Scholars Journal of Physics, Mathematics and Statistics
OPEN ACCESS
Abbreviated Key Title: Sch J Phys Math Stat
ISSN 2393-8056 (Print) | ISSN 2393-8064 (Online)
Journal homepage: http://saspjournals.com/sjpms

2-D Discrete Open Quantum Walk and Its Entropy

Qi Han, Zhihe Chen*, Ziqiang Lu

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, Gansu P. R. China

DOI: 10.21276/sjpms.2019.6.8.2 | **Received:** 15.08.2019 | **Accepted:** 24.08.2019 | **Published:** 30.08.2019

***Corresponding author:** Zhihe Chen

Abstract **Review Article**

In this paper, the two-dimensional discrete time open quantum walk and its quantum entropy are studied. The connection and difference between classical random walk and open quantum random walk are introduced through two examples. Furthermore, the quantum entropy and other properties of these two examples have also been studied.
Keywords: Two-dimensional, quantum entropy, open quantum walk.

INTRODUCTION

Quantum computation and quantum information is an interdisciplinary subject of mathematics, physics, computer science and information theory. In 2000, Nielsen *et al.* [1] systematically introduced quantum computation and quantum information.

Quantum walk is one of the hot issues in quantum computation algorithms in recent years, which was first introduced by Aharonov [2]. It is the quantum version of the classical random walk. A classical random walk is defined by a walker moving left or right by transition probability, while a quantum walk describes the position where a walker may exist by probability amplitude. In 2012, Venegas-Andraca [3] gave a comprehensive review about quantum walk. By the category of system time, quantum walk can be divided into discrete time quantum walks and continuous time quantum walks. By the influence of the system environment, quantum walk can be divided into closed quantum walks and open quantum walks (For briefly, we short it as OQWs). In 2012, Attal *et al.* [4] were first detailly introduced OQWs. Since then, many scholars have become interested in OQWs, which has been studied such as quantum Bernoulli noise [5], quantum Markov semigroups [6], central limit theorems [7].

Quantum entropy is one of the key concepts in quantum information theory, which is used to measure the uncertainty in the state of a physical system. It is well-known that the Shannon entropy associated with probability distribution is defined by

$$H(x) \equiv H(p_1, p_2, \cdots, p_n) \equiv -\sum_x p_x \log_2 p_x.$$

However, the entropy in quantum state is Von Neumann entropy associated with density operators (positive semidefinite operators with unit trace) replacing probability distributions. In 1993, Ohya and Petz[8] introduced entropies for finite quantum systems and general quantum systems, which let people understand the concept and use of quantum entropy more concretely.

By considering the uncertainty in two-dimensional open quantum walk, we combine quantum information with quantum computation. Therefore, in this paper, we mainly study discrete time OQWs in two dimensions and its quantum entropy. We use density operators in OQWs to replace probability distribution in closed quantum walk. Our work is as follows. In section 2, we briefly recall the basic properties of OQWs and give two examples about OQWs with different transfer operators. In section 3, we define the quantum entropy in OQWs and introduce some properties about it. At last, we give some conclusions.

Open quantum walks

Let $\mathcal{H}$ be a separable Hilbert space, which stands for the space of degrees of freedom. Let $\mathcal{K}$ be the state space of a quantum system, which consists of $\mathcal{K}_x$ and $\mathcal{K}_y$, that is, $\mathcal{K}=\mathcal{K}_x \otimes \mathcal{K}_y$. Consider a bounded operator B_i^j on H, which stands for the effect of passing from j to i. We assume that, for each

$$\sum_i B_i^{j*} B_i^j = I,$$

where the above series is strongly convergent (if infinite).

Consider the space $\mathcal{H}\otimes\mathcal{K}$. We shall especially be interested in density matrices on $\mathcal{H}\otimes\mathcal{K}$ with the form

$$\rho = \sum_{x,y} \rho_{xy} \, |x\rangle\langle x|y\rangle\langle y|, \tag{1}$$

where each ρ_{xy} is a positive and trace-class operator, and satisfied the following conditions

$$\begin{cases} \operatorname{Tr}(\rho_{xy}) \neq 1, \\ \sum_{x,y} \operatorname{Tr}(\rho_{xy}) = 1. \end{cases} \tag{2}$$

ρ_{xy} is not exactly a density matrix on $\mathcal{H}$. It is same as transition probability matrix in Markov chain.

Let R, L, B, F be four bounded operators on $\mathcal{H}$ such that

$$R^*R + L^*L = I, \qquad B^*B + F^*F = I.$$

Then we can define an open quantum random walk on $\mathbb{Z}^2$ by saying that one can only jump to nearest neighbors: a jump to the left is given by L, a jump to the right is given by R, a jump to the behind is given by B, a jump to the forward is given by F. In other words, we put

$$R = B_x^{x+1}, L = B_x^{x-1}, B = B_y^{y+1}, F = B_y^{y-1}$$

for all $x, y \in \mathbb{Z}$, all the others B_i^j being equal to 0.

Starting with an initial state $\rho^{(0)} = \rho_{00}\otimes|0\rangle\langle 0|0\rangle\langle 0|$, after one step we have the state

$$\begin{aligned}\rho^{(1)} &= FR\rho_{00}R^*F^*\otimes|1\rangle\langle 1|1\rangle\langle 1| + BR\rho_{00}R^*B^*\otimes|1\rangle\langle 1|-1\rangle\langle -1| \\ &\quad + FL\rho_{00}L^*F^*\otimes|-1\rangle\langle -1|1\rangle\langle 1| + BL\rho_{00}L^*B^*\otimes|-1\rangle\langle -1|-1\rangle\langle -1|.\end{aligned}$$

The probability of presence in $|1\rangle_x|1\rangle_y$, $|1\rangle_x|-1\rangle_y$, $|-1\rangle_x|1\rangle_y$, $|-1\rangle_x|-1\rangle_y$ are $\operatorname{Tr}(FR\rho_{00}R^*F^*)$, $\operatorname{Tr}(BR\rho_{00}R^*B^*)$, $\operatorname{Tr}(FL\rho_{00}L^*F^*)$, $\operatorname{Tr}(BL\rho_{00}L^*B^*)$ respectively.

After the second step, the state of the system is

$$\begin{aligned}\rho^{(2)} &= FRFR\rho_{00}R^*F^*R^*F^*\otimes|2\rangle\langle 2|2\rangle\langle 2| + BLBL\rho_{00}L^*B^*L^*B^*\otimes|-2\rangle\langle -2|-2\rangle\langle -2| \\ &+ BRBR\rho_{00}R^*B^*R^*B^*\otimes|2\rangle\langle 2|-2\rangle\langle -2| + FLFL\rho_{00}L^*F^*L^*F^*\otimes|-2\rangle\langle -2|2\rangle\langle 2| \\ &+ (BRFR\rho_{00}R^*F^*R^*B^* + FRBR\rho_{00}R^*B^*R^*F^*)\otimes|2\rangle\langle 2|0\rangle\langle 0| \\ &+ (FLFR\rho_{00}R^*F^*L^*F^* + FRFL\rho_{00}L^*F^*R^*F^*)\otimes|0\rangle\langle 0|2\rangle\langle 2| \\ &+ (BLFL\rho_{00}L^*F^*L^*B^* + FLBL\rho_{00}L^*B^*L^*F^*)\otimes|-2\rangle\langle -2|0\rangle\langle 0| \\ &+ (BLBR\rho_{00}R^*B^*L^*B^* + BRBL\rho_{00}L^*B^*R^*B^*)\otimes|0\rangle\langle 0|-2\rangle\langle -2| \\ &+ (BLFR\rho_{00}R^*F^*L^*B^* + FLBR\rho_{00}R^*B^*L^*F^* + BRFL\rho_{00}L^*F^*R^*B^* \\ &+ FRBL\rho_{00}L^*B^*R^*F^*)\otimes|0\rangle\langle 0|0\rangle\langle 0|.\end{aligned}$$

The probability of presence in $|2\rangle_x|2\rangle_y$, $|2\rangle_x|-2\rangle_y$, $|-2\rangle_x|2\rangle_y$, $|-2\rangle_x|-2\rangle_y$, $|2\rangle_x|0\rangle_y$, $|0\rangle_x|2\rangle_y$, $|-2\rangle_x|0\rangle_y$, $|0\rangle_x|-2\rangle_y$ and $|0\rangle_x|0\rangle_y$ are then

$\operatorname{Tr}(FRFR\rho_{00}R^*F^*R^*F^*)$, $\operatorname{Tr}(BRBR\rho_{00}R^*B^*R^*B^*)$, $\operatorname{Tr}(FLFL\rho_{00}L^*F^*L^*F^*)$, $\operatorname{Tr}(BLBL\rho_{00}L^*B^*L^*B^*)$, $\operatorname{Tr}(BRFR\rho_{00}R^*F^*R^*B^* + FRBR\rho_{00}R^*B^*R^*F^*)$, $\operatorname{Tr}(FLFR\rho_{00}R^*F^*L^*F^* + FRFL\rho_{00}L^*F^*R^*F^*)$, $\operatorname{Tr}(BLFL\rho_{00}L^*F^*L^*B^* + FLBL\rho_{00}L^*B^*L^*F^*)$, $\operatorname{Tr}(BLBR\rho_{00}R^*B^*L^*B^* + BRBL\rho_{00}L^*B^*R^*B^*)$,

and

$$\operatorname{Tr}(BLFR\rho_{00}R^*F^*L^*B^* + FLBR\rho_{00}R^*B^*L^*F^* + BRFL\rho_{00}L^*F^*R^*B^* + FRBL\rho_{00}L^*B^*R^*F^*),$$

respectively.

One can iterate the above procedure and generate our open quantum random walk on $\mathbb{Z}^2$.

Assume that $\rho_{xy} = \mathcal{M}\rho_{00}\mathcal{M}^*$, where $\mathcal{M}$ represents the effect of the combination of operators R, L, B, F such as $\mathcal{M}=FR$ and $\mathcal{M}^* = R^*F^*$ in one step at position $(1,1)$.

Theorem 1. ρ_{xy} is a normal operator, and it can be represented as

$$\rho_{xy} = \sum_i p_i \, |i\rangle\langle i| \tag{3}$$

Proof. Let first prove that ρ_{xy} is a normal operator.

$$\rho_{xy}^*\rho_{xy} = (\mathcal{M}\rho_{00}\mathcal{M}^*)^*\mathcal{M}\rho_{00}\mathcal{M}^* = (\mathcal{M}^*)^*\rho_{00}^*\mathcal{M}^*\mathcal{M}\rho_{00}\mathcal{M}^* = \mathcal{M}\rho_{00}^*\mathcal{M}^*\mathcal{M}\rho_{00}\mathcal{M}^*=\mathcal{M}\rho_{00}^2\mathcal{M}^*,$$

$$\rho_{xy}\rho_{xy}^* = \mathcal{M}\rho_{00}\mathcal{M}^*(\mathcal{M}\rho_{00}\mathcal{M}^*)^* = \mathcal{M}\rho_{00}\mathcal{M}^*(\mathcal{M}^*)^*\rho_{00}^*\mathcal{M}^* = \mathcal{M}\rho_{00}\mathcal{M}^*\mathcal{M}\rho_{00}^*\mathcal{M}^* = \mathcal{M}\rho_{00}^2\mathcal{M}^*,$$

Then we have $\rho_{xy}^*\rho_{xy} = \rho_{xy}\rho_{xy}^*$, which means ρ_{xy} is a normal operator. It is well-known that the spectral decomposition is an extremely useful representation theorem for normal operators. Thus, we have

$$\rho_{xy} = \sum_{x,y} p(x,y)\, |i\rangle\langle i|,$$

where $p(x,y)$ are the eigenvalues of ρ_{xy}, $|i\rangle$ is an orthonormal basis for $\mathcal{H}$, and each $|i\rangle$ an eigenvector of ρ_{xy} with eigenvalue $p(x,y)$.

In other words, $\rho_{xy} = \sum_{x,y} p(x,y)\, |i\rangle\langle i|$ also means the system is in the state $|i\rangle$ with probability$p(x,y)$.

Example 1

If we take

$$R = F = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } L = B = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

which the operators R, L, B, F do satisfy $R^*R + L^*L = I,\ B^*B + F^*F = I$. Let us consider the associated open quantum random walk on $\mathbb{Z}^2$. Starting with the state

$$\rho^{(0)} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes |0\rangle\langle 0|0\rangle\langle 0|,$$

We find the following probabilities for the 2 first steps

Table-1

(x,y)	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
Probability	$1/4$	$1/4$	$1/4$	$1/4$

Table-2

(x,y)	$(2,2)$	$(2,-2)$	$(2,0)$	$(-2,0)$	$(0,0)$	$(0,-2)$	$(0,2)$	$(-2,2)$	$(-2,-2)$
Probability	$1/16$	$1/16$	$2/16$	$2/16$	$4/16$	$2/16$	$2/16$	$1/16$	$1/16$

Table 1 shows the probabilities for the 2 first steps.

From Table 1 and Figure 1, we can clearly see that it is a symmetric Gaussian distribution, which means that the random walk in the classical case can be represented under certain conditions by quantum random walk. In the classical random walk, as time tends to infinity, the one will eventually return to origin, which can be clearly seen in Figure 1.

Example 2

If we take

$$R = F = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \text{ and } L = B = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix},$$

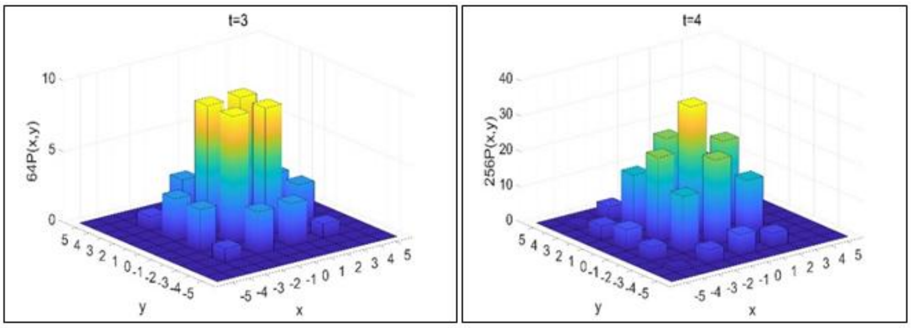

Fig-1: Shows the probability distributions for the third and fourth steps

which the operators R, L, B, F do satisfy $R^*R + L^*L = I,\ B^*B + F^*F = I$. Let us consider the associated open quantum random walk on $\mathbb{Z}^2$. Starting with the state

$$\rho^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |0\rangle\langle 0|0\rangle\langle 0|,$$

we find the following probabilities for the 3 first steps:

Table-1

(x, y)	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
Probability	$1/9$	$2/9$	$1/9$	$5/9$

Table-2

(x, y)	$(2,2)$	$(2,-2)$	$(2,0)$	$(-2,0)$	$(0,0)$	$(0,-2)$	$(0,2)$	$(-2,2)$	$(-2,-2)$
Probability	$1/81$	$1/81$	$4/81$	$14/81$	$24/81$	$12/81$	$6/81$	$2/81$	$17/81$

Table-3

(x, y)	$(3,3)$	$(3,-3)$	$(3,1)$	$(3,-1)$	$(1,3)$	$(1,-3)$	$(1,1)$	$(1,-1)$
Probability	$1/729$	$2/729$	$6/729$	$9/729$	$7/729$	$29/729$	$45/729$	$81/729$
(x, y)	$(-3,3)$	$(-3,-3)$	$(-3,1)$	$(-3,-1)$	$(-1,3)$	$(-1,-3)$	$(-1,1)$	$(-1,-1)$
Probability	$2/729$	$85/729$	$24/729$	$87/729$	$8/729$	$94/729$	$72/729$	$177/729$

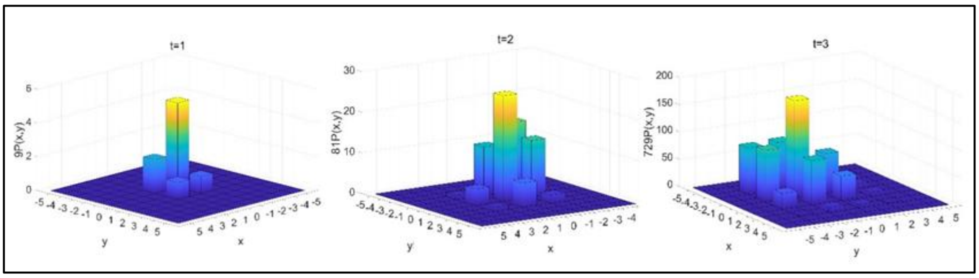

Fig-2

From Table 2 and Figure 2, it is obvious that the distribution is asymmetric, but it will gradually form a Gaussian distribution with the number of steps tends to infinity.

Quantum entropy

Definition 1. [1] Von Neumann defined the entropy of a quantum state ρ by the formula

$$S(\rho) \equiv -Tr(\rho \log \rho), \tag{4}$$

Where the logarithms are taken to base two and ρ actually is a density operator.
Without confusion, all logarithms throughout this article are taken to base two.

Theorem 2. The quantum entropy presents by 2-D open quantum walk is

$$\mathrm{S}(\rho) = -\sum_{x,y} \mathrm{Tr}(\rho_{xy}\log\rho_{xy}). \tag{5}$$

Proof. From Equation (1) we have

$$\rho = \sum_{x,y} \rho_{xy} \, |x\rangle\langle x|y\rangle\langle y|.$$

Bringing it into Equation (4)

$$\begin{aligned}
\mathrm{S}(\rho) &\equiv -Tr(\rho\log\rho) \\
&= -Tr\left(\sum_{x,y} \rho_{xy} \, |x\rangle\langle x|y\rangle\langle y|\log\sum_{x,y} \rho_{xy} \, |x\rangle\langle x|y\rangle\langle y|\right) \\
&= -\sum_{x,y} Tr\big(\rho_{xy}|x\rangle\langle x|y\rangle\langle y|\log\rho_{xy}|x\rangle\langle x|y\rangle\langle y|\big) \\
&= -\sum_{x,y} Tr\big(\rho_{xy}\log\rho_{xy}\big)\,\langle x|x\rangle\langle y|y\rangle \\
&= -\sum_{x,y} Tr\big(\rho_{xy}\log\rho_{xy}\big)
\end{aligned}$$

Theorem 3. The quantum entropy $\mathrm{S}(\rho) = -\sum_{x,y}\mathrm{Tr}(\rho_{xy}\log\rho_{xy})$ in 2-D open quantum walk can be presented as Shannon joint entropy of random variables X and Y

$$\mathrm{S}(\rho) = \mathrm{H}(X,Y) = -\sum_{x,y} p(x,y)\log p(x,y) \tag{6}$$

where $p(x,y)$ are the eigenvalues of ρ_{xy}.

Proof. From Equation (3) we have

$$\rho_{xy} = \sum_{x,y} p(x,y) \, |i\rangle\langle i|.$$

Bringing it into Equation (5)

$$\begin{aligned}
\mathrm{S}(\rho) &= -\sum_{x,y} \mathrm{Tr}(\rho_{xy}\log\rho_{xy}) \\
&= -\sum_{x,y} Tr\big(p(x,y)|i\rangle\langle i|\log p(x,y)|i\rangle\langle i|\big) \\
&= -\sum_{x,y} p(x,y)\log p(x,y)\,\langle i|i\rangle\langle i|i\rangle \\
&= -\sum_{x,y} p(x,y)\log p(x,y) \\
&= \mathrm{H}(X,Y)
\end{aligned}$$

It is obvious that the space of degrees of freedom of separable Hilbert space $\mathcal{H}$ is $(t+1)^2$, where t stands for the number of steps. Therefore, our quantum entropy has its range.

Theorem 4. The quantum entropy $\mathrm{S}(\rho)$ in 2-D open quantum walk has its range that

$$0 \le \mathrm{S}(\rho) \le 2\log(t+1), \tag{7}$$

where t represent the number of steps.

We briefly calculate the first 3 steps of two examples as follows.

Quantum entropy of Example 1

$$\mathrm{S}\big(\rho^{(1)}\big) = -\sum_{x,y} p(x,y)\log p(x,y) = 2 \le 2,$$

$$\mathrm{S}\big(\rho^{(2)}\big) = -\sum_{x,y} p(x,y)\log p(x,y) = 3 \le 2\log 3$$

$$S(\rho^{(3)}) = -\sum_{x,y} p(x,y)\log p(x,y) = \frac{2409}{665} \approx 3.6226 \leq 4$$

Quantum entropy of Example 2

$$S(\rho^{(1)}) = -\sum_{x,y} p(x,y)\log p(x,y) = \frac{867}{523} \approx 1.6577 \leq 2,$$

$$S(\rho^{(2)}) = -\sum_{x,y} p(x,y)\log p(x,y) = \frac{757}{289} \approx 2.6194 \leq 2\log 3,$$

$$S(\rho^{(3)}) = -\sum_{x,y} p(x,y)\log p(x,y) = \frac{1134}{353} \approx 3.2125 \leq 4.$$

By comparing the two examples, it can be found that the quantum entropy of Example 1 in each step is generally larger than that of Example 2 in corresponding step, and both Examples are in the normal range $[0,2\log(t+1)]$. We all know that the larger the entropy, the greater the uncertainty, which shows that the uncertainty of Example 1 is obviously greater than that of Example 2.

CONCLUSIONS

In this paper, we mainly consider the two-dimensional discrete time OQWs. It has widely studied in limit distribution, central limit theorem and large deviation principle. However, we are more interested in quantum Bernoulli noise. This work will be prepared for two-dimensional OQWs in terms of quantum Bernoulli noise.

REFERENCES

1. Nielsen MA, Chuang IL. Quantum computation and quantum information. Cambridge: Cambridge University Press. 2000.
2. Aharonov Y, Davidovich L, Zagury N. Quantum random walks. PHYSICAL REVIEW A. 1993, 48(2): 1687-1690
3. Venegas-Andraca SE. Quantum walk:a comprehensive review. Quantum Information Processing. 2012, 11: 1015-1106
4. Attal S, Petruccione F, Sabot C, Sinayskiy I. Open Quantum Random Walks. J Stat Phys. 2012: 147:832–852
5. Caishi W, Ce W, Suling R, Yuling T. Open quantum random walk in terms of quantum Bernoulli noise. Quantum Inf Process. 2018; 17:46
6. Yuanbao K. Quantum Markov semigroups for continuous-time open quantum random walk. Quantum Information Processing. 2019, 18:196
7. Chul Ki Ko, Norio Konno, Etsuo Segawa, Hyun Jae Yoo. Central Limit Theorems for Open Quantum Random Walks on the Crystal Lattices. Journal of Statistical Physics. 2019, 176:710–735
8. Masanori Ohya, Dénes Petz. Quantum Entropy and Its Use. Springer-Verlag Berlin Heidelberg New York.1993.