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New Composition Theorem for $\mathbb{S}_\gamma^p$-(Pseudo) Almost Automorphic Functions and its Application

Hou Rong[1*], Tang Chao[2]

1College of Mathematics, Sichuan University, Chengdu 610065, China
2College of Information Science and Engineering, Chengdu 610065, China

***Corresponding author:** Hou Rong
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Abstract **Review Article**

In this paper, we prove new composition theorems for generalized Stepanov-like almost automorphic functions and generalized Stepanov-like pseudo almost automorphic functions. And applying the Banach fixed point theorem, we study the existence and uniqueness of pseudo almost automorphic solutions to a class of semilinear integral equations.
Keywords: composition theorems, generalized Stepanov-like pseudo almost automorphic, Banach fixed point theorem.
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INTRODUCTION

The concept of almost automorphic functions, which was introduced by Bochner [1, 2], is an important generalization of the classical almost periodicity in the sense of Bohr. The concept of pseudo almost automorphy was introduced by Liang *et al*. [10], and the authors studied the composition theorem under uniformly continuous condition. The concept of $\mathbb{S}^p$-pseudo almost automorphy was introduced by Diagana [16], and obtained the existence of pseudo-almost automorphic solutions to some differential equations with $\mathbb{S}^p$-pseudo almost automorphic coefficients. In 2012, Diagana [17] introduced the concept of $\mathbb{S}_\gamma^p$-pseudo almost automorphy, and investigated the existence and uniqueness of solutions to some classes of nonautonomous differential equations of sobolev type.

Fan *et al*.[7] proved new composition theorems for Stepanov-like almost automorphic functions and Stepanov-like pseudo almost automorphic functions under locally integrable Lipschitz conditions, and given application to a class of evolution equations with Stepanov-like pseudo almost automorphic coefficients. Rui *et al*. [5] established some new composition theorems for Stepanov-like weighted pseudo almost automorphic functions, and investigated the existence and uniqueness of weighted pseudo almost automorphic mild solutions to a class of nonautonomous evolution equations with $\mathbb{S}^p$-weighted pseudo almost automorphic coefficients. More investigation of new composition theorems under different Lipschitz condition, one can see [3-10] and the references therein.

In recent years, the existence of (pseudo) almost automorphic, Stepanov-like (pseudo) almost automorphic, generalized Stepanov-like (pseudo) almost automorphic, Stepanov-like weighted (pseudo) almost automorphic solutions on kinds of differential equations has been extensively investigated. Such as Diagana and G. M. N'Guérékata [15] studied the existence and uniqueness of an almost automorphic solution to the semilinear equation with $\mathbb{S}^p$-almost automorphic coefficients. One can see [11-25] and the references therein.

In this paper, we study and obtain the existence and uniqueness of pseudo almost automorphic solutions to a class of semilinear integral equations given by

$$Y(t)=\int_{-\infty}^{t} a(t-s)[AY(s)+f(s,Y(s))]ds\,,\ \ t\in\mathbb{R}$$

where $a\in L^1(\mathbb{R}_+)$, $A\colon D(A)\subset\mathbb{X}\mapsto\mathbb{X}$ is the generator of an integral resolvent family defined on a complex Banach space $\mathbb{X}$, and $f\colon\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$ is a $\mathbb{S}_\gamma^p$-pseudo almost automorphic function satisfying suitable Lipschitz conditions.

We establish new composition theorems for generalized Stepanov-like almost automorphic functions and generalized Stepanov-like pseudo almost automorphic functions under suitable Lipschitz conditions. And using the Banach fixed point theorem and new composition theorems proved in this paper, we study the existence and uniqueness of pseudo almost automorphic solutions to the class of semilinear integral equations with generalized Stepanov-like pseudo almost automorphic coefficients.

Preliminaries

Throughout this paper, let $p \in [1, \infty)$, and $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, |\cdot|)$ are two Banach space. Let $C(\mathbb{R}, \mathbb{X})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all continuous functions(respectively, the collection of all jointly continuous functions). Furthermore, $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all bounded continuous functions with supremum norm(respectively, the collection of all jointly bounded continuous functions).

2. 1 $\mathbb{S}_\gamma^p$-almost automorphy

Definition 2.1 A continuous function $Y: \mathbb{R} \mapsto \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\tilde{Y}(t) := \lim_{n\to\infty} Y(t+s_n); \quad Y(t) := \lim_{n\to\infty} \tilde{Y}(t-s_n) \quad \text{for each } t \in \mathbb{R}$$

The collection of all almost automorphic functions $Y: \mathbb{R} \mapsto \mathbb{X}$ is denoted by $AA(\mathbb{R}, \mathbb{X})$.

Definition 2.2[23] The Bochner transform$Y^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$ of a function $Y: \mathbb{R} \mapsto \mathbb{X}$ is defined by

$$Y^b(t,s) := Y(t+s)$$

Remark 2.1 (i)A function $\varphi(t,s), t \in \mathbb{R}, s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(t,s)$ for all $t \in \mathbb{R}, s \in [0,1]$ and $\tau \in [s-1, s]$;

(ii)Note that if $f = l + m$, then $f^b = l^b + m^b$. Moreover, $(\lambda f)^b = \lambda f^b$for each scalar λ.

Definition 2.3[15] The Bochner transform$f^b(t,s,u)$,$t \in \mathbb{R}$, $s \in [0,1]$, $u \in \mathbb{Y}$of a function $f: \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u)$$

for each $u \in \mathbb{Y}$.

Definition 2.4[23] The space $BS^p(\mathbb{R}, \mathbb{X})$ of all Stepanov bounded functions consists of all measurable functions Y on $\mathbb{R}$ with values in $\mathbb{X}$ such that $Y^b \in L^\infty(\mathbb{R}, L^p(0,1; \mathbb{X}, ds))$. This is a Banach space when it is equipped the norm defined by

$$\| Y \|_{\mathbb{S}^p} = \| Y^b \|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \| Y(s) \|^p \, ds\right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}} \left(\int_0^1 \| Y(t+s) \|^p \, ds\right)^{\frac{1}{p}}$$

Definition 2.5[21] A function $Y \in BS^p(\mathbb{R}, \mathbb{X})$ is called Stepanov-like almost automorphic (or $\mathbb{S}^p$-almost automorphic) if $Y^b \in AA(\mathbb{R}, L^p(0,1; \mathbb{X}))$. In other words, a function $Y \in L^p_{loc}(\mathbb{R}, \mathbb{X}, ds)$ is called $\mathbb{S}^p$-almost automorphic if its Bochner transform $Y^b: \mathbb{R} \mapsto L^p(0,1; \mathbb{X}, ds)$ is almost automorphic in the sense that for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $\tilde{Y} \in L^p_{loc}(\mathbb{R}, \mathbb{X}, ds)$ such that

$$\left(\int_t^{t+1} \| Y(s+s_n) - \tilde{Y}(s) \|^p \, ds\right)^{\frac{1}{p}} \to 0 \text{ and } \left(\int_t^{t+1} \| \tilde{Y}(s-s_n) - Y(s) \|^p \, ds\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$. The collection of such functions is denoted by $AS^p(\mathbb{R}, \mathbb{X})$.

Let $\mathbb{U}$ denote the collection of all measurable (weights) functions$\gamma: (0, \infty) \mapsto (0, \infty)$, satisfying the following condition:

$$\gamma_0 := \lim_{\varepsilon\to 0} \int_\varepsilon^1 \gamma(s) \, ds = \int_0^1 \gamma(s) \, ds < \infty \text{ and } m_0 := \inf_{s\in(0,\infty)} \gamma(s) > 0$$

Let $\mathbb{U}_\infty$ denote the collection of all functions$\gamma \in \mathbb{U}$, which are differentiable.

Definition 2.6 Let $\gamma \in \mathbb{U}$. The space $B\mathbb{S}_\gamma^p(\mathbb{R}, \mathbb{X})$ denote all generalized Stepanov spaces consists of all γds-measurable functions Y on $\mathbb{R}$ with values in $\mathbb{X}$ such that its Bochner transform $Y^b \in L^\infty(\mathbb{R}, L^p(0,1; \mathbb{X}, ds))$. This is a Banach space when it is equipped the norm defined by

$$\| Y \|_{\mathbb{S}_\gamma^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \gamma(s-t) \| Y(s) \|^p \, ds\right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}} \left(\int_0^1 \gamma(s) \| Y(t+s) \|^p \, ds\right)^{\frac{1}{p}}$$

Definition 2.7[21] Let $\gamma \in \mathbb{U}$. The space $AS_\gamma^p(\mathbb{R},\mathbb{X})$ of all generalized Stepanov-like almost automorphic (or $\mathbb{S}_\gamma^p$-almost automorphic) functions consists of all $Y \in BS_\gamma^p(\mathbb{R},\mathbb{X})$ such that for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $\tilde{Y} \in L_{loc}^p(\mathbb{R},\mathbb{X},$
$\gamma ds)$ such that

$$(\int_t^{t+1} \gamma(s-t) \parallel Y(s+s_n) - \tilde{Y}(s) \parallel^p ds)^{\frac{1}{p}} \to 0$$
$$(\int_t^{t+1} \gamma(s-t) \parallel \tilde{Y}(s-s_n) - Y(s) \parallel^p ds)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ for each $t \in \mathbb{R}$.

Remark 2.2 Let $\gamma \in \mathbb{U}$. if $1 \le p < q < \infty$ and $Y \in L_{loc}^q(\mathbb{R},\mathbb{X},\gamma ds)$ is $\mathbb{S}_\gamma^q$-almost automorphic, then Y is $\mathbb{S}_\gamma^p$-almost automorphic.

Definition 2.8[15] Let $\gamma \in \mathbb{U}$. A function$f: \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$, $(t,Y) \mapsto f(t,Y)$ with $f(\cdot,Y) \in L_{loc}^p(\mathbb{R},\mathbb{X},$
$\gamma ds)$ for each $Y \in \mathbb{Y}$, is called $\mathbb{S}_\gamma^p$-almost automorphic in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{Y}$ if $t \mapsto f(t,Y)$ is $\mathbb{S}_\gamma^p$-almost automorphic for each $Y \in \mathbb{Y}$, that is, for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $\tilde{f}(\cdot,Y) \in L_{loc}^p(\mathbb{R},\mathbb{X},\gamma ds)$ such that

$$(\int_t^{t+1} \gamma(s-t) \parallel f(s+s_n,Y) - \tilde{f}(s,Y) \parallel^p ds)^{\frac{1}{p}} \to 0$$
$$(\int_t^{t+1} \gamma(s-t) \parallel \tilde{f}(s-s_n,Y) - f(s,Y) \parallel^p ds)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$ for each $Y \in \mathbb{Y}$. The collection of all $\mathbb{S}_\gamma^p$-almost automorphic functions is denoted by $AS_\gamma^p(\mathbb{R} \times \mathbb{Y},\mathbb{X})$.

Theorem 2.1 Let $f \in AS_\gamma^p(\mathbb{R} \times \mathbb{X},\mathbb{X})$ and suppose f satisfies the Lipschitz condition as follows, that is, there exits $q \in [1,p)$ and $L(t) \in AS_\gamma^q(\mathbb{R},\mathbb{R})$, such that for all $u,v \in L_{loc}^p(\mathbb{R},\mathbb{X},\gamma ds)$, $t \in \mathbb{R}$

$$\parallel f(t,u) - f(t,v) \parallel \le L(t) \parallel u - v \parallel \qquad (H1)$$

and $Y \in AS_\gamma^p(\mathbb{R},\mathbb{X})$ such that $K = \overline{\{Y(t): t \in \mathbb{R}\}} \subset \mathbb{X}$ is a compact subset. Then the function $D: \mathbb{R} \mapsto \mathbb{X}$ given by $D(t) = f\big(t,Y(t)\big)$ is $AS_\gamma^q(\mathbb{R},\mathbb{X})$.

Proof. First, we prove that $f(\cdot,Y(\cdot)) \in L_{loc}^q(\mathbb{R},\mathbb{X},\gamma ds)$.

Since $K = \overline{\{Y(t): t \in \mathbb{R}\}}$ is compact, we can find finite open ball $O_k (k = 1,2,\dots,n)$ with center $Y_k \in K$ and radius ε small enough such that $\{Y(t); t \in \mathbb{R}\} \subset \bigcup_{k=1}^n O_k$, $B_k = \{s \in \mathbb{R}; Y(s) \in O_k\}$ and $\mathbb{R} = \bigcup_{k=1}^n B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j$ $(2 \le j \le n)$, then $E_i \cap E_j = \emptyset$, when $i \ne j$, $1 \le i,j \le n$.

Define the step function $\overline{Y}: \mathbb{R} \mapsto \mathbb{X}$ by $\overline{Y}(s) = Y_k$, $s \in E_k$, $k = 1,2,\dots,n$. It's easy to obtain that $\parallel Y(s) - \overline{Y}(s) \parallel \le \varepsilon$ for all $s \in \mathbb{R}$. Using Hölder inequality (where $p' = \frac{p}{p-q}$, $q' = \frac{p}{q}$), we can obtain

$$\begin{aligned}
&\int_t^{t+1} \gamma(s-t) \parallel f(s,Y(s)) \parallel^q ds \\
\le\ & \int_t^{t+1} \gamma(s-t) \parallel f(s,Y(s)) - f(s,\overline{Y}(s)) \parallel^q ds + \int_t^{t+1} \gamma(s-t) \parallel f(s,\overline{Y}(s)) \parallel^q ds \\
\le\ & \int_t^{t+1} \gamma(s-t) L(s)^q \parallel Y(s) - \overline{Y}(s) \parallel^q ds + \sum_{k=1}^n \int_{[t,t+1]\cap E_k} \gamma(s-t)^{(1-\frac{1}{q'})} \cdot \gamma(s-t)^{\frac{1}{q'}} \parallel f(s,Y_k) \parallel^q ds \\
\le\ & \varepsilon^q \parallel L \parallel_{\mathbb{S}_\gamma^q}^q + \gamma_0^{\frac{1}{p'}} \sum_{k=1}^n (\int_{[t,t+1]\cap E_k} \gamma(s-t) \parallel f(s,Y_k) \parallel^p ds)^{\frac{q}{p}}
\end{aligned}$$

Since the arbitrariness of ε and $f(\cdot, Y_k) \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$, for every $k = 1,2,\dots,n$, we get $f(\cdot, Y(\cdot)) \in L^q_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$.

Next, we prove that $D(t)$ is $AS^q_\gamma(\mathbb{R}, \mathbb{X})$.

Let $(s'_n)_{n\in\mathbb{N}}$ be an arbirary sequence of real numbers. Since $Y \in AS^p_\gamma(\mathbb{R}, \mathbb{X})$ and $f \in AS^p_\gamma(\mathbb{R}\times\mathbb{X}, \mathbb{X})$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and functions $\tilde{Y} \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$, $\tilde{f}(\cdot, Y) \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$ such that

$$(\int_t^{t+1} \gamma(s-t) \parallel Y(s+s_n) - \tilde{Y}(s) \parallel^p ds)^{\frac{1}{p}} \to 0 \qquad (2.1)$$

$$(\int_t^{t+1} \gamma(s-t) \parallel \tilde{Y}(s-s_n) - Y(s) \parallel^p ds)^{\frac{1}{p}} \to 0 \qquad (2.2)$$

as $n \to \infty$ for each $t \in \mathbb{R}$.

$$(\int_t^{t+1} \gamma(s-t) \parallel f(s+s_n, Y) - \tilde{f}(s, Y) \parallel^p ds)^{\frac{1}{p}} \to 0 \qquad (2.3)$$

$$(\int_t^{t+1} \gamma(s-t) \parallel \tilde{f}(s-s_n, Y) - f(s, Y) \parallel^p ds)^{\frac{1}{p}} \to 0 \qquad (2.4)$$

as $n \to \infty$ pointwise on $\mathbb{R}$ for each $Y \in \mathbb{X}$.

Let us consider the function $\tilde{D}: \mathbb{R} \mapsto \mathbb{X}$ defined by $\tilde{D}(t) = \tilde{f}(t, \tilde{Y}(t))$. Note that

$$D(s+s_n) - \tilde{D}(s) = f(s+s_n, Y(s+s_n)) - f(s+s_n, \tilde{Y}(s)) + f(s+s_n, \tilde{Y}(s)) - \tilde{f}(s, \tilde{Y}(s))$$

Let $r_0 := \frac{pq}{p-q}$, obviously $1 \le q < r_0 < \infty$. By Remark 2.2, $L(t) \in AS^{r_0}_\gamma(\mathbb{R}, \mathbb{R})$. Using Hölder inequality, we can obtain

$$\begin{aligned}
&\int_t^{t+1} \gamma(s-t) \parallel D(s+s_n) - \tilde{D}(s) \parallel^q ds \\
\le\ &\int_t^{t+1} \gamma(s-t) \parallel f(s+s_n, Y(s+s_n)) - f(s+s_n, \tilde{Y}(s)) \parallel^q ds \\
&+ \int_t^{t+1} \gamma(s-t) \parallel f(s+s_n, \tilde{Y}(s)) - \tilde{f}(s, \tilde{Y}(s)) \parallel^q ds \\
\le\ &\int_t^{t+1} \gamma(s-t) L(s+s_n)^q \parallel Y(s+s_n) - \tilde{Y}(s) \parallel^q ds \\
&+ \int_t^{t+1} \gamma(s-t) \parallel f(s+s_n, \tilde{Y}(s)) - \tilde{f}(s, \tilde{Y}(s)) \parallel^q ds \\
\le\ &\parallel L \parallel^q_{\mathbb{S}^{r_0}_\gamma} (\int_t^{t+1} \gamma(s-t) \parallel Y(s+s_n) - \tilde{Y}(s) \parallel^p ds)^{\frac{q}{p}} \\
&+ \gamma_0^{\frac{1}{p'}} (\int_t^{t+1} \gamma(s-t) \parallel f(s+s_n, \tilde{Y}(s)) - \tilde{f}(s, \tilde{Y}(s)) \parallel^p ds)^{\frac{q}{p}}
\end{aligned}$$

We can deduce from (2.1), (2.3) that

$$\int_t^{t+1} \gamma(s-t) \parallel D(s+s_n) - \tilde{D}(s) \parallel^q ds \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Similarly we can deduct from (2.2), (2.4) that

$$\int_t^{t+1} \gamma(s-t) \parallel \tilde{D}(s-s_n) - D(s) \parallel^q ds \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$. Thus $D(t) = f(t, Y(t))$ is $AS^q_\gamma(\mathbb{R}, \mathbb{X})$. The proof is complete.

2.2 $\mathbb{S}^p_\gamma$-pseudo almost automorphy

Define the classes of functions:

$$PAP_0(\mathbb{R}, \mathbb{X}) := \{Y \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \parallel Y(t) \parallel dt = 0\}$$

$$PAP_0(\mathbb{R}\times\mathbb{Y}, \mathbb{X}) := \{f \in BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \parallel f(t, Y) \parallel dt = 0, f(\cdot, Y) \text{ is bounded for each } Y \in \mathbb{Y}\}$$

Definition 2.9[25] A function $Y \in C(\mathbb{R}, \mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $Y = x + z$, where $x \in AA(\mathbb{R}, \mathbb{X})$ and $z \in PAP_0(\mathbb{R}, \mathbb{X})$. The collection of such functions is denoted by $PAA(\mathbb{R}, \mathbb{X})$.

Proposition 2.1[25] The space $PAA(\mathbb{R}, \mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.

Definition 2.10[25] A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $f = l + m$, where $l \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $m \in PAP_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$. The collection of such functions is denoted by $PAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Define the following collections

$PAP_0(\mathbb{R}, L^p(0,1; \mathbb{X}, \gamma ds)) \coloneqq \{Y(\cdot) \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\int_t^{t+1} \gamma(s-t) \parallel Y(s) \parallel^p ds)^{\frac{1}{p}} dt = 0\}$$

$PAP_0(\mathbb{R} \times \mathbb{Y}, L^p(0,1; \mathbb{X}, \gamma ds)) \coloneqq \{f(\cdot, Y) \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$:

$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\int_t^{t+1} \gamma(s-t) \parallel f(s,Y) \parallel^p ds)^{\frac{1}{p}} dt = 0,\ f(\cdot, Y)$ is bounded for each $Y \in \mathbb{Y}\}$

Definition 2.11[17] Let $\gamma \in \mathbb{U}_\infty$. A function $Y \in BS^p_\gamma(\mathbb{R}, \mathbb{X})$ is called $\mathbb{S}^p_\gamma$-pseudo almost automorphic (or generalized Stepanov-like pseudo almost automorphic) if it can be expressed as $Y = x + z$, where $x^b \in AA(\mathbb{R}, L^p(0,1; \mathbb{X}, \gamma ds))$ and $z^b \in PAP_0(\mathbb{R}, L^p(0,1; \mathbb{X}, \gamma ds))$. The collection of such functions is denoted by $PAA^p_\gamma(\mathbb{R}, \mathbb{X})$.

Remark 2.3 By definition, the decomposition of $\mathbb{S}^p_\gamma$-pseudo almost automorphic functions is unique. Furthermore, $\mathbb{S}^p_\gamma$-pseudo almost automorphic spaces are translation-invariant.

Proposition 2.2[17] If $Y \in PAA(\mathbb{R}, \mathbb{X})$, then $Y \in PAA^p_\gamma(\mathbb{R}, \mathbb{X})$. In other words, $PAA(\mathbb{R}, \mathbb{X}) \subset PAA^p_\gamma(\mathbb{R}, \mathbb{X})$.

Proposition 2.3[17]Let $\gamma \in \mathbb{U}_\infty$. The space $PAA^p_\gamma(\mathbb{R}, \mathbb{X})$ equipped with the norm $\|\cdot\|_{\mathbb{S}^p_\gamma}$ is a Banach space.

Definition 2.12[17] Let $\gamma \in \mathbb{U}_\infty$. A function $f: \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$, $(t, Y) \mapsto f(t, Y)$ with $f(\cdot, Y) \in L^p_{loc}(\mathbb{R}, \mathbb{X}, \gamma ds)$ for each $Y \in \mathbb{Y}$, is called $\mathbb{S}^p_\gamma$-pseudo almost automorphic if there exist functions $l, m: \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that $f = l + m$ where $l^b \in AA(\mathbb{R} \times \mathbb{Y}, L^p(0,1; \mathbb{X}, \gamma ds))$ and $m^b \in PAP_0(\mathbb{R} \times \mathbb{Y}, L^p(0,1; \mathbb{X}, \gamma ds))$. The collection of such functions is denoted by $PAA^p_\gamma(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Theorem 2.2 Let $f = l + m \in PAA^p_\gamma(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ such that its Bochner transform $l^b \in AA(\mathbb{R} \times \mathbb{X}, L^p(0,1; \mathbb{X}, \gamma ds)$ and $m^b \in PAP_0\big(\mathbb{R} \times \mathbb{X}, L^p(0,1; \mathbb{X}, \gamma ds)\big)$. Assume that l and m satisfy (H1). Furthermore, if $Y = x + z \in PAA^p_\gamma(\mathbb{R}, \mathbb{X})$ with $x^b \in AA(\mathbb{R}, L^p(0,1; \mathbb{X}, \gamma ds)$ and $z^b \in PAP_0(\mathbb{R}, L^p($
$0,1; \mathbb{X}, \gamma ds))$ and such that $K = \overline{\{x(t): t \in \mathbb{R}\}}$ is compact, then there exits $q \in [1, p)$ such that $f(\cdot, Y(\cdot))$ belongs to $PAA^q_\gamma(\mathbb{R}, \mathbb{X})$.

Proof. We have

$$\begin{aligned} f(t, Y(t)) &= l(t, x(t)) + f(t, Y(t)) - l(t, x(t)) \\ &= l(t, x(t)) + f(t, Y(t)) - f(t, x(t)) + m(t, x(t)) \end{aligned}$$

Denote by

$$\begin{aligned} \Gamma^b(\cdot) &= f^b(\cdot, Y^b(\cdot) \\ &= l^b(\cdot, x^b(\cdot)) + f^b(\cdot, Y^b(\cdot)) - f^b(\cdot, x^b(\cdot)) + m^b(\cdot, x^b(\cdot)) \end{aligned}$$

Next, we shall show that $\Gamma^b(\cdot) \in PAP(\mathbb{R}, L^q(0,1; \mathbb{X}, \gamma ds))$ by several steps.

Step 1: we claim that $l^b\big(\cdot, x^b(\cdot)\big) \in AA\big(\mathbb{R}, L^q(0,1; \mathbb{X}, \gamma ds)\big)$. In fact, we know that the function l satisfies (H1) and its Bochner transform $l^b \in AA\big(\mathbb{R} \times \mathbb{X}, L^p(0,1; \mathbb{X}, \gamma ds)\big)$, $x^b \in AA(\mathbb{R}, L^p(0,1; \mathbb{X},$
$\gamma ds)$. Moreover, $K = \overline{\{x(t): t \in \mathbb{R}\}}$ is compact. Thus, by Theorem 2.1, we have $l^b(\cdot, x^b(\cdot)) \in AA(\mathbb{R}, L^q(0,1; \mathbb{X}, \gamma ds))$.

Step 2: we claim that $\varphi^b(\cdot) = f^b(\cdot, Y^b(\cdot)) - f^b(\cdot, x^b(\cdot)) \in PAP_0(\mathbb{R}, L^q(0,1; \mathbb{X}, \gamma ds))$. Since l, m satisfy (H1), using Hölder inequality (where $p' = \frac{p}{p-q}, q' = \frac{p}{q}$), we can obtain

$$\begin{aligned}\int_0^1 \gamma(s) \parallel \varphi(t+s) \parallel^q ds &= \int_t^{t+1} \gamma(s-t) \parallel \varphi(s) \parallel^q ds \\ &= \int_t^{t+1} \gamma(s-t) \parallel f(s,Y(s)) - f(s,x(s)) \parallel^q ds \\ &\leq \int_t^{t+1} \gamma(s-t) \parallel l(s,Y(s)) - l(s,x(s)) \parallel^q ds \\ &\quad + \int_t^{t+1} \gamma(s-t) \parallel m(s,Y(s)) - m(s,x(s)) \parallel^q ds \\ &\leq 2\int_t^{t+1} \gamma(s-t)^{(1-\frac{1}{q'})} L(s)^q \cdot \gamma(s-t)^{\frac{1}{q'}} \parallel Y(s) - x(s) \parallel^q ds \\ &\leq 2(\int_t^{t+1} \gamma(s-t) L(s)^{p'q} ds)^{\frac{1}{p'}} \cdot (\int_t^{t+1} \gamma(s-t) \parallel z(s) \parallel^p ds)^{\frac{q}{p}} \\ &\leq 2 \parallel L \parallel^q_{\mathbb{S}^{r_0}_\gamma} (\int_t^{t+1} \gamma(s-t) \parallel z(s) \parallel^p ds)^{\frac{q}{p}}\end{aligned}$$

where $r_0 := \frac{pq}{p-q}$. Thus, for any $T > 0$

$$\frac{1}{2T}\int_{-T}^{T}(\int_0^1 \gamma(s) \parallel \varphi(t+s) \parallel^q ds)^{\frac{1}{q}} dt \leq \frac{\parallel L \parallel_{\mathbb{S}^{r_0}_\gamma}}{T}\int_{-T}^{T}(\int_t^{t+1} \gamma(s-t) \parallel z(s) \parallel^p ds)^{\frac{1}{p}} dt$$

Since $z^b \in PAP_0(\mathbb{R}, L^p(0,1;\mathbb{X},\gamma ds))$, we can obtain

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}(\int_0^1 \gamma(s) \parallel \varphi(t+s) \parallel^q ds)^{\frac{1}{q}} dt = 0$$

which implies $\varphi^b(\cdot) \in PAP_0(\mathbb{R}, L^q(0,1;\mathbb{X},\gamma ds))$.

Step 3: we also claim that $m^b(\cdot, x^b(\cdot)) \in PAP_0(\mathbb{R}, L^q(0,1;\mathbb{X},\gamma ds))$. Since $K = \overline{\{x(t): t \in \mathbb{R}\}}$ is compact, we can find finite open ball $O_k (k = 1,2,\ldots,n)$ with center $x_k \in K$ and radius ε small enough such that $\{x(t); t \in \mathbb{R}\} \subset \bigcup_{k=1}^n O_k$, $B_k = \{s \in \mathbb{R}; x(s) \in O_k\}$ and $\mathbb{R} = \bigcup_{k=1}^n B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j$ $(2 \leq j \leq n)$, when $i \neq j, 1 \leq i,j \leq n$.

Define the step function $\overline{x}: \mathbb{R} \mapsto \mathbb{X}$ by $\overline{x}(s) = x_k$, $s \in E_k, k = 1,2,\ldots,n$. It's easy to obtain that $\parallel x(s) - \overline{x}(s) \parallel \leq \varepsilon$ for all $s \in \mathbb{R}$. It follows that

$$\begin{aligned}&\int_t^{t+1} \gamma(s-t) \parallel m(s,x(s)) \parallel^q ds \\ \leq &\int_t^{t+1} \gamma(s-t) \parallel m(s,x(s)) - m(s,\overline{x}(s)) \parallel^q ds + \int_t^{t+1} \gamma(s-t) \parallel m(s,\overline{x}(s)) \parallel^q ds \\ \leq &\int_t^{t+1} \gamma(s-t) L(s)^q \parallel x(s) - \overline{x}(s) \parallel^q ds + \sum_{k=1}^n \int_{[t,t+1]\cap E_k} \gamma(s-t) \parallel m(s,x_k) \parallel^q ds \\ \leq &\ \varepsilon^q \parallel L \parallel^q_{\mathbb{S}^q_\gamma} + \gamma_0^{\frac{1}{p'}} \sum_{k=1}^n (\int_{[t,t+1]\cap E_k} \gamma(s-t) \parallel m(s,x_k) \parallel^p ds)^{\frac{q}{p}}\end{aligned}$$

Thus, for any $T > 0$

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T}(\int_t^{t+1} \gamma(s-t) \parallel m(s,x(s)) \parallel^q ds)^{\frac{1}{q}} dt \\ \leq &\frac{1}{2T}\int_{-T}^{T}\{\varepsilon^q \parallel L \parallel^q_{\mathbb{S}^q_\gamma} + \gamma_0^{\frac{1}{p'}} \sum_{k=1}^n (\int_{[t,t+1]\cap E_k} \gamma(s-t) \parallel m(s,x_k) \parallel^p ds)^{\frac{q}{p}}\}^{\frac{1}{q}} dt\end{aligned}$$

Since the arbitrariness of ε and $m^b \in PAP_0(\mathbb{R} \times \mathbb{X}, L^p(0,1;\mathbb{X},\gamma ds))$, we can obtain

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\int_{t}^{t+1}\gamma(s-t)\parallel m(s,x(s))\parallel^{q}ds)^{\frac{1}{q}}dt=0$$

which implies $m^b(\cdot,x^b(\cdot))\in PAP_0(\mathbb{R},L^q(0,1;\mathbb{X},\gamma ds))$. The proof is complete.

Pseudo almost automorphic solution

In this section, we study the semilinear equation

$$Y(t)=\int_{-\infty}^{t}a(t-s)\left[AY(s)+f\big(s,Y(s)\big)\right]ds,\quad t\in\mathbb{R} \qquad (3.1)$$

Definition 3.1 Let A be the generator of an integral resolvent family $S(t)_{t\geq 0}$. A continuous function $Y:\mathbb{R}\mapsto\mathbb{X}$ is called a mild solution of equation (3.1) if it satisfies the integral equation

$$Y(t)\quad=\quad\int_{-\infty}^{t}S(t-s)f(s,Y(s))ds,\quad \text{for all } t\in\mathbb{R}$$

Theorem 3.1 Assume that A generates an integral resolvent family $S(t)_{t\geq 0}$ such that

$$\parallel S(t)\parallel\leq\Psi(t),\ \text{for all } t\geq 0,\ \text{ with } \Psi\in L^1(\mathbb{R}_+)$$

where Ψ is a decreasing function such that $\Psi_0:=\sum_{k=1}^{\infty}\Psi(k)<\infty$. Let $f:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$, $f\in PAA_{\gamma}^{p}(\mathbb{R}\times\mathbb{X},\mathbb{X})\cap C(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and f satisfies the Lipschitz condition

$$\parallel f(t,u)-f(t,v)\parallel\leq L(t)\parallel u-v\parallel,\ \text{for all } u,v\in\mathbb{X},t\in\mathbb{R}$$

where $L\in L^1(\mathbb{R}_+)$, $C_L:=\sup_{t\in\mathbb{R}}\parallel\int_t^{t+1}L(s)ds<\infty$. Then equation (3.1) has a unique pseudo almost automorphic mild solution whenever $C_L\Psi_0<1$.

Proof. First we prove that the integral operator Λ defined by

$$\Lambda Y(t):=\int_{-\infty}^{t}S(t-s)f(s,Y(s))ds$$

maps $PAA(\mathbb{R},\mathbb{X})$ into $PAA(\mathbb{R},\mathbb{X})$.

Since $f\in PAA_{\gamma}^{p}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $Y\in PAA(\mathbb{R},\mathbb{X})\subset PAA_{\gamma}^{p}(\mathbb{R},\mathbb{X})$. By Theorem 2.2, it follows that $D(t):=f(t,Y(t))\in PAA_{\gamma}^{q}(\mathbb{R},\mathbb{X})$. Now let $D=l+m$, where their Bochner transform $l^b\in AA(\mathbb{R},L^q(0,1;\mathbb{X},\gamma ds)$ and $m^b\in PAP_0(\mathbb{R},L^q(0,1;\mathbb{X},\gamma ds))$. Consider for each $k=1,2,\ldots$, the integral

$$\begin{aligned}\Phi_k(t)\quad&=\quad\int_{k-1}^{k}S(\xi)\,D(t-\xi)d\xi\\&=\quad\int_{k-1}^{k}S(\xi)l(t-\xi)d\xi+\int_{k-1}^{k}S(\xi)m(t-\xi)d\xi\end{aligned}$$

and set $X_k(t):=\int_{k-1}^{k}S(\xi)l(t-\xi)d\xi$ and $Z_k(t):=\int_{k-1}^{k}S(\xi)m(t-\xi)d\xi$.

Let us show that $X_k\in AA(\mathbb{R},\mathbb{X})$. For that, letting $\sigma=t-\xi$, we obtain

$$X_k(t)=\int_{t-k}^{t-k+1}S(t-\sigma)l(\sigma)d\sigma,t\in\mathbb{R}$$

Using the Hölder inequality (where $q':=q,\frac{1}{p'}+\frac{1}{q'}=1$) and mean value theorem of integrals, it follows that

$$\begin{aligned}\parallel X_k(t)\parallel\quad&=\quad\parallel\int_{t-k}^{t-k+1}S(t-\sigma)l(\sigma)d\sigma\parallel\\&\leq\quad\int_{t-k}^{t-k+1}\parallel S(t-\sigma)\parallel\parallel l(\sigma)\parallel d\sigma\\&\leq\quad\int_{t-k}^{t-k+1}\Psi(t-\sigma)\gamma(\sigma-t+k)^{-\frac{1}{q'}}\cdot\gamma(\sigma-t+k)^{\frac{1}{q'}}\parallel l(\sigma)\parallel d\sigma\\&\leq\quad(\int_{k}^{k-1}\Psi(\tau)^{p'}\gamma(k-\tau)^{-\frac{p'}{q'}}d\tau)^{\frac{1}{p'}}\cdot(\int_{t-k}^{t-k+1}\gamma(\sigma-t+k)\parallel l(\sigma)\parallel^{q'}d\sigma)^{\frac{1}{q'}}\\&\leq\quad m_0^{-\frac{1}{q}}\parallel l\parallel_{\mathbb{S}_{\gamma}^{q}}(\int_{k}^{k-1}\Psi(\tau)^{p'}d\tau)^{\frac{1}{p'}}\end{aligned}$$

Since $\Psi\in L^1(\mathbb{R}_+)$ and $L^{p'}(\mathbb{R}_+)\subset L^1(\mathbb{R}_+)$, then

$$m_0^{-\frac{1}{q}}\parallel l\parallel_{\mathbb{S}_{\gamma}^{q}}\sum_{k=1}^{\infty}(\int_{k}^{k-1}\Psi(\tau)^{p'}d\tau)^{\frac{1}{p'}}=m_0^{-\frac{1}{q}}\parallel l\parallel_{\mathbb{S}_{\gamma}^{q}}\parallel\Psi\parallel_{L^{p'}(\mathbb{R}_+)}<\infty$$

We deduce from the well-known Weirstrass theorem that the series $\sum_{k=1}^{\infty} X_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} S(t-\sigma) l(\sigma) d\sigma = \sum_{k=1}^{\infty} X_k(t)$$

$X(t) \in C(\mathbb{R}, \mathbb{X})$, and

$$\| X(t) \| \quad = \quad \| \sum_{k=1}^{\infty} X_k(t) \| \le \sum_{k=1}^{\infty} \| X_k(t) \| \} < \infty$$

We deduce from the well-known Weirstrass theorem that $X(t)$ is bounded.

Since $l \in AS_{\gamma}^{q}(\mathbb{R}, \mathbb{X})$, then for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $\tilde{l} \in AS_{\gamma}^{q}(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{n\to\infty} \| l(t+s_n) - \tilde{l}(t) \|_{\mathbb{S}_{\gamma}^{q}} = 0 \text{ and } \lim_{n\to\infty} \| \tilde{l}(t-s_n) - l(t) \|_{\mathbb{S}_{\gamma}^{q}} = 0 \qquad (3.2)$$

Let

$$\tilde{X}_k(t) := \int_{k-1}^{k} S(\xi) \tilde{l}(t-\xi) d\xi$$

Then, using the Hölder inequality and mean value theorem of integrals, it follows that

$$\begin{aligned} & \| X_k(t+s_n) - \tilde{X}_k(t) \| \\ = \ & \| \int_{k-1}^{k} S(\xi)[l(t+s_n-\xi) - \tilde{l}(t-\xi)] d\xi \| \\ \le \ & \int_{k-1}^{k} \Psi(\xi) \gamma(\xi-k+1)^{-\frac{1}{q'}} \cdot \gamma(\xi-k+1)^{\frac{1}{q'}} \| l(t+s_n-\xi) - \tilde{l}(t-\xi) \| d\xi \\ \le \ & \left(\int_{k-1}^{k} \Psi(\xi)^{p'} \gamma(\xi-k+1)^{-\frac{p'}{q'}} d\xi\right)^{\frac{1}{p'}} \cdot \left(\int_{k-1}^{k} \gamma(\xi-k+1) \| l(t+s_n-\xi) - \tilde{l}(t-\xi) \|^{q} d\xi\right)^{\frac{1}{q}} \\ \le \ & m_0^{-\frac{1}{q}} \| \Psi \|_{L^{p'}} \| l(t+s_n) - \tilde{l}(t) \|_{\mathbb{S}_{\gamma}^{q}} \end{aligned}$$

Now using (3.2) and Lebesgue dominated convergence theorem, it follows that

$$\| X_k(t+s_n) - \tilde{X}_k(t) \| \to 0, \ \text{ as } n \to \infty$$

Similarly, using (3.2) and Lebesgue dominated convergence theorem, it follows that

$$\| \tilde{X}_k(t-s_n) - X_k(t) \| \to 0, \ \text{ as } n \to \infty$$

Thus, each $X_k \in AA(\mathbb{R}, \mathbb{X})$ for each k. Hence their uniform limit $X(t) \in AA(\mathbb{R}, \mathbb{X})$.

Let us show that $Z_k \in PAP_0(\mathbb{R}, \mathbb{X})$. Using the Hölder inequality and mean value theorem of integrals, we can obtain

$$\begin{aligned} \| Z_k(t) \| \quad &= \quad \| \int_{t-k}^{t-k+1} S(t-\sigma) m(\sigma) d\sigma \| \\ &\le \quad m_0^{-\frac{1}{q}} \| \Psi \|_{L^{p'}} \left(\int_{t-k}^{t-k+1} \gamma(\sigma-t+k) \| m(\sigma) \|^{q} d\sigma\right)^{\frac{1}{q}} \end{aligned}$$

Thus, for any $T > 0$

$$\frac{1}{2T} \int_{-T}^{T} \| Z_k(t) \| dt \le \frac{m_0^{-\frac{1}{q}} \| \Psi \|_{L^{p'}}}{2T} \int_{-T}^{T} \left(\int_{t-k}^{t-k+1} \gamma(\sigma-t+k) \| m(\sigma) \|^{q} d\sigma\right)^{\frac{1}{q}} dt$$

Since $m^b \in PAP_0(\mathbb{R}, L^q(0,1; \mathbb{X}, \gamma ds))$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \| Z_k(t) \| dt = 0$$

that is, $Z_k \in PAP_0(\mathbb{R}, \mathbb{X})$. Furthermore,

$$Z(t) := \int_{-\infty}^{t} S(t-\sigma)m(\sigma)d\sigma = \sum_{k=1}^{\infty} Z_k(t)$$

$Z(t) \in C(\mathbb{R}, \mathbb{X})$, and

$$\| Z(t) \| = \| \sum_{k=1}^{\infty} Z_k(t) \| \leq \sum_{k=1}^{\infty} \| Z_k(t) \|$$

Then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \| Z(t) \| dt = 0$$

Consequently the uniform limit $Z(t) \in PAP_0(\mathbb{R}, \mathbb{X})$. Thus, $\Lambda Y(t) = X(t) + Z(t) \in PAA(\mathbb{R}, \mathbb{X})$.

Next, we prove that the existence and uniqueness of pseudo almost automorphic solution applying the Banach fixed point theorem.

Since Ψ is a decreasing function such that$\Psi_0 := \sum_{k=1}^{\infty} \Psi(k) < \infty$, and let $Y_1, Y_2 \in PAA(\mathbb{R}, \mathbb{X})$. Using mean value theorem of integrals, we have

$$\begin{aligned} E \| \Lambda Y_1(t) - \Lambda Y_2(t) \|_\infty &= \sup_{t\in\mathbb{R}} \| \int_{-\infty}^{t} S(t-s)[f(s, Y_1(s)) - f(s, Y_2(s))]ds \| \\ &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \| S(t-s) \| \| f(s, Y_1(s)) - f(s, Y_2(s)) \| ds \\ &\leq \sup_{t\in\mathbb{R}} \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} \Psi(t-s) L(s) \| Y_1(s) - Y_2(s) \| ds \\ &\leq C_L \Psi_0 \| Y_1 - Y_2 \|_\infty \end{aligned}$$

Since $C_L \Psi_0 < 1$, hence by the Banach fixed point theorem, Λ has a fixed point$Y \in PAA(\mathbb{R}, \mathbb{X})$. The proof is complete.

CONCLUSION

This paper mainly studied new composition theorems for $\mathbb{S}_\gamma^p$-(pseudo) almost automorphic functions. By using Hölder inequality, mean value theorem of integrals and the Banach fixed point theorem, we obtained the existence and uniqueness of pseudo almost automorphic solutions to a class of semilinear integral equations, under some suitable conditions.

There are two direct issues which require further study. We will study the existence of pseudo almost automorphic solutions of a class of stochastic differential equations perturbed noise, under $\mathbb{S}_\gamma^p$ -pseudo almost automorphic coefficients. Also, we will investigate generalized Stepanov-like weighted pseudo almost automorphic functions, which are more generalized than $\mathbb{S}_\gamma^p$-pseudo almost automorphic functions, and study theirs new composition theorems and applications.

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