Vertex Distinguishing General-total Coloring of $K_{2,5, p}$

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Abstract

With the wide application of graph coloring in real life, it has gradually become one of the important fields studied by many scholars. A general total coloring of graph $G$ refers to mapping $f : V(G) \cup E(G) \rightarrow \{1, k\}$. For any $x \in V(G)$, let $C_x(x)$ or $C(x)$ be the set of colors of vertex $x$ and edges incident with $x$ under $f$, which is called the color set of point $x$ under $f$. For any $u, v \in V(G)$, if $C(u) \neq C(v)$, then $f$ is called a $k$-vertex distinguishing general-total coloring of graph $G$ (k-GVDTC). The minimum number of colors required for a VDT coloring of $G$ is denoted by $\chi_{gvt}(G)$. The vertex distinguishing general-total chromatic number of graph $G$ is discussed in this paper by using the methods of distributing the color sets in advance, constructing the colorings and contradiction. The vertex distinguishing general-total chromatic numbers of $K_{2,5,p}$ are determined.

Keywords: complete tripartite graphs; general-total coloring; vertex distinguishing general-total coloring; vertex distinguishing general-total chromatic number.

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INTRODUCTION

The point distinguishing general edge coloring of graphs are raised by Harary F in [1] and then be studied in [1-6] deeply. For a total coloring (proper or not) $f$ of $G$ and a vertex $x$ of $G$, let $C_x(x)$ be the set of colors of vertex $x$ and edges incident with $x$ under $f$. For a proper total coloring, if $C(u) \neq C(v)$ for any two distinct vertices $u$ and $v$, then the coloring is called a vertex distinguishing (proper) total coloring, or a VDT coloring of $G$ for short. The minimum number of colors required for a VDT coloring of $G$ is denoted by $\chi_{vt}(G)$. The vertex distinguishing (proper) total coloring of graph was introduced and studied in [7]. In the following we consider not necessarily proper general total coloring which are vertex distinguishing. A general total coloring $f$ of $G$ is an assignment of some colors to the vertices and edges of $G$, for any $u, v \in V(G)$, $u \neq v$, we have $C(u) \neq C(v)$, then $f$ is called a vertex distinguishing general total coloring or a GVDTC briefly. Vertex distinguishing general-total coloring was presented in [8]. The minimum number of colors required for a GVDTC of graph $G$ is denoted by $\chi_{gvt}(G)$.

In this paper, we consider vertex distinguishing general coloring of $K_{2,5,p}$, its general vertex distinguishing chromatic of $K_{2,5,p}$ will be determined as well. Let $X = \{x_1,x_2,...,x_n\}, Y = \{y_1,y_2,...,y_n\}, Z = \{z_1,z_2,...,z_p\}, V(K_{m,n,p}) = X \cup Y \cup Z$, and $E(K_{m,n,p}) = \{x_iy_j | i = 1,2,...,m, j = 1,2,...,n\} \cup \{y_iz_t | j = 1,2,...,n, t = 1,2,...,p\} \cup \{x_iz_t | i = 1,2,...,m, t = 1,2,...,p\}$.

For convenience of description, we make the following agreements: When an $l$-GVDTC of a graph is mentioned or is to be given herein, we always think that the $l$ color used is $1,2,...,l$; An $i$-subset of $\{1,2,...,l\}$ is a subset of $\{1,2,...,l\}$ containing $i$ elements; If $A$ is a subset of $\{1,2,...,l\}$, then $\overline{A}$ is used to denote the complement set of $A$. 

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Preliminaries

Lemma 1: If \( k \geq 11 \) and \( p > \frac{k}{m} \left( k \right) \), then \( K_{2,5,p} \) has no \((k-1)\)-GVDTC.

Proof: Suppose \( K_{2,5,p} \) has a \((k-1)\)-GVDTC coloring \( f \).

Claim 1: Any 1-subset of \{1,2,\ldots,k\} cannot be the color sets of vertices in \( X \cup Y \). Otherwise, if \( \{1\} \) is the color sets of the vertices in \( X \) then the subsets of all vertices in \( Z \) contain color "1". The number of the subsets of \{1,2,\ldots,k\} which may become the color sets of vertices in \( Z \) is \( \binom{k-1}{1} + \binom{k-2}{2} + \cdots + \binom{k-2}{7} \). However, \( p > \frac{k}{m} \left( k \right) \). This is a contradiction.

Claim 2: There are at least three 1-subsets of \{1,2,\ldots,k\}, which cannot be the color sets of vertices in \( Z \).

Otherwise, without loss of generality, we may assume that \{3\}, \{4\}, \ldots, \{k\} are all the color sets of vertices in \( Z \). That is to say, \( C(X) \cap C(Y) \supseteq \{3,4,\ldots,k\} \). Thus, the color sets of vertices in \( X \cup Y \) can only be \{3,4,\ldots,k\}, \{1,3,4,\ldots,k\}, \{2,3,4,\ldots,k\} or \{1,2,\ldots,k\}, which cannot distinguish the 7 vertices in \( X \cup Y \). This is a contradiction.

From Claim 2, we can assume that \{1\}, \{2\} or \{3\} cannot be the color sets of vertices in \( Z \). Combining Claim 1, we know \{1\}, \{2\} or \{3\} are not available for any vertex in the graph. Because \( C(X) \) and \( f(i) = 1,2,3,4,5 \), the 7 subsets are different, and at least 3 of them are not \( \emptyset \), \{1\}, \{2\} or \{3\}. We can assume that \( C(Y) \), \( C(Y)_3 \) or \( C(Y)_4 \) are all not \( \emptyset \) in \{1\}, \{2\}, \{3\}.

(i) When \( |C(Y)_j| \geq 3 \), j = 3, 4, 5, we know that \( \{1\}, \{2\}, \{3\}, C(Y)_3 \), \( C(Y)_4 \) and \( C(Y)_5 \) cannot be the color sets of vertices in \( Z \).

Then the available color sets of vertices in \( Z \) are the 1-subset, 2-subset, \ldots, 8-subset of \{1,2,\ldots,k\}, except for \{1\}, \{2\}, \{3\}, \( C(Y)_3 \), \( C(Y)_4 \) and \( C(Y)_5 \). So \( p > \frac{k}{m} \left( k \right) \), a contradiction.

(ii) \( |C(Y)_j| \geq 4 \) for \( j \in \{3,4,5\} \), we know that 1-subset, 2-subset, 3-subset and 4-subset of \( C(Y)_j \) cannot be the color sets of vertices in \( Z \). So \( p > \frac{k}{m} \left( k \right) \), a contradiction.

Lemma 2: If \( k \geq 11 \) and \( p > \frac{k}{m} \left( k \right) \), then \( K_{2,5,p} \) exist a \( k \)-GVDTC.

Proof: If the first, we distribute subsets of \{1,2,\ldots,k\} to the vertices of \( K_{2,5,p} \). Put \( D(x) = D(x) \cup \{2\} \), \( D(y) = D(y) \cup \{1\} \), \( D(y) = D(x) \cup \{3\} \), \( D(Y) = D(x) \cup \{4\} \), \( D(Y) = D(x) \cup \{5\} \), \( D(y) = D(x) \cup \{1\} \), \( D(y) = D(x) \cup \{3\} \), \( D(y) = D(x) \cup \{4\} \), \( D(y) = D(x) \cup \{5\} \). Let \( \mathbb{R} \) be the sequence, which is consist of the subsets of \{1,2,\ldots,k\} with cardinal numbers between 2 and 8, except for \{1\}, \{2\}, \{3\}, \{4\}, \{6\} and \{5,6\}, noticing that \( |\mathbb{R}| = \binom{k-2}{2} + \binom{k-3}{3} + \binom{k-4}{4} + \cdots + \binom{k-8}{8} \), a contradiction.

In the following, we give a \( k \)-GVDTC coloring \( f \) of \( K_{2,5,p} \) using colors 1,2,\ldots,k. Put \( g(x) = g(x) = 1, i = 1,2, g(y) = 1, j = 1,2,\ldots, g(z) = \text{max}(D(z)) \), \( t = 1,2,\ldots, p \) and \( g(x,y) \in X \cup Y \). Let \( g(x,y) = \min[D(x) \cup D(y)] \). When \( u \in X \cup Y \) \( D(z) = 2 \), let \( g(u) = \min[D(u) \cup D(z)] \). When \( |D(z)| = 3 \), we assume that \( D(z) = \{a,b,c\} \) and \( a < b < c \). Color edges \( x_1z_1 \), \( y_1z_1 \) and \( x_2z_2 \) with \( a, b \) and \( c \). When \( u \in \{y_2, y_3, y_4, z_1\} \), \( g(u) = \min[D(u) \cup D(z)] \). When \( |D(z)| = 4 \), we assume that \( D(z) = \{a,b,c,d\} \) and \( a < b < c < d \). Color edges \( x_1z_1 \), \( y_1z_1 \) and \( x_2z_2 \) with \( a, b, c \) and \( d \). When \( u \in \{y_3, y_4, z_2, y_5, z_3\} \), \( g(u) = \min[D(u) \cup D(z)] \). When \( |D(z)| = 5 \), we assume that \( D(z) = \{a,b,c,d,e\} \) and \( a < b < c < d < e \). Color edges \( x_1z_1 \), \( y_1z_1 \), \( x_2z_2 \), \( y_2z_2 \) and \( y_2z_3 \) with \( a, b, c, d \) and \( e \). Then \( u \in \{y_5, z_3\} \), \( g(u) = \min[D(u) \cup D(z)] \). When \( |D(z)| = 6 \), we assume that \( D(z) = \{a_1, a_2, a_3, a_4, a_5, a_6\} \) and \( a_1 < a_2 < a_3 < a_4 < a_5 < a_6 \). Color edges \( x_1z_1 \), \( y_1z_1 \), \( x_2z_2 \), \( y_2z_2 \), \( y_3z_3 \) and \( y_4z_4 \) with \( a_1, a_2, a_3, a_4, a_5 \) and \( a_6 \). Let \( g(z) = \min[D(y)] \).
So and \( y \) \( x \) \( y \) \( x \) \( y \) \( x \) \( y \) \( x \) \( y \) \( x \). From Claim 2 in Lemma 1, when \( y \) \( x \) \( y \), and at least 6 color sets of vertices in \( y \) \( j \) \( i \) \( y \) \( j \) \( i \) \( y \) \( j \), with \( a_1, a_2, a_3, a_4, a_5, a_6 \) and \( a_7 \).

It is not hard to see that \( C(v) = D(v), \forall v \in V(K_{2,5,p}) \). Therefore our coloring \( g \) is a vertex distinguishing general-total coloring.

**MAIN RESULTS AND ITS PROOFS**

**Theorem 1:** For any positive integer \( p \geq 5 \), we have:

\[
\chi_{gt}(K_{2,5,p}) = \begin{cases} 
5, & \text{when } 5 \leq p \leq 18; \\
6, & \text{when } 19 \leq p \leq 50; \\
7, & \text{when } 51 \leq p \leq 114; \\
8, & \text{when } 115 \leq p \leq 242; \\
9, & \text{when } 243 \leq p \leq 498; \\
10, & \text{when } 499 \leq p \leq 1006; \\
\end{cases}
\]

\[
\begin{aligned}
&k, & \text{when } & \frac{3}{11}(k-1) - 6 \leq p \leq \frac{3}{11}(k) - 6, k \geq 11
\end{aligned}
\]

**Proof:** From Lemma 1 and Lemma 2, we know that if \( l \geq 11 \) and \( \sum_{i=1}^{t} \left( \frac{k}{i} \right) - 6 < p \leq \sum_{i=1}^{t} \left( \frac{k}{i} \right) - 6, \) then the conclusion is true. Now we consider the other 6 cases.

**Case 1:** If \( 499 \leq p \leq 1006 \), then \( \chi_{gt}(K_{2,5,p}) = 10 \).

Assume that \( K_{2,5,p} \) has a 9-GVDTC. From Claim 1 in Lemma 1, when \( l = 10 \), we know that any 1-subset of \( \{1,2,\ldots,9\} \) cannot be the color sets of vertices in \( X \cup Y \). From Claim 2 in Lemma 1, when \( l = 10 \), we know that at least three 1-subsets of \( \{1,2,\ldots,9\} \) cannot be any color set of vertex in \( Z \). So we can assume that \{1,2\} or {3} cannot be the color sets of vertices in the graph. Because \( C(x_i) \) and \( C(y_j) \), \( i = 1,2,3,4,5 \), the 7 subsets are different, and at least 3 of them are not \( \emptyset \), \{1\}, \{2\} or \{3\}. We can assume that \( C(Y_3) \), \( C(Y_4) \) or \( C(Y_5) \) are all not \( \emptyset \), \{1\}, \{2\}, \{3\}.

(i) When \( |C(Y_j)| \leq 3, j = 3,4,5 \), the available color sets of vertices in \( Z \) are the 1-subset, 2-subset, \ldots, 8-subset of \( \{1,2,\ldots,9\} \), except for \{1\}, \{2\}, \{3\}, \( C(Y_j) \), \( C(Y_4) \) and \( C(Y_5) \), and at least 6 color sets of vertices in \( X \cup Y \). So \( p \leq \sum_{i=1}^{t} \left( \frac{k}{i} \right) - 12 = 498 \), a contradiction.

(ii) \( |C(Y_j)| \leq 4 \) for \( j = 3,4,5 \). We know that 1-subset, 2-subset, 3-subset and 4-subset of \( C(Y_j) \) cannot be the color set of any vertex in \( Z \). So \( p \leq \sum_{i=1}^{t} \left( \frac{k}{i} \right) - 15 = 496 \), a contradiction.

A 10-GVDTC of \( K_{2,5,p} \) can be obtained by 10-GVDTC of \( K_{2,5,1006} \), which is limited by \( \{x_1,x_2,y_1,\ldots,y_{11},z_1,z_2,\ldots,z_5\} \). Then we give a 10-GVDTC of \( K_{2,5,1006} \). Put \( D(x_1) = \{1,2,\ldots,10\} \), \( D(x_2) = D(x_3) \), \( D(x_4) = D(x_5) \), \( D(y_1) = D(y_2) = D(y_3) \), \( D(y_4) = D(y_5) \), \( D(y_6) = D(y_7) \), \( D(y_8) = D(y_9) \), \( D(y_{10}) = D(y_{11}) \), \( D(z_1) = D(z_2) = D(z_3) \), \( D(z_4) = D(z_5) \), \( D(z_6) = D(z_7) \), \( D(z_8) = D(z_9) \), \( D(z_{10}) = D(z_{11}) \). Put 1-subset, 2-subset, \ldots, 8-subset of \( \{1,2,\ldots,10\} \), except for \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,5\}, \{1,7\}, \{1,9\}, \{1,10\}, \{2,4\}, \{2,8\}, \{2,10\}, \{3,5\}, \{3,9\}, \{3,10\}, \{4,6\}, \{4,8\}, \{4,10\}, \{5,7\}, \{5,9\}, \{5,10\}, \{6,8\}, \{6,10\}, \{7,9\}, \{7,10\}, \{8,9\}, \{8,10\}, \{9,10\} \), as the color sets of vertices in \( Z \). By the second paragraph of Lemma 2, we could get the specific coloring method of \( K_{2,5,1006} \).

Case 2: If \( 243 \leq p \leq 498 \), then \( \chi_{gt}(K_{2,5,p}) = 9 \).

Assume that \( K_{2,5,p} \) has a 8-GVDTC. From Claim 1 in Lemma 1, when \( l = 9 \), we know that any 1-subset of \( \{1,2,\ldots,8\} \) cannot be the color set of vertices in \( X \cup Y \). From Claim 2 in Lemma 1, when \( l = 9 \), we know that at least three 1-subset of \( \{1,2,\ldots,8\} \) cannot be any color set of vertex in \( Z \). So we can assume that \{1\}, \{2\} or \{3\} are not the color sets of vertices in the graph. Because \( C(x_i) \) and \( C(y_j) \), \( i = 1,2,3,4 \),
5, the 7 subsets are different, and at least 3 of them are not $\emptyset, \{1\}, \{2\}$ or $\{3\}$. We can assume that $\overline{C(y_j)}$, $C(y_4)$ or $\overline{C(y_3)}$ are all not $\emptyset, \{1\}, \{2\}, \{3\}$.

(i) When $|\overline{C(y_j)}| \leq 3, j=3,4,5$, the available color set of vertices in Z are the 1-subset, 2-subset, ..., 8-subset of $\{1,2,..,8\}$, except for $\{1\}, \{2\}, \{3\}, \overline{C(Y_3)}, C(Y_4)$ and $\overline{C(Y_3)}$, and all color sets of vertices in $X \cup Y$. So $p \leq \sum_{i=1}^{8} \binom{8}{i} - 13 = 242$, a contradiction.

(ii) $|\overline{C(y_j)}| \leq 4$ for $j \in \{3,4,5\}$, we know that 1-subset, 2-subset, 3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p \leq \sum_{i=1}^{8} \binom{8}{i} - 15 = 240$, a contradiction.

A 9-GVDTC of $K_{2,5,\phi}$ can be obtained by 9-GVDTC of $K_{2,5,498}$, which is limited by $\{x_1, y_2, x_3, \ldots, y_5, z_1, z_2, \ldots, z_\phi\}$. Then we give a 9-GVDTC of $K_{2,5,498}$. Put $D(x_1) = \{1,2, \ldots, 9\}$, $D(x_2) = D(x_1) \setminus \{2\}$, $D(x_3) = D(x_1) \setminus \{1\}$, $D(y_2) = D(x_1) \setminus \{3\}$, $D(y_3) = D(x_1) \setminus \{4\}, D(y_4) = D(x_1) \setminus \{5\}, D(y_5) = D(x_1) \setminus \{1,3\}$. Put 1-subset, 2-subset, ..., 8-subset of $\{1,2,..,9\}$, except for $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}$, we know that the color sets of vertices in Z. By the second paragraph of Lemma 2, we can get the specific coloring method of $K_{2,5,498}$.

**Case 3: If 1155p≤242, then $\chi_C(K_{2,5,\phi})=8$.**

Assume that $K_{2,5,\phi}$ has a 7-GVDTC $\mathcal{C}$. From Claim 1 and Claim 2 in Lemma 1, when $l=8$, we can assume that $\{1\}, \{2\}$ or $\{3\}$ are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(Y_j)}, i=1,2, j=1,2,3,4,5$, the 7 subsets are different, and at least 3 of them are not $\emptyset, \{1\}, \{2\}$ or $\{3\}$. We can assume that $\overline{C(Y_3)}, C(Y_4)$ or $\overline{C(Y_3)}$ are all not $\emptyset, \{1\}, \{2\}, \{3\}$.

(i) When $|\overline{C(y_j)}| \leq 3, j=3,4,5$, the available color set of vertices in Z are the 1-subset, 2-subset, ..., 7-subset of $\{1,2,..,7\}$, except for $\{1\}, \{2\}, \{3\}, \overline{C(Y_3)}, C(Y_4)$ and $\overline{C(Y_3)}$, and all color sets of vertices in $X \cup Y$. So $p \leq \sum_{i=1}^{7} \binom{7}{i} - 13 = 114$, a contradiction.

(ii) $|\overline{C(y_j)}| \leq 4$ for $j \in \{3,4,5\}$, we know that 1-subset, 2-subset, 3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p \leq \sum_{i=1}^{7} \binom{7}{i} - 15 = 112$, a contradiction.

A 8-GVDTC of $K_{2,5,\phi}$ can be obtained by 8-GVDTC of $K_{2,5,242}$, which is limited by $\{x_1, x_2, y_1, y_2, \ldots, y_5, z_1, z_2, \ldots, z_\phi\}$. Then we give a 8-GVDTC of $K_{2,5,242}$. Put $D(x_1) = \{1,2,..,8\}$, $D(x_2) = D(x_1) \setminus \{2\}$, $D(y_2) = D(x_1) \setminus \{1\}$, $D(y_3) = D(x_1) \setminus \{4\}, D(y_4) = D(x_1) \setminus \{5\}, D(y_5) = D(x_1) \setminus \{1,3\}$. Put 1-subset, 2-subset, ..., 8-subset of $\{1,2,..,8\}$, except for $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}$, as the color sets of vertices in Z. By the second paragraph of Lemma 2, we can get the specific coloring method of $K_{2,5,242}$.

**Case 4: If 51\leq\phi\leq114, then $\chi_C(K_{2,5,\phi})=7$.**

Assume that $K_{2,5,\phi}$ has a 6-GVDTC $\mathcal{C}$. From Claim 1 and Claim 2 in Lemma 1, when $l=7$, we can assume that $\{1\}, \{2\}$ or $\{3\}$ are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(Y_j)}, i=1,2, j=1,2,3,4,5$, the 7 subsets are different, and at least 3 of them are not $\emptyset, \{1\}, \{2\}$ or $\{3\}$. We can assume that $\overline{C(Y_3)}, C(Y_4)$ or $\overline{C(Y_3)}$ are all not $\emptyset, \{1\}, \{2\}, \{3\}$.

(i) When $|\overline{C(y_j)}| \leq 3, j=3,4,5$, the available color sets of vertices in Z are the 1-subset, 2-subset, ..., 6-subset of $\{1,2,..,6\}$, except for $\{1\}, \{2\}, \{3\}, \overline{C(Y_3)}, C(Y_4)$ and $\overline{C(Y_3)}$, and all color sets of vertices in $X \cup Y$. So $p \leq \sum_{i=1}^{6} \binom{6}{i} - 13 = 60$, a contradiction.
(ii) $|C(Y_j)| \leq 4$ for $j \in \{3,4,5\}$. We know that $1$-subset, $2$-subset, $3$-subset and $4$-subset of $C(Y_j)$ cannot be the color set of any vertex in $Z$. So $p \leq \frac{\sum_{j=1}^{5} \binom{5}{j}}{m(i)} - 15 = 48$, a contradiction.

A $7$-GVDTC of $K_{2,5,\rho}$ can be obtained by $7$-GVDTC of $K_{2,5,114}$, which is limited by $\{x_1, x_2, y_1, y_2, \ldots, y_5, z_1, z_2, \ldots, z_\rho\}$. Then we give a $7$-GVDTC of $K_{2,5,114}$. Put $D(x_1) = \{1, \ldots, 7\}$, $D(x_2) = D(x_1) \setminus \{2\}$, $D(y_1) = D(x_1) \setminus \{1\}$, $D(y_2) = D(x_1) \setminus \{3\}$, $D(y_3) = D(x_1) \setminus \{4\}$, $D(y_4) = D(x_1) \setminus \{5\}$, $D(y_5) = D(x_1) \setminus \{1, 3\}$. Put $1$-subset, $2$-subset, ..., $7$-subset of $\{1, 2, \ldots, 7\}$ except for $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}$, as the color sets of vertices in $Z$. By the second paragraph of Lemma 2, we could get the specific coloring method of $K_{2,5,114}$.

Case 5: If $19 \leq \rho \leq 50$, then $\chi_{pet}(K_{2,5,\rho}) = 6$.

Assume that $K_{2,5,\rho}$ has a $5$-GVDTC $g$. From Claim 1 and Claim 2 in Lemma 1, when $i = 6$, we can assume that $\{1\}, \{2\} \text{ or } \{3\}$ are not the color sets of vertices in the graph. Because $C(x_i)$ and $C(y_j)$, $i = 1, 2, 3, 4, 5$, the $7$ subsets are different, and at least $3$ of them are not $\emptyset$, $\{1\}, \{2\}$ or $\{3\}$. We can assume that $C(y_3), C(y_4), C(y_5)$ are all not $\emptyset$, $\{1\}, \{2\}, \{3\}$.

(i) When $|C(y_j)| \leq 3$, $j = 3, 4, 5$, the available color sets of vertices in $Z$ are the $1$-subset, $2$-subset, $3$-subset of $\{1, 2, \ldots, 5\}$, except for $\{1\}, \{2\}, \{3\}$, $C(y_j)$ and $C(y_3)$, and $C(y_4)$ and $C(y_5)$, and all color sets of vertices in $X \cup Y$. So $p \leq \frac{\sum_{j=3}^{5} \binom{5}{j}}{m(i)} - 13 = 18$, a contradiction.

(ii) $|C(y_j)| \leq 4$ for $j \in \{3,4,5\}$, we know that $1$-subset, $2$-subset, $3$-subset and $4$-subset of $C(y_j)$ cannot be the color set of any vertex in $Z$. So $p \leq \frac{\sum_{j=3}^{5} \binom{5}{j}}{m(i)} - 15 = 16$, a contradiction.

A $6$-GVDTC of $K_{2,5,\rho}$ can be obtained by $6$-GVDTC of $K_{2,5,50}$, which is limited by $\{x_1, x_2, y_1, y_2, \ldots, y_3, z_1, z_2, \ldots, z_\rho\}$. Then we give a $6$-GVDTC of $K_{2,5,50}$. Put $D(x_1) = \{1, \ldots, 6\}$, $D(x_2) = D(x_1) \setminus \{2\}$, $D(y_1) = D(x_1) \setminus \{1\}$, $D(y_2) = D(x_1) \setminus \{3\}$, $D(y_3) = D(x_1) \setminus \{4\}$, $D(y_4) = D(x_1) \setminus \{5\}$, $D(y_5) = D(x_1) \setminus \{1, 3\}$. Put $1$-subset, $2$-subset, ..., $6$-subset of $\{1, 2, \ldots, 6\}$, except for $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}$, as the color sets of vertices in $Z$. By the second paragraph of Lemma 2, we could get the specific coloring method of $K_{2,5,50}$.

Case 6: If $5 \leq \rho \leq 18$, then $\chi_{pet}(K_{2,5,\rho}) = 5$.

Assume that $K_{2,5,\rho}$ has a $5$-GVDTC $g$.

Claim 3: Any $1$-subset of $\{1, 2, 3, 4\}$ cannot be the color sets of vertices in $X$.

Otherwise, we can assume that $g(x_1) = \{1\}$, then any vertex in $Y \cup Z$ contains color “$1$”. There are only $8$ subsets which contain color “$1$", so they cannot distinguishing at least $11$ vertices in $Y \cup Z \cup \{x_1\}$, a contradiction.

Claim 4: There are at least three $1$-subsets of $\{1, 2, 3, 4\}$, which cannot be the color sets of vertices in $Z$.

Otherwise, we can assume that only $\{1\}$ or $\{2\}$ are not the color sets of vertices in $Z$. $C(x_1) \cup C(y_1) \supseteq \{3, 4\}$, so the color sets which can be assigned are $\{3, 4, 1, 3, 4, 2, 3, 4\}$ and $\{1, 2, 3, 4\}$. That $4$ sets can not distinguish the $7$ vertices in $X \cup Y$. This is a contradiction.

Claim 5: When $p \geq 6$, any $1$-subset of $\{1, 2, 3, 4\}$ cannot be the color sets of vertices in $Y$.

Otherwise, we could assume $g(y_1) = 1$, then any vertex in $X \cup Z$ must contain color “$1$". There are only $8$ subsets which contain color “$1$", so they cannot distinguishing at least $9$ vertices in $X \cup Z \cup \{y_1\}$, a contradiction.

The following are discussed in two cases:

(i) When $p \geq 6$, 

From Claim 2, we can assume \( \{1\}, \{2\}, \{3\} \) cannot be the color sets of vertices in \( Z \). Because \( \overline{C(x)} \) and \( \overline{C(y)} \), \( i=1, j=1, 2, 3, 4, 5 \), the 7 subsets are different, and at least 3 of them are not \( \emptyset, \{1\}, \{2\} \) or \( \{3\} \). We can assume that \( \overline{C(y)} \), \( \overline{C(y)} \) or \( \overline{C(y)} \) are all not \( \emptyset, \{1\}, \{2\}, \{3\} \). Then the available color sets of vertices in \( Z \) are the 1-subset, 2-subset, 3-subset, 4-subset of \( \{1, 2, 3, 4\} \), except for \( \{1\}, \{2\}, \{3\}, \overline{C(y)} \), \( \overline{C(y)} \), \( \overline{C(y)} \) and all color sets of vertices in \( X \cup \overline{Y} \). So \( p \leq \sum_{i=1}^{4} \frac{\delta(X)}{\delta(Y)} - 13 = 2 \), a contradiction.

(ii) When \( p = 5 \),

\( \{1, 2, 3, 4\} \) has 15 subsets, they could be the color sets of vertices in \( K_{2,5,5} \). There are 14 subsets except for \( \{1, 2, 3, 4\} \), which are complementary to each other. So there are 7 pairs of sets which are complementary to each other. The color sets in \( X \) and \( Y \) and \( Z \) and \( X \) must not complement each other. We consider if there are two vertices complementing to each other in \( X \); then any 1-subset cannot be the color sets of vertices in \( Z \). \( p + 7 \leq \sum_{i=1}^{4} \frac{\delta(X)}{\delta(Y)} - 4 \), \( p \leq 4 \), a contradiction. Similarly, the color sets of any two vertices in \( Y \) or \( Z \) cannot complement each other. So there are only 7 subsets which could be the color sets of vertices in \( K_{2,5,5} \), a contradiction.

A 5-GVDTC of \( K_{2,5,5} \) can be obtained by 5-GVDTC of \( K_{2,5,5} \), which is limited by \( \{x_1, x_2, y_1, y_2, \ldots, y_5, z_1, z_2, \ldots, z_p\} \). Then we give a 5-GVDTC of \( K_{2,5,5} \). Put \( D(x_1) = \{1, 2, \ldots, 5\} \), \( D(x_2) = D(x_1) \backslash \{2\} \), \( D(y_1) = D(x_1) \backslash \{1\} \), \( D(y_2) = D(x_1) \backslash \{3\} \), \( D(y_3) = D(x_1) \backslash \{4\} \), \( D(y_4) = D(x_1) \backslash \{5\} \), \( D(y_5) = D(x_1) \backslash \{1, 3\} \). Put 1-subset, 2-subset, ..., 5-subset of \( \{1, 2, \ldots, 5\} \), except for \( \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\} \), as the color sets of vertices in \( Z \). By the second paragraph of Lemma 2, we could get the specific coloring method of \( K_{2,5,5} \).

REFERENCES
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