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Estimation of the Xrama Distribution Parameter Under Complete and Progressive Type-II Censored Schemes

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Abstract **Original Research Article**

In this paper, a two-component mixture of Exponential and Rama distribution named Xrama distribution is introduced and discussed. The distributional properties such as moment and the related measures including other characteristics namely quantile function, mean residual life function, Bonferroni curve, Lorenz curve, order statistics, stochastic ordering, and entropy were derived. Simulation study for both complete and progressive type-2 censored schemes is used to illustrate and analyze the behavior of the model. The proposed model's performance is assessed through a comparison with various other lifetime distributions using negative log-likelihood, Akaike Information Criterion, Corrected Akaike Information Criterion, Bayesian Information Criterion, and Hannan–Quinn information criterion on an actual dataset. The results reveal that, when modeling real lifetime data, the suggested model exhibits greater flexibility than the competing models examined in this study.

Keywords: Binomial removal, efficiency, Rama distribution, XRama distribution.

1. INTRODUCTION

The role and significance of probability models in describing data sets from lifetime events cannot be ignored. Probability distribution plays a pivotal role in modeling lifetime data sets, offering a fundamental framework to understand and analyze the uncertainty associated with events or phenomena that occur over time. Whether in fields like reliability engineering, finance, epidemiology, or countless others, the choice of an appropriate probability distribution is essential for making informed predictions, optimizing resource allocation, and assessing risks. Statisticians engage in the critical task of making informed decisions regarding a population based on a sample taken from it. Evidently, many decision-relevant phenomena are subject to change, and the most effective approach to account for uncertainties surrounding them is through the adoption of probabilistic models. These probability models function as mathematical frameworks for describing physical phenomena. An essential prerequisite for applying probabilistic models to real-world issues is ensuring that the observed sample data conforms to specific probability distributions. Traditional probability distributions, for instance lognormal, exponential, normal, Weibull, gamma, and distributions proposed by Rama Shanker *et al.* (2019), Tesfalem and Shanker (2019), and Alhyasat *et al.* (2021) and many others, are commonly employed for modeling various real-life problems. Regrettably, numerous data sets do not align with existing probability distributions, prompting a demand for alternative distributions, particularly for extending the current ones to better suit the fitting of real-world data. Shanker (2017) suggested a lifetime distribution for modeling data sets that are more flexible in reliability and in terms of hazard rate shapes than the lifetime distributions mentioned in his article. Rama distribution is a one-parameter distribution formed by mixing exponential (θ) and gamma (4, θ), with mixing proportion $p = \frac{\theta^3}{\theta^3+6}$. To provide a more robust and flexible version of Rama distribution, many extensions such as Chrisogonus Kelechi Onyekwere *et al.* (2021) and Enogwe *et al.* (2021), and others, have been created, each with a specific application area. Hence, the chief aim of this article is to suggest a more flexible model that can be used to model data sets in reliability and in terms of hazard rate. We shall equally provide Parameter estimation for the suggested model under a complete sample and under the Progressive Type-II Censoring Scheme (PT2CS), then compare with some existing models by Shanker and Shukla (2017), KK (2018), Chrisogonus K Onyekwere and O. J. Obulezi (2022), and Rama Shanker (2017). Some of these distributions have been extended as well as modified, see O. J. Obulezi *et al.* (2023), Etaga, Celestine, *et al.* (2023), Innocent *et al.* (2023), Oramulu, Igbokwe, *et al.* (2023), O. Obulezi, Igbokwe, and Anabike (2023), Onuoha *et al.* (2023), Musa, Onyeagu, and O. J. Obulezi (2023a), Musa, Onyeagu, and O. J. Obulezi (2023b), Omoruyi *et al.* (2023), B. C. Nwankwo *et al.* (2024), Oramulu, Etaga, *et al.* (2023), and Etaga, M. P. Nwankwo, *et al.* (2023).

The remainder of the paper is organized in the following sequence; section 2 is on the derivation of the proposed distribution. Section 3, dwells on the statistical properties of the Xrama distribution. The parameter of the proposed distribution is estimated under the complete sample in section 4. Section 5, centers on the parameter estimation under a progressive type-2 censoring scheme. We demonstrate the usefulness of the proposed model using simulation and real-life data. The work is concluded in section 6.

2. Xrama Distribution

Let $X \sim$ Xrama (θ), then the pdf and cdf are respectively

$$f(x) = \frac{\theta^4}{(\theta^3+6)^2}\left[\theta^3 + 6x^3 + 12\right] e^{-\theta x}; \quad x>0, \quad \theta>0 \tag{1}$$

and

$$F(x) = 1 - \left\{1 + \frac{1}{(\theta^3+6)^2}\left[6\theta^3x^3 + 18\theta^2x^2 + 36\theta x\right]\right\} e^{-\theta x} \tag{2}$$

XRama distribution is a mixture of exponential (θ) and Shanker(θ) with mixing proportion $p = \frac{\theta^3}{\theta^3+6}$.

The survival and hazard rate functions are respectively

$$S(x) = \left\{1 + \frac{1}{(\theta^3+6)^2}\left[6\theta^3x^3 + 18\theta^2x^2 + 36\theta x\right]\right\} e^{-\theta x} \tag{3}$$

and

$$hrf(x) = \frac{\theta^7 + 6\theta^5x^3 + 12\theta^4}{\theta^6 + 12\theta^3 + 36 + 6\theta^3x^3 + 18\theta^2x^2 + 36\theta x} \tag{4}$$

The limiting values of the XRama hazard function are

$$\lim_{x\to 0} hrf(x) = \frac{\theta^7 + 12\theta^4}{\theta^6 + 12\theta^3 + 36} \quad \text{and} \quad \lim_{x\to\infty} hrf(x) = \theta^2$$

The plots are displayed in the figures 1, 2, 3, and 4 below

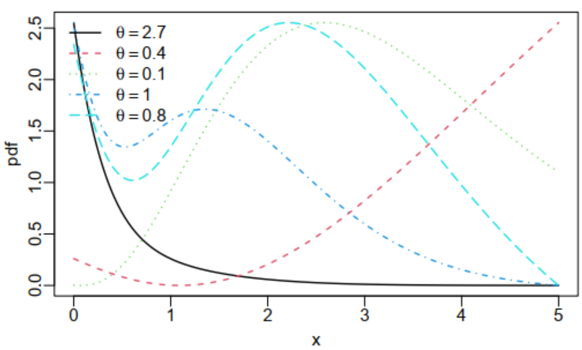

Figure 1: Pdf of XRama distribution

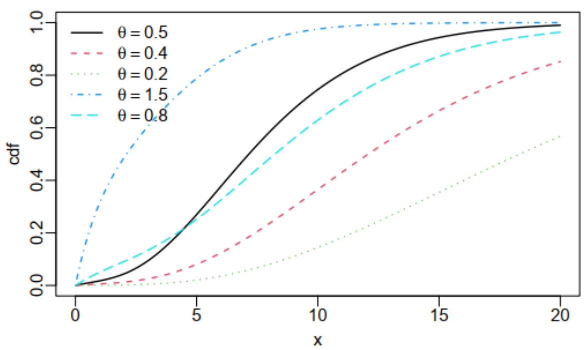

Figure 2: Cdf of Xrama distribution

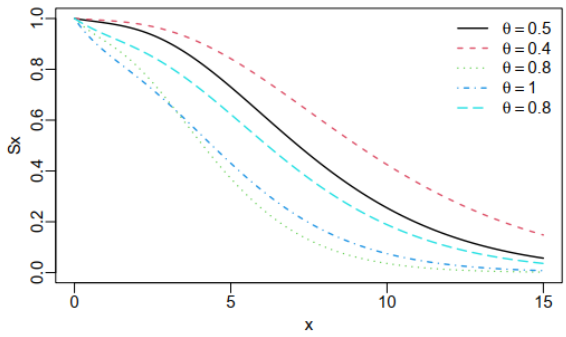

Figure 3: Survival function of Xrama distribution

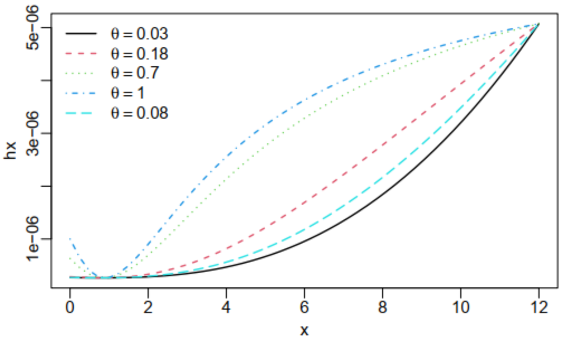

Figure 4: Hazard function of Xrama distribution

3. Statistical Properties of the Xrama Distribution

In this section, some properties of the proposed distribution are derived.

3.1 Quantile function

Let $X \sim$ Xrama (θ) with cdf defined is eq 2. We define the quantile xq for $u \sim U(0,1)$ where $u = F(x)$ hence xq is the inverse of the cdf obtained by transformation

$$\ln\left\{\frac{1-u}{(\theta^3+6)^2}\right\} - \theta x_q = \ln\left(6\theta^3 x_q^3 + 18\theta^2 x_q^2 + 36\theta x_q + 12\theta^3 + \theta^6 + 36\right) \tag{5}$$

Which has no closed-form solution.

3.2 Moment of Xrama Distribution

Let $X \sim$ Xrama (θ) with pdf in eq 1, then the r^{th} non-central moment is obtained as follows;

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \left[\frac{\theta^4}{(\theta^3+6)^2}\left[\theta^3 + 6\theta x^3 + 12\right] e^{-\theta x}\right] dx \\ &= \frac{\theta^4}{(\theta^3+6)^2}\left\{(\theta^3+12)\int_0^\infty x^r e^{-\theta x} dx + 6\int_0^\infty x^{(r+3)} e^{-\theta x} dx\right\} \end{aligned} \tag{6}$$

Recall the Gamma function from basic statistical knowledge

$$\int_0^\infty x^{(r-1)} e^{-\lambda x} dx = \left(\frac{1}{\lambda}\right)^\alpha \Gamma_\alpha$$

Which yields

$$\mu_r' = \frac{\theta^4}{(\theta^3+6)^2}\left\{(\theta^3+12)\int_0^\infty x^{(r+1)-1} e^{-\theta x} dx + 6\int_0^\infty x^{(r+4)-1} e^{-\theta x} dx\right\} = \frac{\theta^4}{(\theta^3+6)^2}\left\{\frac{(\theta^3+12)}{\theta^{(r+1)}}\Gamma_{(r+1)} + \frac{6}{\theta^{(r+4)}}\Gamma_{(r+4)}\right\} \tag{7}$$

The Mean μ of the distribution is obtained from the r^{th} non-central moment by making the substitution $r = 1$

$$\mu = \frac{\theta^6 + 12\theta^3 + 144}{\theta(\theta^3+6)^2} \tag{8}$$

The second, third, and fourth non-central moments are obtained by making the substitutions $r = 2,3,$ and 4 in eq 7 respectively

$$\mu_2' = \frac{2\left(\theta^6+12\theta^3+360\right)}{\theta^2\left(\theta^3+6\right)^2};\quad \mu_3' = \frac{6\left(\theta^6+12\theta^3+720\right)}{\theta^3\left(\theta^3+6\right)^2};\quad \mu_4' = \frac{24\left(\theta^6+12\theta^3+1255\right)}{\theta^4\left(\theta^3+6\right)^2} \tag{9}$$

The Variance (σ^2), Skewness (β), Kurtosis (ζ) and Coefficient of Variation (η) are respectively;

$$\sigma^2 = \frac{\left(\theta^{12}+240\theta^9+648\theta^6+6048\theta^3+5184\right)}{\theta^2\left(\theta^3+6\right)^4};\quad \beta = \frac{6\left(\theta^6+12\theta^3+720\right)\left(\theta^3+6\right)^4}{\left(5184+\theta^3\left(12+\theta^3\right)\left(504+12\theta^3+6\theta^6\right)\right)^{\frac{3}{2}}};$$
$$\zeta = \frac{24\left(6+\theta^3\right)^6\left(1260+12\theta^3+\theta^6\right)}{\left(5184+\theta^3\left(12+\theta^3\right)\left(504+12\theta^3+\theta^6\right)\right)^2} \quad \text{and} \quad \eta = \frac{\sqrt{\theta^{12}+240\theta^9+648\theta^6+6048\theta^3+5184}}{144+12\theta^3+\theta^6}\times\frac{100}{1} \tag{10}$$

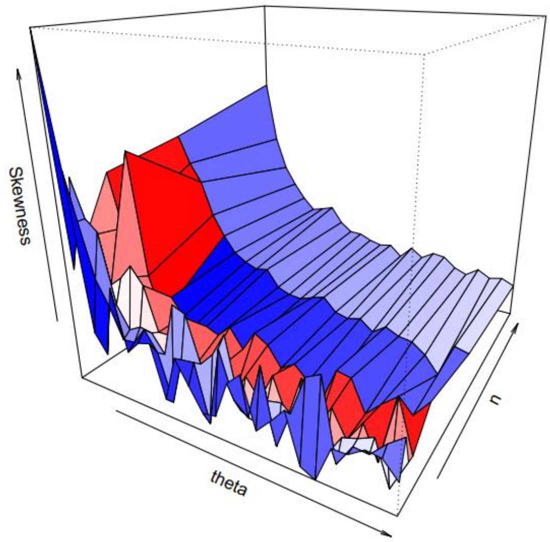

Figure 5: Surface plot of the skewness of Xrama distribution

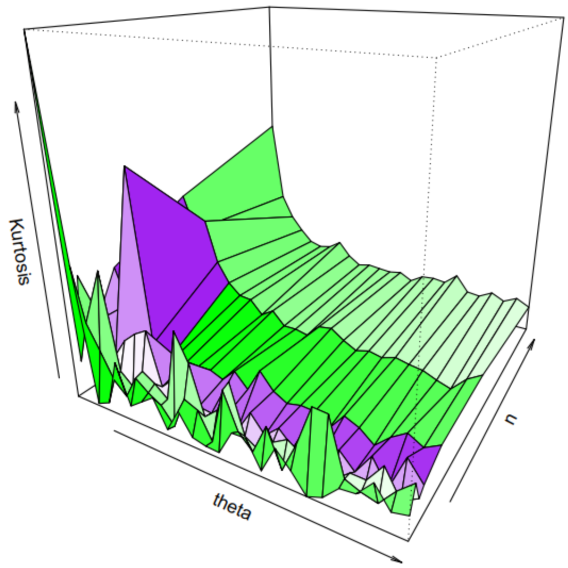

Figure 6: Surface plot of the Kurtosis of Xrama distribution

3.3 Information Measure

A popular measure of information in the statistical literature is the Reny entropy. For a non-negative number $\omega \neq 1$, the Reny entropy for Xrama distribution can be defined as

$$R_E = \frac{1}{1-\omega}\log\int_0^\infty f^\omega(x)dx = \frac{1}{1-\omega}\log\frac{\theta^{4\omega}}{(\theta^3+6)^{2\omega}}\int_0^\infty\left(\theta^3+6x^3+12\right)^\omega e^{-\theta\omega x}dx$$
$$= \frac{1}{1-\omega}\log\frac{\theta^{4\omega}}{(\theta^3+6)^\omega}\sum_{j=0}^{\omega}\binom{\omega}{j}\left(\theta^3+12\right)^j 6^{\omega-j}\int_0^\infty x^{\omega-j}e^{-\theta\omega x}dx \quad (11)$$
$$= \frac{1}{1-\omega}\log\left\{\frac{1}{(\theta^3+6)^\omega}\sum_{j=0}^{\omega}\binom{\omega}{j}\left(\theta^3+12\right)^j 6^{\omega-j}\theta^{\omega+3j-1}\omega^{3j-3\omega-1}\Gamma_{(3\omega-3j+1)}\right\}$$

3.4 Bonferroni and Lorenz curve

$$B(p) = \frac{1}{p\mu_1\prime}\int_0^q xf(x)dx = \frac{\theta(\theta^3+6)^2}{p(\theta^6+12\theta^3+144)}\int_0^q x\left(\frac{\theta^4}{(\theta^3+6)^2}\left(\theta^3+6\theta x^3+12\right)e^{-\theta x}\right)dx$$
$$= \frac{\theta^5}{p(\theta^6+12\theta^3+144)}\left(\theta^3+12\left(\frac{\gamma(2,q)}{\theta^2}\right)+6\left(\frac{\gamma(4,q)}{\theta^3}\right)\right) \quad (12)$$

3.5 Mean Residual Life Function

$$M(x) = E(X-x|X>x) = \frac{1}{1-F(x)}\int_x^\infty(1-F(t))dt$$
$$= \frac{(\theta^3+6)^2}{(\theta^3+6)^2+(6\theta^3x^3+18\theta^2x^2+36\theta x)e^{-\theta x}}\int_x^\infty\frac{\left((\theta^3+6)^2+6\theta^3t^3+18\theta^2t^2+36\theta t\right)e^{-\theta t}}{(\theta^3+6)^2}dt \quad (13)$$

Solving and simplifying the equation will yield:

$$M(x) = \frac{\left((\theta^3+6)^2e^{-\theta x}+6\theta^3x^2e^{-\theta x}+18\theta^2x^2e^{-\theta x}+36\theta xe^{-\theta x}+18x^2e^{-\theta x}+72xe^{-\theta x}+108e^{-\theta x}\right)}{\theta(\theta^3+16)^2+(6\theta^3x^3+18\theta^2x^2+36\theta x)e^{-\theta x}} \quad (14)$$

The mean residual life function when evaluated at $x=0$ gives the mean of the distribution.

Table 1: Mean (μ), Variance (σ^2), Skewness (β), Kurtosis (ζ) and Coefficient of Variation (η)

θ	n	μ	σ^2	β	ζ	η
	20	0.000034	0.000000	2.908842	12.247931	0.096067
	50	0.000035	0.000000	2.792892	10.399366	0.212711
0.1	100	0.000033	0.000000	-4.025562	22.727475	0.138165
	200	0.000033	0.000000	-3.173774	24.233221	0.093069
	500	0.000033	0.000000	-3.597249	41.946953	0.160814
	20	0.001208	0.000000	-1.754938	5.548409	0.219976
	50	0.001304	0.000000	0.035623	1.532055	0.094959
0.25	100	0.001181	0.000001	-8.563875	81.892819	0.877509
	200	0.001098	0.000004	-12.903956	176.598500	1.851769
	500	0.001191	0.000001	-10.046018	125.377037	0.707186
	20	-0.077939	0.561932	-4.114158	17.972811	-9.618001
	50	-0.041747	0.214164	-3.712640	16.454528	-11.085433
0.75	100	-0.000334	0.352227	-8.536583	78.845506	-1775.502676
	200	0.035790	0.070516	-5.376874	34.096663	7.419593
	500	-0.046507	0.908323	-11.795030	163.710781	-20.492777
	20	0.180051	12.848621	-3.863139	16.703073	19.908272
	50	-1.579857	86.316015	-4.645073	25.555355	-5.880683
1.5	100	-1.421620	115.908573	-6.015761	40.712219	-7.573109
	200	-2.905337	554.686134	-7.998299	70.586398	-8.106383
	500	-5.151618	3315.070302	-17.937670	359.560411	-11.176421
	20	-4.441863	285.244763	-2.881232	10.699035	-3.802276
	50	-181.620235	673350.913594	-4.567238	22.177749	-4.518107
2	100	-213.700156	1249013.753344	-6.047887	39.555064	-5.229724
	200	-33.815365	38805.032229	-6.985226	55.726996	-5.825456
	500	-152.601375	8674209.710800	-22.206390	495.374507	-19.299965

3.6 The Shape of Xrama Distribution

Let $X \sim$ Xrama (θ) with pdf in eq 1, the mode is obtained as follows Differentiate eq 1 with respect to x, which is

$$\frac{df(x)}{dx} = \frac{\theta^4}{(\theta^3+6)^2}\left\{-\theta^4 - 12\theta + 18x^2 - 6\theta x^3\right\}e^{-\theta x} \tag{15}$$

set $\frac{df(x)}{dx} = 0$ and solve for x to obtain

$$x = \pm\sqrt{\theta^4 + 12\theta} \quad \text{or} \quad \frac{18 - 12\theta - \theta^4}{6\theta}$$

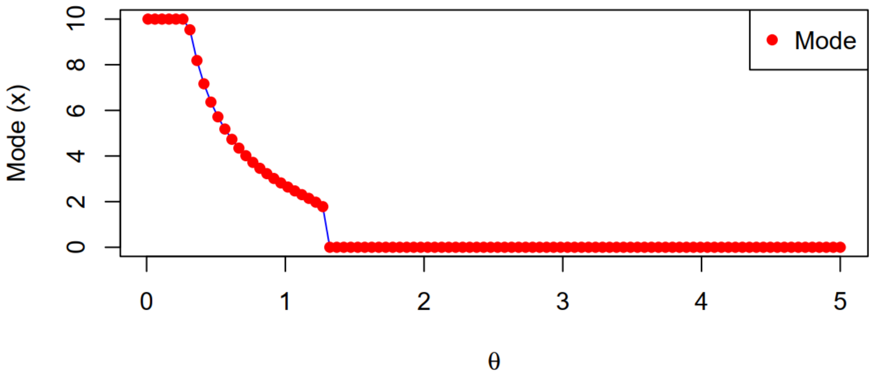

Figure 7: Mode of Xrama distribution for different values of θ

From fig 7, it is easy to see that for $0 \leq \theta < 0.5$, the proposed distribution attains its maximum with Mode $(X) = 10$.

3.7 Moment generating function

Let $X \sim$ Xrama (θ) with pdf in eq 1, the moment generating function is

$$M_X(t) = E(e^{xt}) = \int_0^\infty e^{xt} f(x)dx = \frac{\theta^4}{(\theta^3+6)^2}\left[\frac{\theta^3}{\theta - t} + \frac{12}{\theta - t} + \frac{36}{(\theta - t)^4}\right] \tag{16}$$

3.8 Characteristic function

Let $X \sim$ Xrama (θ) with pdf in eq 1, the characteristic function is

$$\Phi_X(it) = E(e^{itx}) = \int_0^\infty e^{itx} f(x)dx = \frac{\theta^4}{(\theta^3+6)^2}\left[\frac{\theta^3}{\theta - it} + \frac{12}{\theta - it} + \frac{36}{(\theta - it)^4}\right] \tag{17}$$

3.9 Distribution of the Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of $X(r)$; $(r = 1,2,..n)$ and the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude from the Xrama distribution is

$$f_{r:n}(x;\theta) = \binom{n}{r} f(x)\{F(x)\}^{r-1}\{1 - F(x)\}^{n-r} = \binom{n}{r}\frac{\theta^4}{(\theta^3+6)^2}(\theta^3 + 6x^3 + 12)e^{-\theta x}\left\{1 - \left\{1 + \frac{6\theta x}{(\theta^3+6)^2}\left[\theta^2 x^2 + 3\theta x + 6\right]\right\}e^{-\theta x}\right\}^{r-1}$$
$$\times \left\{\left\{1 + \frac{6\theta x}{(\theta^3+6)^2}\left[\theta^2 x^2 + 3\theta x + 6\right]\right\}e^{-\theta x}\right\}^{n-r} \tag{18}$$

The pdf of the largest order statistics is obtained when $r = n$ and that is;

$$f_{n:n}(x;\theta) = \frac{n\theta^4}{(\theta^3+6)^2}(\theta^3 + 6x^3 + 12)e^{-\theta x}\left\{1 - \left\{1 + \frac{6\theta x}{(\theta^3+6)^2}\left[\theta^2 x^2 + 3\theta x + 6\right]\right\}e^{-\theta x}\right\}^{n-1} \tag{19}$$

The pdf of the smallest order statistics is obtained when $r = 1$ and it is

$$f_{1:n}(x;\theta) = \frac{n\theta^4}{(\theta^3+6)^2}(\theta^3+6x^3+12)\left\{1+\frac{6\theta x}{(\theta^3+6)^2}\left[\theta^2x^2+3\theta x+6\right]\right\}^{n-1} e^{-\theta n x} \quad (20)$$

4. Estimation of Xrama Distribution Parameter Under Complete Sample

In this section, the maximum likelihood estimation under a complete data scheme will be discussed.

Definition 4.1 (Maximum Likelihood Estimation). Suppose n independent random samples $(x_1, x_2, \ldots, x_n)$ are drawn from Xrama distribution, the likelihood function is

$$\begin{aligned}\ell(f(x,\theta) &= \prod_{i=1}^{n} \frac{\theta^4}{\left(\theta^3+6\right)^2}\left[\theta^3+6x^3+12\right]e^{-\theta x} \\ &= \frac{\theta^{4n}}{(\theta^3+6)^{2n}} e^{-\theta\sum x}\prod_{i=1}^{n}\{\theta^3+6x^3+12\}\end{aligned} \quad (21)$$

Taking the log of ℓ and differentiating with respect to θ yields the following results

$$\begin{aligned}\psi &= 4n\ln\theta - 2n\ln(\theta^3+6) - \theta\sum x + \sum_{i=1}^{n}\ln(\theta^3+6x^3+12) \\ \frac{d\psi}{d\theta} &= \frac{4n}{\theta} - \frac{6n\theta^2}{\theta^3+6} - \sum_{i=1}^{n} x + \sum_{i=1}^{n}\frac{3\theta^2}{\theta^3+6x^3+12}\end{aligned} \quad (22)$$

set $\frac{d\psi}{d\theta} = 0$, one obtains

$$\frac{4n}{\theta} - \frac{6n\theta^2}{\theta^3+6} - \sum_{i=1}^{n} x + \sum_{i=1}^{n}\frac{3\theta^2}{\theta^3+6x^3+12} = 0 \quad (23)$$

which has no closed-form solution, hence will be implemented using **optim()** function in R.

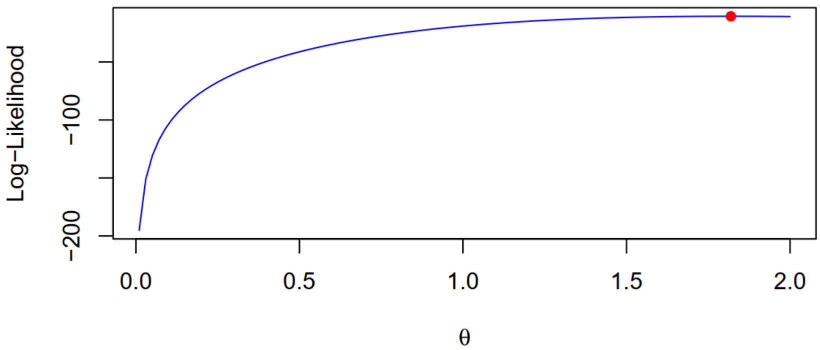

Figure 8: Profile of the Log-likelihood of Xrama distribution under complete sample

5. Estimation of Xrama Distribution Parameter Under Progressive Type-II Censoring Scheme (PT2CS)

Cheng and Amin (1983) are the first to use the Maximum product spacing estimation method and its competitiveness with the Likelihood estimator was later studied by Ranneby (1984). Balakrishnan and Cramer (2014) showed that under a progressive type-II censoring scheme, the maximum product spacing estimation has a competitive advantage over the likelihood estimation. Further, Anatolyev and Kosenok (2005) also claimed that for skewed models or in small-sample cases for heavy-tailed models, maximum product spacing estimations are more efficient than maximum likelihood estimations. Almetwally *et al.*, (2023) therefore concluded that Maximum product spacing possesses most of the properties of the maximum likelihood and a product of spacing is bounded, hence, even when likelihood breaks down due to the unbounded likelihood function, the maximum product spacing method can provide asymptotically optimal estimates.

In Ng (2005) study, the most common censoring schemes are Type-I and Type-II censoring, but these do not have the flexibility of allowing the removal of units at points other than the terminal point of the experiment. For this reason, a more general censoring scheme called Progressive Type-II censoring is considered. With a progressive type-II censoring scheme obtaining good estimators with the lifetime experiment and keeping some experimental units from failure is achieved. A frequent definition of the Progressive Type-II Censoring Scheme in the literature is as follows; consider an experiment in which units are placed on a life-test. At the time $x_{(1)}$ of the first failure, R_1 units are randomly removed from the remaining $n-1$ surviving units. At the time $x_{(2)}$ of the second failure, R_2 units are randomly removed from the remaining $n-R_1-2$ surviving units. This experiment continues in this fashion and is terminated when m^{th} failure occurs at time x_m, and $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving units are removed from the test. We call $\underset{\sim}{R} = (R_1, R_2, ..., R_m)$, the progressive Type-II censoring scheme. R_i are fixed prior to the experiment. The progressive Type-II right censoring is a special case of this scheme where the scheme is $R = (0, 0, ..., n-m)$, see Albert (2009).

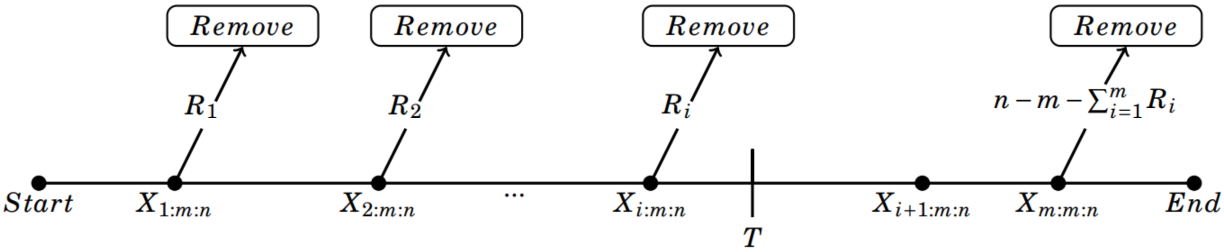

Figure 9: Progressive Type-II Censoring Scheme Illustration

Let $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$; $1 \leq m \leq n$ be a progressively type-II censored sample observed from a lifetime test involving n units and $R_1, R_2, ..., R_m$ being the censoring scheme. The joint PDF of a progressive type-II censored sample is given by

$$f(x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n}) = \Theta \prod_{i=1}^{m} f(x_{i:m:n})[1-F(x_{i:m:n})]^{R_i} \tag{24}$$

Where $\Theta = n\left(n-R_1-1\right)...\left(n-\sum_{i=1}^{m-1}\left(R_i+1\right)\right)$

Where $\Theta = n\left(n-R_1-1\right)...\left(n-\sum_{i=1}^{m-1}\left(R_i+1\right)\right)$

Definition 5.1 (Maximum likelihood Estimation under PT2CS). Let $X_{1:m:n}^{R_1,R_2,...,R_m}, X_{2:m:n}^{R_1,R_2,...,R_m}, ..., X_{m:m:n}^{R_1,R_2,...,R_m}$, $1 \leq m \leq n$ be a progressively type-II censored sample observed from a life test involving n units taken from a population with PDF $f(x)$ and CDF $F(x)$ given in eq 1 and 2, and the censoring scheme $\underset{\sim}{R} = (R_1, R_2, ...R_m)$.

Given that the unit of the item being removed from the test is independent of other units but has the same removal probability p, then the number of units removed at every failure time assumes the binomial distribution. That is $R_i \sim Binom(n-m-\sum_{i=1}^{m-1} R_i, p)$. The joint likelihood based on the PT2CS with binomial removal is

$$\ell(x_{i:m:n};\theta) = L(x_{i:m:n},\theta)P(R=r) \tag{25}$$

where $L(x_{i:m:n},\theta)$ is already provided in eq 27 while

$$P(R=r) = \frac{(n-m)!}{(n-m-\sum_{j=1}^{m-1} r_j)! \prod_{j=1}^{m-1} r_j} p^{\sum_{j=1}^{m-1} r_j} (1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \tag{26}$$

To maximize p, it follows easily from the maximizing $P(R=r)$ in eq 29 and because $L(x_{i:m:n},\theta)$ is independent of p, therefore the MLE of p is

$$\hat{p} = \frac{\sum_{j=1}^{m-1} r_j}{(m-1)(n-m) - \sum_{j=1}^{m-1}(m-j)r_j)! \prod_{j=1}^{m-1} r_j} \tag{27}$$

Also, $P(R=r)$ is independent of θ, hence, the likelihood function of the Xrama distribution parameter θ under PT2SC is

$$\begin{aligned} &\frac{1}{(\theta^3+6)}\sum_{i=1}^{m}\frac{R_i\left\{(\theta^3+6)(6\theta^5+36\theta^2+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}^2+36x_{i:m:n})\right\}}{\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}} \\ &-\frac{1}{(\theta^3+6)}\sum_{i=1}^{m}\frac{R_i\left\{6\theta^2(\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n})\right\}}{\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}} \\ &+\frac{4m}{\theta}-\frac{6m\theta^2}{\theta^3+6}+\sum_{i=1}^{m}\frac{3\theta^2}{\theta^3+12+6x_{i:m:n}^3}-\sum_{i=1}^{m}(1+R_i)x_{i:m:n}=0 \end{aligned} \tag{28}$$

Proof. Given the progressive type II censoring scheme sample $X_{1:m:n}^{R_1,R_2,\ldots,R_m}, X_{2:m:n}^{R_1,R_2,\ldots,R_m}, \ldots., X_{m:m:n}^{R_1,R_2,\ldots,R_m}$, $1 \le m \le n$ with joint pdf defined in equation 1, the likelihood function of the Xrama distribution parameter θ is

$$L(\theta|\underset{\sim}{x}) \propto \Theta\frac{\theta^{4m}}{(\theta^3+6)^{2m}}\prod_{i=1}^{m}(\theta^3+12+6x_{i:m:n}^3)\left[1+\frac{1}{(\theta^3+6)^2}\left(6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}\right)\right]^{R_i} e^{-\theta x_{i:m:n}(1+R_i)} \tag{29}$$

where $\underset{\sim}{x} = x_{1:m:n}, x_{2:m:n}, \ldots., x_{m:m:n}$.

The log-likelihood for the parameter θ is

$$\begin{aligned} \ell &= \log\Theta + 4m\log\theta - 2m\log(\theta^3+6) + \sum_{i=1}^{m}\log(1+\theta x_{i:m:n}^2) + \sum_{i=1}^{m}\log(\theta^3+12+6x_{i:m:n}^3) \\ &+ \sum_{i=1}^{m}R_i\log\left\{1+\frac{1}{(\theta^3+6)^2}(6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}\right\} - \theta\sum_{i=1}^{m}(1+R_i)x_{i:m:n} \end{aligned} \tag{30}$$

Differentiating ℓ with respect to θ and equating the result to zero yields the following

$$\begin{aligned} &\frac{1}{(\theta^3+6)}\sum_{i=1}^{m}\frac{R_i\left\{(\theta^3+6)(6\theta^5+36\theta^2+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}^2+36x_{i:m:n})\right\}}{\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}} \\ &-\frac{1}{(\theta^3+6)}\sum_{i=1}^{m}\frac{R_i\left\{6\theta^2(\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n})\right\}}{\theta^6+12\theta^3+36+6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}} \\ &+\frac{4m}{\theta}-\frac{6m\theta^2}{\theta^3+6}+\sum_{i=1}^{m}\frac{3\theta^2}{\theta^3+12+6x_{i:m:n}^3}-\sum_{i=1}^{m}(1+R_i)x_{i:m:n}=0 \end{aligned} \tag{31}$$

Equation 7 does not have a closed-form solution hence Newton-Raphson iterative method is implemented in R using the "maxLik" package to reach convergence for θ. Under asymptotic condition, a $100(1-\alpha)\%$ confidence interval for θ is easily obtained as

$$\hat{\theta}_i \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\theta}_i}, \quad i=1,2,\ldots,m \tag{32}$$

Definition 5.2 (Maximum Product Spacing Estimation under PT2CS). Let $X_{1:m:n}^{R_1,R_2,\ldots,R_m}$, $X_{2:m:n}^{R_1,R_2,\ldots,R_m}, \ldots., X_{m:m:n}^{R_1,R_2,\ldots,R_m}$, $1 \le m \le n$ be a progressively type-II censored sample observed from a life test involving n units taken from a population with PDF $f(x)$ and CDF $F(x)$, Ng, Chan, and N19100180993 Balakrishnan 2002 derived the Maximum Product Spacing function as follows

$$D_{i:m:n}(\theta) = \prod_{i=1}^{m+1}\left(F(x_{i:m:n},\theta) - F(x_{i-1:m:n},\theta)\right)\prod_{i=1}^{m}\left(1-F(x_{i:m:n},\theta)\right)^{R_i} \tag{33}$$

Substituting equation 3 into equation 10, we obtain the product spacing function of the Xrama parameter θ, which is

$$D_{i:m:n}(\theta)=\prod_{i=1}^{m+1}\left[\left(1+\frac{6\theta^3x_{i-1:m:n}^3+18\theta^2x_{i-1:m:n}^2+36\theta x_{i-1:m:n}}{(\theta^3+6)^2}\right)e^{-\theta x_{i-1:m:n}}-\left(1+\frac{6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}}{(\theta^3+6)^2}\right)e^{-\theta x_{i:m:n}}\right]\times\prod_{i=1}^{m}\left[\left(1+\frac{6\theta^3x_{i:m:n}^3+18\theta^2x_{i:m:n}^2+36\theta x_{i:m:n}}{(\theta^3+6)^2}\right)e^{-\theta x_{i:m:n}}\right]^{R_i} \quad (34)$$

The log-product spacing of θ is

$$\ell_D=\sum_{i=1}^{m+1}\log\left[\left(1+\frac{6\theta^3x_{i-1}^3+18\theta^2x_{i-1}^2+36\theta x_{i-1}}{(\theta^3+6)^2}\right)e^{-\theta x_{i-1}}-\left(1+\frac{6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i}{(\theta^3+6)^2}\right)e^{-\theta x_i}\right]+\sum_{i=1}^{m}R_i\log\left(1+\frac{6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i}{(\theta^3+6)^2}\right)-\theta\sum_{i=1}^{m}R_ix_i \quad (35)$$

For simplicity, we suppressed the following notations $_{1:m:n}\equiv_i$ and $_{i-1:m:n}\equiv_{i-1}$. Differentiating ℓ concerning θ and equating the result to zero yields the following maximum product spacing estimate without a closed form.

$$\sum_{i=1}^{m+1}\frac{1}{\Omega}\Big(-\theta e^{-\theta x_{i-1}}+\theta e^{-\theta x_i}+\left\{\frac{(\theta^3+6)\left[18\theta^2x_{i-1}^3-6\theta^4x_{i-1}^3+36\theta x_{i-1}^2-18\theta^3x_{i-1}^2+36x_{i-1}-36\theta^2x_{i-1}\right]-6\theta^2\left[6\theta^3x_{i-1}^3+18\theta^2x_{i-1}^2+36\theta x_{i-1}\right]}{(\theta^3+6)^3}\right\}e^{-\theta x_{i-1}}-\left\{\frac{(\theta^3+6)\left[18\theta^2x_i^3-6\theta^4x_i^3+36\theta x_i^2-18\theta^3x_i^2+36x_{i-1}-36\theta^2x_i\right]+6\theta^2\left[6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i\right]}{(\theta^3+6)^3}\right\}e^{-\theta x_i}+\sum_{i=1}^{m}\frac{R_i}{\Delta}-\theta e^{-\theta x_i}+\left\{\frac{(\theta^3+6)\left[18\theta^2x_i^3-6\theta^4x_i^3+36\theta x_i^2-18\theta^3x_i^2+36x_i-36\theta^2x_i\right]-6\theta^2\left[6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i\right]}{(\theta^3+6)^3}\right\}e^{-\theta x_i}=0 \quad (36)$$

where $\Omega=\left[\left(1+\frac{6\theta^3x_{i-1}^3+18\theta^2x_{i-1}^2+36\theta x_{i-1}}{(\theta^3+6)^2}\right)e^{-\theta x_{i-1}}-\left(1+\frac{6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i}{(\theta^3+6)^2}\right)e^{-\theta x_i}\right]$ and $\Delta=1+\frac{6\theta^3x_i^3+18\theta^2x_i^2+36\theta x_i}{(\theta^3+6)^2}$.

Again, the Newton-Raphon iterative method in R will be used to implement the convergence of the maximum product spacing estimator of the parameter θ since has no explicit form.

5.1 Bootstrap Confidence Intervals

When effective sample sizes required for asymptotic confidence interval construction are small, Asymptotic Confidence Intervals will perform poorly hence re-sampling technique namely the bootstrap method provides precise estimates Almetwally et al. (2023). We consider two popular bootstrap techniques in the literature proposed by Efron and Tibshirani (1994) namely *Boot- t* and *Boot- p*. Here are the steps;

- Given n, m and progressive censoring scheme $(R_1, R_2, \ldots, R_m)$, compute the maximum likelihood estimate of the unknown parameter θ based on the progressive type-II censoring scheme sample
- Using the maximum likelihood estimate obtained in step 5.1, generate bootstrap sample $(X_1^*, X_2^*, \ldots, X_m^*)$ from Xrama(θ)
- Using outputs of step 5.1, compute the bootstrap estimate of the θ say $\hat{\Phi}$ where $\hat{\Phi}_r^b=\hat{\theta}^b$ is the bootstrap estimate of the unknown parameter Φ.
- Repeat steps 5.1- 5.1 β times to get the bootstrap estimates $\hat{\Phi}_r^b, \quad r=1,2,\ldots,\beta$
- Compute the bootstrap confidence intervals of Φ_{as}

1. $Boot-p$ method
   (a) Arrange the bootstrap estimates in ascending order obtained $(\hat{\Phi}^b_{|1|}, \quad \hat{\Phi}^b_{|2|} \quad ... \quad \hat{\Phi}^b_{|\beta|})$
   (b) A $100(1-\alpha)\%$ $Boot-p$ confidence interval of Φ is given by

$$\left\{\hat{\Phi}^b_{|\beta(\frac{\alpha}{2})|}, \quad \hat{\Phi}^b_{|\beta(1-\frac{\alpha}{2})|}\right\}$$

2. $Boot-t$ method
   (a) Obtain T-statistic for the unknown parameter

$$T^{\Phi}_r = \frac{\hat{\Phi}^b_r - \hat{\Phi}}{\sqrt{V\hat{a}r(\hat{\Phi}^b)}} \qquad r = 1,2,...,\beta$$

   . where $\hat{\Phi}$ is the maximum likelihood estimate of Φ and $V\hat{a}r(\hat{\Phi}^b)$ is the bootstrap variance of $\hat{\Phi}^b$
   (b) Arrange the T-statistic to obtain

$$(T^{\Phi}_{|1|}, \quad T^{\Phi}_{|2|} \quad ... \quad T^{\Phi}_{|\beta|})$$

   (c) A $100(1-\alpha)\%$ $Boot-t$ confidence interval of Φ is given as

$$\left\{\hat{\Phi} - T^{\Phi}_{|\beta\frac{\alpha}{2}|}\sqrt{V\hat{a}r(\hat{\Phi}^b)}, \quad \hat{\Phi} + T^{\Phi}_{|\beta(1-\frac{\alpha}{2})|}\sqrt{V\hat{a}r(\hat{\Phi}^b)}\right\}$$

6. Applications

In this section, we illustrate the usefulness of the proposed distribution with a simulation study and real-life data applications.

6.1 Simulation Study for Complete Sample

We draw $N = 1000$ samples of sizes ($n = 25, 50, 75, 100, 200, 500$ and 1000) assuming the parameter values ($\theta = 0.1, 0.25, 1.5, 1.75$ and 2.0) from Xrama distribution. The following performance indices (Bias, Mean Square Error (MSE), Root Mean Square Error (RMSE), Coverage probability (CP), and Average length of the confidence intervals (AL)) are computed including the estimated mean of the parameter.

$$\begin{aligned}
Bias_\theta(n) &= \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta) \\
RMSE_\theta(n) &= \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta)^2} \\
MSE_\theta(n) &= \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta)^2 \\
CP_\theta(n) &= \sum_{i=1}^{1000} I\left(\hat{\theta}_i - 1.95996 S_{\hat{\theta}_i}, \hat{\theta}_i + 1.95996 S_{\hat{\theta}_i}\right) \\
AL_\theta(n) &= \frac{3.919928}{1000}\sum_{i=1}^{1000} S_{\hat{\theta}_i}
\end{aligned} \qquad (37)$$

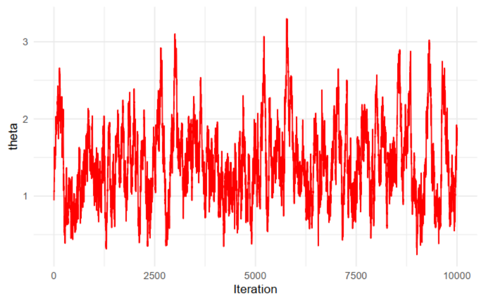

Figure 10: MCMC number of iterations for the estimation of θ

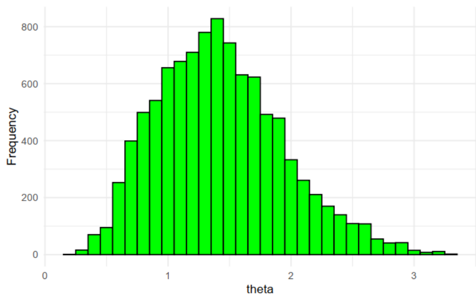

Figure 11: Histogram of θ values generated by MCMC algorithm

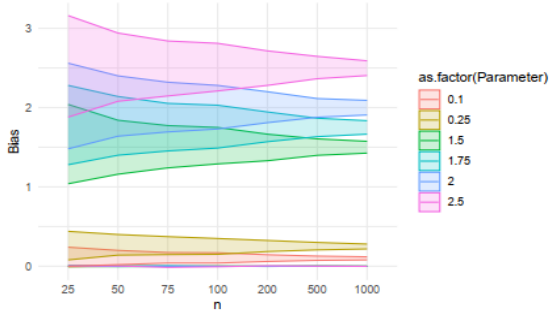

Figure 12: Bias of the estimated θ

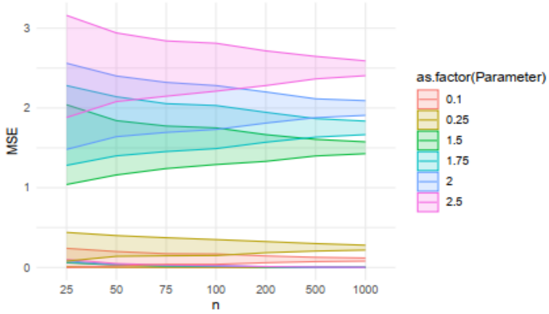

Figure 13: MSE of the estimated θ

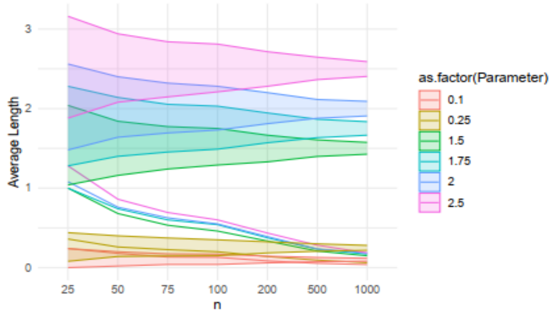

Figure 14: Average Length of the Confidence Intervals

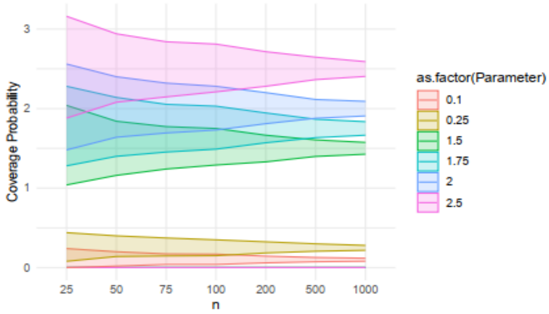

Figure 15: Coverage Probability

Table 2: Simulation study with the performance metrics

n	θ	μ	$Bias_\theta(n)$	$MSE_\theta(n)$	$RMSE_\theta(n)$	LowerCI	UpperCI	$AL_\theta(n)$	$CP_\theta(n)$
25	0.1	0.09500	-0.00500	0.00398	0.06312	0.00000	0.24000	0.24000	0.001
	0.25	0.24552	-0.00448	0.00988	0.09940	0.08000	0.44000	0.36000	0.001
	1.5	1.49696	-0.00304	0.05866	0.24220	1.04000	2.04000	1.00000	0.001
	1.75	1.76000	0.01000	0.06429	0.25355	1.28000	2.28000	1.00000	0.001
	2	2.00284	0.00284	0.08235	0.28696	1.48000	2.56000	1.08000	0.001
50	0.1	0.09878	-0.00122	0.00206	0.04537	0.02000	0.20000	0.18000	0.001
	0.25	0.25458	0.00458	0.00488	0.06987	0.14000	0.40000	0.26000	0.001
	1.5	1.49610	-0.00390	0.03065	0.17507	1.16000	1.84050	0.68050	0.001
	1.75	1.75182	0.00182	0.03592	0.18953	1.39950	2.14000	0.74050	0.001
	2	1.99788	-0.00212	0.03786	0.19459	1.63950	2.40000	0.76050	0.001
75	0.1	0.09965	-0.00035	0.00130	0.03601	0.04000	0.17333	0.13333	0.001
	0.25	0.24811	-0.00189	0.00350	0.05913	0.14667	0.37333	0.22667	0.001
	1.5	1.49787	-0.00213	0.01827	0.13515	1.24000	1.77333	0.53333	0.001
	1.75	1.75720	0.00720	0.02232	0.14941	1.45333	2.05333	0.60000	0.001
	2	1.99960	-0.00040	0.02598	0.16119	1.69333	2.32000	0.62667	0.001
100	0.1	0.09989	-0.00011	0.00100	0.03160	0.04000	0.17000	0.13000	0.001
	0.25	0.25028	0.00028	0.00277	0.05266	0.15000	0.35000	0.20000	0.001
	1.5	1.50143	0.00143	0.01427	0.11947	1.29000	1.75000	0.46000	0.001
	1.75	1.74850	-0.00150	0.01907	0.13808	1.48975	2.03025	0.54050	0.001
	2	2.00033	0.00033	0.02058	0.14345	1.73000	2.28000	0.55000	0.001
200	0.1	0.09995	-0.00005	0.00050	0.02234	0.06000	0.14513	0.08512	0.001
	0.25	0.24944	-0.00056	0.00130	0.03603	0.18500	0.32500	0.14000	0.001
	1.5	1.49818	-0.00182	0.00717	0.08468	1.33000	1.66500	0.33500	0.001
	1.75	1.74977	-0.00023	0.00871	0.09334	1.57000	1.94500	0.37500	0.001
	2	1.99836	-0.00164	0.01011	0.10054	1.81000	2.20013	0.39013	0.001
500	0.1	0.09932	-0.00068	0.00019	0.01388	0.07400	0.12800	0.05400	0.001
	0.25	0.25165	0.00165	0.00055	0.02351	0.20600	0.30000	0.09400	0.001
	1.5	1.50121	0.00121	0.00297	0.05446	1.39800	1.60605	0.20805	0.001
	1.75	1.74904	-0.00096	0.00342	0.05848	1.63400	1.86610	0.23210	0.001
	2	2.00076	0.00076	0.00374	0.06116	1.87800	2.11405	0.23605	0.001
1000	0.1	0.09973	-0.00028	0.00010	0.00989	0.08000	0.11903	0.03903	0.001
	0.25	0.24962	-0.00038	0.00025	0.01567	0.21800	0.28003	0.06203	0.001
	1.5	1.49964	-0.00036	0.00148	0.03853	1.42598	1.57500	0.14903	0.001
	1.75	1.74986	-0.00014	0.00183	0.04278	1.66498	1.83500	0.17003	0.001
	2	1.99912	-0.00088	0.00211	0.04593	1.90700	2.09103	0.18403	0.001

6.2 Simulation Study for PT2CS

We generate 10000 random samples for sizes $n = 50, 100, 200, 500, 1000$ with different number of failures m and removal probabilities $p = 0.20$ and $p = 0.80$ to obtain the number of units removed R_i at each failure time. Between MLE and MPSE, the method that gives the minimum value of the Biases, MSEs, and length of confidence interval (L.CI) of the estimate is considered the best. The table contains the Biases, L.CI, $Boot-p$, and $Boot-t$ for the MLEs and MPSEs.

- Case I: for $\theta = 0.5$
- Case II: for $\theta = 1.0$
- Case III: for $\theta = 2.5$

From tables 3 to 5, the following remarks can be deduced;

1. The $Boot-t$ estimates are high, indicating that the parameter estimates for both MLE and MPSE over all the sample sizes n are subject to significant variability. This is to say that the estimates may be less reliable due to sampling fluctuations or inherent variability in the simulated data.
2. The $Boot-p$ estimates are less than 0.05 suggesting that the estimates obtained from the resampling are significant.
3. The MSEs, Bias, and AL.CI decreases as the sample size increases for all parameter combinations.
4. The bias, MSEs, and AL.CI decreases as the number of removals (m) increases.
5. The MPSE estimates are more efficient than the maximum likelihood (ML) estimates for most studied cases of the XRama distribution under PT2CS with binomial removals.
6. The MSEs of MPSEs are smaller than the MSEs of MLEs in most stages hence the MPSE method outperforms the MLE method under a PT2CS.
7. The length of confidence intervals of the MPSEs are smaller than those of the MLE for most considered cases.

Table 3: MLEs and MPSEs of the Xrama parameter under the PT2CS with binomial removals for Case I.

Method	n	p	m	θ	Bias	MSE	AL.CI	$Boot-p$	$Boot-t$
MLE	50	0.2	30	0.57419	0.86221	0.74340	1.72441	0.00000	12.34339
MLE			45	0.57894	-0.00807	0.00007	0.01614	0.00000	12.32354
MPSE			30	0.50832	0.66848	0.44686	0.42182	0.00000	12.34339
MPSE			45	0.51325	-0.03767	0.00142	0.43262	0.00000	12.32354
MLE		0.8	30	0.12699	-0.40134	0.16107	0.80268	0.00000	12.32541
MLE			45	0.12690	-0.38021	0.14456	0.76043	0.00000	12.33069
MPSE			30	0.12383	-0.40313	0.16252	0.74603	0.00000	12.32541
MPSE			45	0.12374	-0.38329	0.14691	0.74620	0.00000	12.33069
MLE	100	0.2	60	0.53298	0.23187	0.05377	0.46375	0.00000	17.38579
MLE			85	0.53643	-0.02601	0.00068	0.05202	0.00000	17.36830
MPSE			60	0.50458	0.18752	0.03516	0.27021	0.00000	17.38579
MPSE			85	0.50778	-0.04545	0.00207	0.27414	0.00000	17.36830
MLE		0.8	60	0.12592	-0.36477	0.13306	0.72954	0.00000	17.37225
MLE			85	0.12595	-0.37177	0.13822	0.74355	0.00000	17.36746
MPSE			60	0.12435	-0.36659	0.13439	0.74816	0.00000	17.37225
MPSE			85	0.12438	-0.37354	0.13953	0.74811	0.00000	17.36746
MLE	200	0.2	130	0.51493	-0.12954	0.01678	0.25908	0.00000	24.52520
MLE			180	0.51507	-0.07160	0.00513	0.14320	0.00000	24.54529
MPSE			130	0.50171	-0.13527	0.01830	0.17947	0.00000	24.52520
MPSE			180	0.50185	-0.08138	0.00662	0.17947	0.00000	24.54529
MLE		0.8	130	0.12539	-0.38551	0.14862	0.77101	0.00000	24.53197
MLE			180	0.12541	-0.36110	0.13039	0.72220	0.00000	24.52291
MPSE			130	0.12461	-0.38612	0.14909	0.74921	0.00000	24.53197
MPSE			180	0.12463	-0.36226	0.13124	0.74918	0.00000	24.52291
MLE	500	0.2	210	0.50598	0.01163	0.00014	0.02325	0.00000	38.73522
MLE			300	0.50595	-0.07111	0.00506	0.14221	0.00000	38.73530
MPSE			210	0.50087	0.00583	0.00003	0.11241	0.00000	38.73522
MPSE			300	0.50084	-0.07411	0.00549	0.11368	0.00000	38.73530
MLE		0.8	210	0.12526	-0.37360	0.13958	0.74720	0.00000	38.74633
MLE			300	0.12519	-0.36881	0.13602	0.73763	0.00000	38.74729
MPSE			210	0.12495	-0.37393	0.13982	0.74948	0.00000	38.74633
MPSE			300	0.12487	-0.36913	0.13626	0.74963	0.00000	38.74729
MLE	1000	0.2	350	0.50338	0.06945	0.00482	0.13891	0.00000	54.77910
MLE			400	0.50238	-0.06409	0.00411	0.12817	0.00000	54.79663
MPSE			350	0.50085	0.06628	0.00439	0.07850	0.00000	54.77910
MPSE			400	0.49986	-0.06624	0.00439	0.07947	0.00000	54.79663
MLE		0.8	350	0.12507	-0.38062	0.14487	0.76125	0.00000	54.80475
MLE			400	0.12510	-0.38156	0.14559	0.76312	0.00000	54.78031
MPSE			350	0.12491	-0.38077	0.14498	0.74987	0.00000	54.80475
MPSE			400	0.12494	-0.38169	0.14569	0.74980	0.00000	54.78031

Table 4: MLEs and MPSEs of the Xrama parameter under the PT2CS with binomial removals for Case II.

Method	n	p	m	θ	Bias	MSE	AL.CI	$Boot-p$	$Boot-t$
MLE	50	0.2	30	0.57171	-0.22684	0.05146	0.45367	0.00000	12.33579
MLE			45	0.57827	-0.09647	0.00931	0.19294	0.00000	12.33020
MPSE			30	0.50567	-0.28276	0.07996	1.00649	0.00000	12.33579
MPSE			45	0.51083	-0.22611	0.05113	1.00218	0.00000	12.33020
MLE		0.8	30	0.12695	-0.87544	0.76640	1.75089	0.00000	12.31682
MLE			45	0.12672	-0.87507	0.76576	1.75015	0.00000	12.33355
MPSE			30	NA	-0.87833	0.77146	1.74610	0.00000	12.31682
MPSE			45	0.12357	-0.87812	0.77110	1.74656	0.00000	12.33355
MLE	100	0.2	60	0.53417	-0.21666	0.04694	0.43332	0.00000	17.37565
MLE			85	0.53027	-0.59324	0.35193	1.18648	0.00000	17.41083
MPSE			60	0.50586	-0.28110	0.07902	0.95657	0.00000	17.37565
MPSE			85	0.50201	-0.61883	0.38295	0.96390	0.00000	17.41083
MLE		0.8	60	0.12579	-0.87900	0.77264	1.75800	0.00000	17.38584
MLE			85	0.12578	-0.86694	0.75158	1.73388	0.00000	17.38696
MPSE			60	0.12422	-0.88051	0.77530	1.74842	0.00000	17.38584
MPSE			85	0.12421	-0.86896	0.75509	1.74844	0.00000	17.38696
MLE	200	0.2	130	0.51420	-0.42269	0.17867	0.84539	0.00000	24.53060
MLE			180	0.51538	-0.39289	0.15437	0.78579	0.00000	24.53445
MPSE			130	0.50106	-0.44183	0.19521	0.97283	0.00000	24.53060
MPSE			180	0.50215	-0.40797	0.16644	0.97038	0.00000	24.53445
MLE		0.8	130	0.12532	-0.87545	0.76641	1.75089	0.00000	24.54313
MLE			180	0.12532	-0.86811	0.75362	1.73622	0.00000	24.55043
MPSE			130	0.12454	-0.87624	0.76780	1.74935	0.00000	24.54313
MPSE			180	0.12454	-0.86891	0.75500	1.74935	0.00000	24.55043
MLE	500	0.2	210	0.50565	-0.31024	0.09625	0.62049	0.00000	38.75536
MLE			300	0.50497	-0.51946	0.26984	1.03892	0.00000	38.78083
MPSE			210	0.50055	-0.32018	0.10252	0.98870	0.00000	38.75536
MPSE			300	0.49987	-0.52342	0.27397	0.99006	0.00000	38.78083
MLE		0.8	210	0.12514	-0.87448	0.76472	1.74896	0.00000	38.75708
MLE			300	0.12515	-0.88165	0.77731	1.76331	0.00000	38.76178
MPSE			210	0.12483	-0.87478	0.76524	1.74972	0.00000	38.75708
MPSE			300	0.12483	-0.88191	0.77777	1.74971	0.00000	38.76178
MLE	1000	0.2	350	0.50239	-0.47621	0.22678	0.95242	0.00000	54.79451
MLE			400	0.50190	-0.48891	0.23903	0.97782	0.00000	54.79495
MPSE			350	0.49987	-0.47899	0.22943	0.99522	0.00000	54.79451
MPSE			400	0.49938	-0.49177	0.24184	0.99620	0.00000	54.79495
MLE		0.8	350	0.12506	-0.88278	0.77931	1.76557	0.00000	54.79812
MLE			400	0.12510	-0.87493	0.76550	1.74986	0.00000	54.79296
MPSE			350	0.12491	-0.88292	0.77954	1.74987	0.00000	54.79812
MPSE			400	0.12495	-0.87508	0.76577	1.74979	0.00000	54.79296

Table 5: MLEs and MPSEs of the Xrama parameter under the PT2CS with binomial removals for Case III.

Method	n	p	m	θ	Bias	MSE	AL.CI	$Boot-p$	$Boot-t$
MLE	50	0.2	30	0.58643	-1.97386	3.89614	3.94773	0.00000	12.33832
MLE			45	0.57536	-2.12757	4.52655	4.25514	0.00000	12.32643
MPSE			30	0.51416	-2.03541	4.14290	3.87781	0.00000	12.33832
MPSE			45	0.50799	-2.14890	4.61775	3.87534	0.00000	12.32643
MLE		0.8	30	0.12697	-2.38007	5.66472	4.76013	0.00000	12.31646
MLE			45	0.12698	-2.33861	5.46908	4.67721	0.00000	12.31948
MPSE			30	0.12381	-2.38298	5.67861	4.74606	0.00000	12.31646
MPSE			45	NA	-2.34448	5.49658	4.74604	0.00000	12.31948
MLE	100	0.2	60	0.53188	-1.44236	2.08041	2.88472	0.00000	17.39341
MLE			85	0.53000	-1.90622	3.63366	3.81243	0.00000	17.38818
MPSE			60	0.50366	-1.48864	2.21605	3.93649	0.00000	17.39341
MPSE			85	0.50207	-1.92751	3.71528	3.94000	0.00000	17.38818
MLE		0.8	60	0.12566	-2.37253	5.62888	4.74505	0.00000	17.38673
MLE			85	0.12581	-2.37303	5.63129	4.74607	0.00000	17.35697
MPSE			60	0.12410	-2.37417	5.63671	4.74867	0.00000	17.38673
MPSE			85	0.12424	-2.37473	5.63933	4.74839	0.00000	17.35697
MLE	200	0.2	130	0.51165	-2.06472	4.26307	4.12944	0.00000	24.55739
MLE			180	0.51579	-1.72880	2.98873	3.45759	0.00000	24.53156
MPSE			130	0.49865	-2.07218	4.29391	3.97669	0.00000	24.55739
MPSE			180	0.50252	-1.75351	3.07481	3.96842	0.00000	24.53156
MLE		0.8	130	0.12547	-2.37852	5.65736	4.75704	0.00000	24.51831
MLE			180	0.12534	-2.38124	5.67031	4.76248	0.00000	24.54720
MPSE			130	0.12468	-2.37925	5.66084	4.74906	0.00000	24.51831
MPSE			180	0.12455	-2.38192	5.67354	4.74933	0.00000	24.54720
MLE	500	0.2	210	0.50610	-2.09227	4.37759	4.18454	0.00000	38.75301
MLE			300	0.50670	-1.90679	3.63585	3.81358	0.00000	38.73544
MPSE			210	0.50099	-2.09598	4.39311	3.98780	0.00000	38.75301
MPSE			300	0.50159	-1.91401	3.66345	3.98659	0.00000	38.73544
MLE		0.8	210	0.12513	-2.37676	5.64898	4.75352	0.00000	38.76350
MLE			300	0.12525	-2.37915	5.66037	4.75831	0.00000	38.74441
MPSE			210	0.12482	-2.37706	5.65041	4.74974	0.00000	38.76350
MPSE			300	0.12493	-2.37944	5.66173	4.74951	0.00000	38.74441
MLE	1000	0.2	350	0.50243	-1.97214	3.88934	3.94428	0.00000	54.79035
MLE			400	0.50353	-1.94866	3.79729	3.89733	0.00000	54.77031
MPSE			350	0.49991	-1.97457	3.89891	3.99514	0.00000	54.79035
MPSE			400	0.50100	-1.95190	3.80992	3.99293	0.00000	54.77031
MLE		0.8	350	0.12507	-2.37236	5.62808	4.74471	0.00000	54.78511
MLE			400	0.12505	-2.37073	5.62037	4.74147	0.00000	54.78874
MPSE			350	0.12492	-2.37252	5.62883	4.74985	0.00000	54.78511
MPSE			400	0.12489	-2.37090	5.62118	4.74991	0.00000	54.78874

6.3 Real Data Applications

The first data is on Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L) studied by Tolba et al. 2023 in table 6.

Table 6: Vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L)

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2

The measures of model performance for the distributions are the negative Log-Likelihood (NLL), Akaike Information Criterion (AIC), Corrected AIC (CAIC), Bayesian Information Criterion (BIC), Hannan–Quinn information criterion (HQIC), Cramer von Mises (W^*), Anderson Darling (A^*), while the Kolmogorov-Smirnov (K-S) statistic and the p-value determine the fitness of the distribution to the data.

Table 7: MLE estimates, measures model performance and fitness for the fitted distributions

Dist	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value	θ
XRama	57.28	116.556	116.681	118.083	117.077	0.111	0.703	0.127	0.647	1.349
Ishita	57.3	116.506	116.731	118.132	117.126	0.095	0.604	0.140	0.514	1.157
Akash	57.57	117.149	117.274	118.676	117.670	0.099	0.630	0.157	0.376	1.166
Pranav	58.34	118.672	118.797	120.198	119.192	0.136	0.847	0.146	0.466	1.466
Chris-Jerry	57.93	117.854	117.979	119.380	118.374	0.103	0.655	0.178	0.230	1.165
Rama	59.34	120.683	120.808	122.210	121.204	0.154	0.952	0.177	0.238	1.531
Rani	59.88	121.752	121.877	123.278	122.272	0.194	1.165	0.152	0.415	1.784

From table 7, the proposed distribution has the largest p-value among others indicating the best fit for the data and the criteria for model performance in estimating the parameter also indicate that the proposed distribution competes favorably with the other distributions.

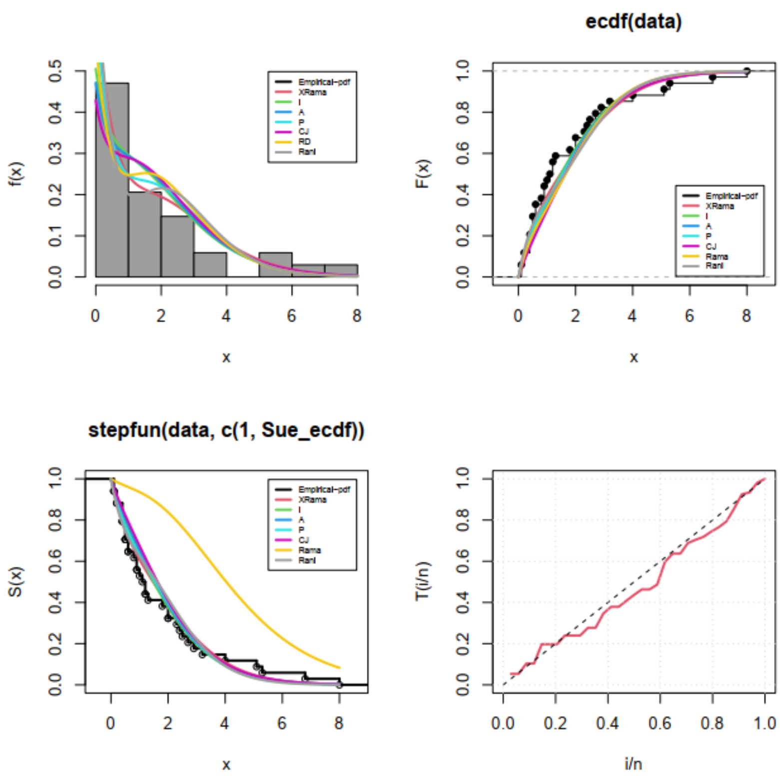

Figure 16: The density, CDF, empirical reliability, and TTT plots for the Vinyl Chloride data

Figure 16 reveals also that Xrama is best for modeling the Vinyl Chloride data among the competing distributions.

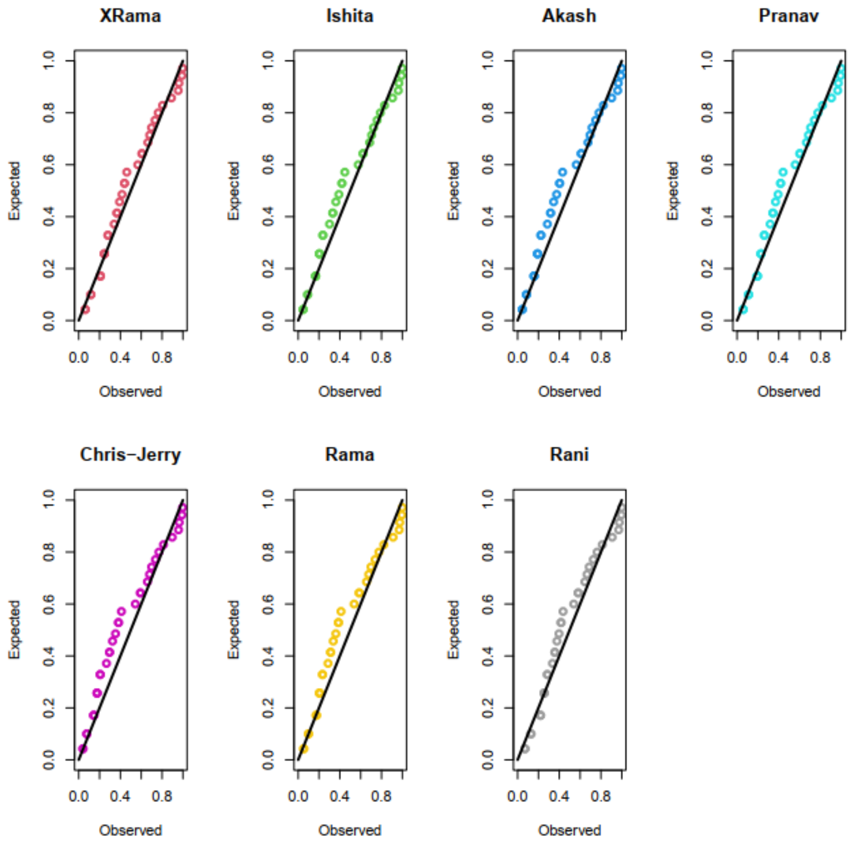

Figure 17: PP plots for the Vinyl Chloride data

7. CONCLUDING REMARKS

This paper introduces a two-component mixture of Exponential and Rama distribution named Xrama distribution. The mathematical and statistical properties for this distribution were derived. The reliability, hazard rate, reversed hazard rate, and odd functions are detailed. Parameter estimation was derived under a complete sample and under the Progressive Type-II Censoring Scheme (PT2CS) using the maximum likelihood estimation and maximum product spacing estimation. The study includes simulation analysis and a comparison of the model's performance in fitting real data against some commonly used models for describing lifetime data, evaluated through Bias, Mean Square Error (MSE), Root Mean Square Error (RMSE), etc. The results suggest that the Xrama presents a better alternative when modeling lifetime data.

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