varphi_t(y)|, (y,t) \in \mathbb{R}^{n+1}_+.$ The β -order intrinsic square function is defined as follows:

$$S_\beta f(x) = (\iint_{\Gamma(x)} A_\beta f(y,t)^2 \frac{dydt}{t^{n+1}})^{\frac{1}{2}}.$$

For any $x \in \mathbb{R}^n$, suppose $\vec{f} = (f_1, f_2, \ldots)$ is a sequence of locally integrable function in $\mathbb{R}^n$. Then the vector-valued intrinsic square function is defined as

$$S_\beta(\vec{f})(x) = (\sum_{j=1}^{\infty} |S_\beta(f_j)(x)|^2)^{\frac{1}{2}}.$$

PRINCIPAL LEMMA

Lemma 2.1 [2] (Generalized Hölder inequality) Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and the following inequalities

$$\int_{\mathbb{R}^n}|f(x)g(x)|\,dx\le r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},\qquad(2.1)$$

hold, where $r_p=1+1/p_- -1/p_+$.

Lemma 2.2 [4] If $p(\cdot)\in\mathcal{P}\cap LH(\mathbb{R}^n)$, then $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.

According to [21], if ω is a weighted function, then $(L^{p(\cdot)}(\mathbb{R}^n,\omega))'=L^{p'(\cdot)}(\mathbb{R}^n,\omega^{-1})$, where $p'(\cdot)$ denotes conjugate index of $p(\cdot)$. There are two following lemmas.

Lemma 2.3 [21] Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n),\ \omega\in A_{p(\cdot)}$. Then there exists a constant $C>0$, such that for any ball B in $\mathbb{R}^n$, we have

$$C^{-1}\le\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\omega)}\|\chi_B\|_{(L^{p(\cdot)}(\omega))'}\le C.\qquad(2.2)$$

Lemma 2.4 [21] Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n),\omega\in A_{p(\cdot)}$. Then there exist a positive constant C, such that for any ball B in $\mathbb{R}^n$ and any measurable set $E\subset B$, we have

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\omega)}}{\|\chi_E\|_{L^{p(\cdot)}(\omega)}}\le C\frac{|B|}{|E|}\qquad(2.3)$$

and

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\omega)}}{\|\chi_B\|_{L^{p(\cdot)}(\omega)}}\le C\left(\frac{|E|}{|B|}\right)^{\delta},\qquad(2.4)$$

where δ is a constant which satisfies $0<\delta<1$.

For the intrinsic square function, the following results are obtained.

Lemma 2.5 [21] Let $0<\beta\le 1,\ \ p(\cdot)\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n),\ \ \omega\in A_{p(\cdot)}$. Then the intrinsic square function S_β is bounded on the weighted exponential Lebesgue space $L^{p(\cdot)}(\omega)$, that is

$$\|S_\beta f\|_{L^{p(\cdot)}(\omega)}\le C\|f\|_{L^{p(\cdot)}(\omega)}.$$

Lemma 2.6 [19] Let $0<\beta\le 1,\ \ 1<p<\infty,\ \ \omega\in A_p$. Then there exists a constants C which is independent of $\vec{f}=(f_1,f_2,\ldots)$, such that $\|(\sum_j|S_\beta(f_j)|^2)^{\frac12}\|_{L^p(\omega)}\le C\|(\sum_j|f_j|^2)^{\frac12}\|_{L^p(\omega)}$.

Lemma 2.7 [9] (Extrapolation theorem) Assuming that $p_0>0$, any $\omega_0\in A_1$ and (f,g) is included in a family of non-negative ordered pairs $\mathcal{F}$. Then

$$\|f\|_{L^{p_0}(\omega_0)}\le C\|g\|_{L^{p_0}(\omega_0)},(f,g)\in\mathcal{F}..$$

Let $p(\cdot)\in\mathcal{P}_0$, such that $p_-\ge p_0$ and M be bounded on $L^{(p(\cdot)/p_0)'}(\omega^{-p_0})$. Then

$$\|f\|_{L^{p(\cdot)}(\omega)}\le C\|g\|_{L^{p(\cdot)}(\omega)}.$$

Combining Lemma 2.6 and the Extrapolation Theorem, the following conclusions can be easily drawn.

Lemma 2.8 Let $0<\beta\le 1, p(\cdot)\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n),\omega\in A_{p(\cdot)}$. Then there exists a constant C which is independent of $\vec{f}=(f_1,f_2,\ldots)$, such that

$$\|(\sum_j |S_\beta(f_j)|^2)^{\frac{1}{2}}\|_{L^{p(\cdot)}(\omega)} \le C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_{L^{p(\cdot)}(\omega)}.$$

Main Result

Theorem 3.1 Let $0<\beta\le 1, q(\cdot)\in LH(\mathbb{R}^n)\cap \mathcal{P}(\mathbb{R}^n)$, $0<p<\infty, \lambda>0, \frac{1}{q_-}<r<1$, $\omega\in A_{rq(\cdot)}$ and $\lambda-n\delta<\alpha<n(1-r)$, $(0<\delta<1)$. Then there exists a constant C which is independent of $\vec{f}=(f_1,f_2,\ldots)$, such that

$$\|(\sum_j |S_\beta(f_j)|^2)^{\frac{1}{2}}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)} \le C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}. \tag{3.1}$$

Proof For any $(\sum_j |f_j|^2)^{\frac{1}{2}}\in M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)$, $i,j\in\mathbb{Z}$, we denote that

$$f_j(x)=\sum_{i=-\infty}^{+\infty} f_j(x)\cdot\chi_i(x)=\sum_{i=-\infty}^{+\infty} f_j^i(x).$$

By the definition of the weighted Herz-Morrey space with variable exponents, we have

$$\begin{aligned}
\|S_\beta(\vec{f})\|^p_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|(\sum_{j=1}^{\infty}|S_\beta(f_j)|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)})\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\sum_{i=-\infty}^{k-2}\|(\sum_{j=1}^{\infty}|S_\beta(f_j^i)|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)}))\\
&\quad + C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\sum_{i=k-1}^{k+1}\|(\sum_{j=1}^{\infty}|S_\beta(f_j^i)|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)}))\\
&\quad + C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\sum_{i=k+2}^{\infty}\|(\sum_{j=1}^{\infty}|S_\beta(f_j^i)|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)}))\\
&= F_1+F_2+F_3.
\end{aligned}$$

Firstly, we estimate F_2. Using lemma 2.8, we obtain

$$\begin{aligned}
F_2 &\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\sum_{i=k-1}^{k+1}\|\%(\sum_{j=1}^{\infty}|S_\beta(f_j^i)|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)}))\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\sum_{i=k-1}^{k+1}\|(\sum_{j=1}^{\infty}|f_j^i|^2)^{\frac{1}{2}}\chi_k\|^p_{L^{q(\cdot)}(\omega)}))\\
&\le C\|(\sum_{j=1}^{\infty}|f_j|^2)^{\frac{1}{2}}\|^p_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}.
\end{aligned}$$

Next, we estimate F_1. By $i\le k-2, x\in A_k$ and $(y,t)\in\Gamma(x)$, for any $\varphi\in\mathcal{C}_\beta$, we have

$$|(f_j^i)*\varphi_t(y)|=|\int_{A_i}\varphi_t(y-z)f_j(z)dz|\le Ct^{-n}\int_{\{z\in A_i:|y-z|<t\}}|f_j(z)|dz. \tag{3.2}$$

For $z\in A_i$, and $|y-z|<t$, we obtain

$$t>\frac{1}{2}(|x-y|+|y-z|)\ge\frac{1}{2}|x-z|\ge\frac{1}{2}(|x|-2^i)\ge\frac{1}{2}(|x|-2^{k-2})\ge\frac{|x|}{4}. \tag{3.3}$$

Using (3.2) and (3.3), we have

$$|S_\beta(f_j^i)(x)|=(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\beta}|f_j^i*\varphi_t(y)|^2\frac{dydt}{t^{n+1}})^{\frac{1}{2}}$$

$$\leq C\left(\int_{\frac{|x|}{4}}^{\infty}\int_{\{y:|x-y|<t\}}\left(\frac{1}{t^n}\int_{\{z\in A_i:|y-z|<t\}}|f_j(z)|\,dz\right)^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_{A_i}|f_j(z)|\,dz\right)\left(\int_{\frac{|x|}{4}}^{\infty}\left(\int_{\{y:|x-y|<t\}}dy\right)\frac{dt}{t^{3n+1}}\right)^{\frac{1}{2}}=C\left(\int_{A_i}|f_j(z)|\,dz\right)\left(\int_{\frac{|x|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}$$

$$\leq C|x|^{-n}\cdot\int_{A_i}|f_j(z)|\,dz. \qquad (3.4)$$

By the duality and Cauchy-Schwarz inequality, there is

$$\left(\sum_j|S_\beta(f_j^i)(x)|^2\right)^{\frac{1}{2}}\leq C\left(\sum_j\left||x|^{-n}\cdot\int_{A_i}|f_j(z)|\,dz\right|^2\right)^{\frac{1}{2}}\leq C\sup_{(\sum_j|\xi_j|^2)^{\frac{1}{2}}\leq 1}\sum_j\left(|x|^{-n}\int_{A_i}|f_j(z)|\,dz\cdot\xi_j\right)$$

$$\leq C|x|^{-n}\int_{A_i}\sup_{(\sum_j|\xi_j|^2)^{\frac{1}{2}}\leq 1}\left(\sum_j|f_j(z)|\cdot\xi_j\right)dz\leq C|x|^{-n}\cdot\int_{A_i}\left(\sum_j|f_j(z)|^2\right)^{\frac{1}{2}}dz. \quad (3.5)$$

Next, by using (2.1), lemma 2.2 and (2.2), notice that for any $x\in A_k$, $|x|^{-n}\leq C|B_k|^{-1}$. Then

$$\left(\sum_j|S_\beta(f_j^i)(x)|^2\right)^{\frac{1}{2}}\leq C|x|^{-n}\cdot\int_{A_i}\left(\sum_j|f_j(z)|^2\right)^{\frac{1}{2}}dz$$

$$\leq C|x|^{-n}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}\left\|\omega^{-\frac{1}{q(\cdot)}}\chi_{B_i}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}$$

$$\leq C|x|^{-n}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}|B_i|\cdot\|\chi_{B_i}\|_{L^{q(\cdot)}(\omega)}^{-1}$$

$$\leq C\frac{|B_i|}{|B_k|}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}\|\chi_{B_i}\|_{L^{q(\cdot)}(\omega)}^{-1}. \qquad (3.6)$$

By (3.6), we obtain

$$F_1\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left(\sum_{i=-\infty}^{k-2}\frac{|B_i|}{|B_k|}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(\omega)}}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}\right)^p\right). \qquad (3.7)$$

For $k,i\in\mathbb{Z}$ and $k\geq i+2$, we have $B_i\subset B_k$. For $rq(\cdot)\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$, then (2.3) in Lemma 2.4, can tell us

$$\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(\omega)}}=\left(\frac{\|\chi_{B_k}\|_{L^{rq(\cdot)}(\omega)}}{\|\chi_{B_i}\|_{L^{rq(\cdot)}(\omega)}}\right)^r\leq C\left(\frac{|B_k|}{|B_i|}\right)^r=C2^{(k-i)nr}. \qquad (3.8)$$

On the other hand, notice that

$$\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}\leq 2^{-j\alpha}\left(\sum_{l=-\infty}^{i}2^{l\alpha p}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_l\right\|_{L^{q(\cdot)}(\omega)}^p\right)^{\frac{1}{p}}$$

$$\leq C2^{i(\lambda-\alpha)}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\omega)}. \qquad (3.9)$$

By combining (3.7), (3.8) , (3.9) and $\alpha<n(1-r)$, we obtain

$$F_1\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left(\sum_{i=-\infty}^{k-2}2^{n(r-1)(k-i)}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\chi_i\right\|_{L^{q(\cdot)}(\omega)}\right)^p\right)$$

$$\leq C\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\omega)}^p\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\left(\sum_{i=-\infty}^{k-2}2^{(i-k)(\lambda+n(1-r)-\alpha)}\right)^p\right)$$

$$\leq C\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\omega)}^p\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\right)$$

$$\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|^p_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}. \qquad (3.10)$$

At last, we estimate F_3. For any $k\in\mathbb{Z}, x\in A_k, i\geq k+2, (y,t)\in\Gamma(x), z\in A_i$ and $|y-z|<t$, we obtain that $t>\frac{1}{2}(|x-y|+|y-z|)\geq\frac{1}{2}(|z|-|x|)\geq 2^{i-3}$.

Similar to (3.4), there is $|S_\beta(f_j^i)(x)|\leq C|B_i|^{-1}\int_{A_i}|f_j(z)|dz$. Therefore,

$$(\sum_j |S_\beta(f_j^i)(x)|^2)^{\frac{1}{2}}\leq C|B_i|^{-1}\int_{A_i}(\sum_j |f_j(z)|^2)^{\frac{1}{2}}dz. \qquad (3.11)$$

By using (3.11), Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}(\sum_j |S_\beta(f_j^i)(x)|^2)^{\frac{1}{2}} &\leq C|B_i|^{-1}\int_{\mathbb{R}^n}(\sum_j |f_j(z)|^2)^{\frac{1}{2}}\cdot\omega^{\frac{1}{p(\cdot)}}\cdot\omega^{-\frac{1}{p(\cdot)}}|\cdot(\chi_{A_i})^2dz\\
&\leq C|B_i|^{-1}\|(\sum_j |f_j|^2)^{\frac{1}{2}}\cdot\omega^{\frac{1}{q(\cdot)}}\chi_{A_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\omega^{-\frac{1}{q(\cdot)}}\chi_{A_i}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\chi_i\|_{L^{q(\cdot)}(\omega)}\cdot|B_i|^{-1}\cdot\|\omega^{-\frac{1}{q(\cdot)}}\chi_{B_i}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\chi_i\|_{L^{q(\cdot)}(\omega)}\cdot\|\chi_{B_i}\|^{-1}_{L^{q(\cdot)}(\omega)}. \qquad (3.12)\end{aligned}$$

From (2.4) of Lemma 2.4 and $\lambda-n\delta<\alpha$, similar to (3.10) as above, we can get that

$$\begin{aligned}F_3 &\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0}2^{k\alpha p}(\sum_{i=k+2}^{\infty}\|(\sum_j |f_j|^2)^{\frac{1}{2}}\chi_i\|_{L^{q(\cdot)}(\omega)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(\omega)}})^p)\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0}2^{k\alpha p}(\sum_{i=k+2}^{\infty}\|(\sum_j |f_j|^2)^{\frac{1}{2}}\chi_i\|_{L^{q(\cdot)}(\omega)}2^{\delta n(k-i)})^p)\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0}2^{k\alpha p}(\sum_{i=k+2}^{\infty}2^{i(\lambda-\alpha)}\cdot 2^{\delta n(k-i)}\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)})^p)\\
&\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|^p_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p}(\sum_{k=-\infty}^{k_0}2^{k\lambda p}(\sum_{i=k+2}^{\infty}2^{(i-k)(\lambda-\alpha-n\delta)})^p)\\
&\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|^p_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}.\end{aligned}$$

In combination with the estimation of F_1, F_2, F_3, we obtain

$$\|(\sum_j |S_\beta(f_j)|^2)^{\frac{1}{2}}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}\leq C\|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)}. \qquad \square$$

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