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Theoretical Analysis of Fuzzy Network Graph via Connectivity Indices

Shu Gong[1*], Guang Hua[2], Wei Gao[3]

[1]Department of Computer Science, Guangdong University of Science and Technology, Dongguan 523083, China
[2]School of Information and Control Engineering, China University of Mining and Technology, Xuzhou 221116, China
[3]School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China

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***Corresponding author:** Shu Gong
Department of Computer Science, Guangdong University of Science and Technology, Dongguan 523083, China

Abstract **Original Research Article**

In recent years, the study on topological indices related to network security has attracted attention from mathematics and computer science fields. Although several gratifying theoretical results exist, definitions and conclusions under most mathematical frameworks need to be expanded. The focus topic of this paper is to use a graph model to represent the network structure, and the membership functions (MFs) of vertices and edges describe the uncertain nature of stations and channels. The connectivity index is used to describe the robustness and the stability of the fuzzy graph of the corresponding network. In this paper, the definition of the connectivity index and the derived concept are given in a bipolar intuitionistic fuzzy graph (BIFG) setting, and the characteristics of connectivity indices are given from the theoretical point of view. These results have potential applications in the field of computer security, and we analyzed topics for future research.
Keywords: Network, fuzzy graph, connectivity index, bipolar intuitionistic fuzzy graph.

1. INTRODUCTION

Graph models are often used in computer frameworks. For example, in a network, vertices are used to represent sites, and (directed or undirected) edges represent (directed or undirected) channels between vertices. Then the entire computer network can be shown by graph structures.

In recent years, graph network models and fuzzy graph (FG) network frameworks have raised ample attention of researchers [1], considered the relationship between toughness and fractional flow in network when certain number of vertices is removed [2]. Discussed the toughness condition for computer communication networks when channels are attacked, and its corresponding isolated toughness analysis was given by [3-5] focused on the parameter condition for data transform path in networks [6]. Raised new complex wave patterns to the electrical transmission line model arising in the network systems [7]. Studied the degree-based topological indices of some derived networks, and topological indices of certain OTIS Interconnection networks are determined in [8]. More related results can refer to [9, 10].

This article considers the following two backgrounds of the network:

- Stations and channels may have many uncertain attributes, and definite quantities cannot represent these uncertain characteristics. Therefore, tools of fuzzy mathematics are needed to express these uncertain quantities. Therefore, when we need to deal with the uncertainty in the network structure, we use fuzzy graphs instead of traditional graphs to represent the network. That is, each vertex and each edge in the network graph has a corresponding membership function. When there are multiple uncertain attributes that need to be represented, the multiple membership functions of vertices or edges can be defined.
- Network security has received extensive attention in recent years, the current popular research topics are network attacks, defense, privacy protection, data security, etc. Under the graph model, researchers have worked out how to characterize the vulnerability of the network and the security of data transmission between vertices from the perspective of modern graph theory. Some graph topology indicators are defined, such as toughness, isolated toughness, the connectivity.

In this paper, the above two issues are considered together, that is, to study the security-related indexes of network fuzzy graphs. The object of our investigation is the bipolar intuitionistic fuzzy graph (BIFG), and the corresponding index is the connectivity index of this kind of FG. The paper is roughly organized as follows: First, we give the existing concept of the BIFG; then, the new connectivity index on the BIFG is defined, and some theoretical features are obtained as well.

2. Preliminary Definitions

This section aim is to introduce the extant concepts on BIFG and connective indices on the intuitionistic fuzzy graph.

2.1. Bipolar Fuzzy Graph

Let V be a universal set. The set $A=\{(v,\mu_A^P(v),\mu_A^N(v)):v\in V\}$ is a bipolar fuzzy set (BFS) in V if two maps satisfy $\mu_A^P:V\to[0,1]$ and $\mu_A^N:V\to[-1,0]$. If $A=\{(v,\mu_A^P(v),\mu_A^N(v)):v\in V\}$ is a bipolar fuzzy set on an underlying set V and $B=(\mu_B^P,\mu_B^N)$ is a BFS in $\tilde{V}^2$ where $\mu_B^P(v,v')\le\min\{\mu_A^P(v),\mu_A^P(v')\}$, $\mu_B^N(v,v')\ge\max\{\mu_A^N(v),\mu_A^N(v')\}$ for any $(v,v')\in\tilde{V}^2$, and $\mu_B^P(v,v')=\mu_B^N(v,v')=0$ for any $(v,v')\in\tilde{V}^2-E$, then $G=(V,A,B)$ is a bipolar fuzzy graph (BFG) of the graph $G^*=(V,E)$. In the whole paper, $\wedge$ and $\vee$ mean minimum and maximum operations, respectively.

The order, neighborhood, neighborhood degree, irregular and subdigraph of bipolar fuzzy graphs were recently revised and defined by Poulik and Ghorai [11] which were firstly defined by Akram [12].

2.2. Bipolar Intuitionistic Fuzzy Graph

Shannon and Atanassov [13, 14] introduced the intuitionistic fuzzy graph (IFG), and Ezhilmaran and Sankar [15] introduced bipolar intuitionistic fuzzy set (BIFS) and BIFG. A BIFS on universal set V is denoted by;

$A=\{(v,\mu_A^P(v),\mu_A^N(v),\eta_A^P(v),\eta_A^N(v)):v\in V\}$,

Where $\mu_A^P:V\to[0,1]$, $\mu_A^N:V\to[-1,0]$, $\eta_A^P:V\to[0,1]$, $\eta_A^N:V\to[-1,0]$. Furthermore, we have $0\le\mu_A^P(v)+\eta_A^P(v)\le1$ and $-1\le\mu_A^N(v)+\eta_A^N(v)\le0$ for any $v\in V$. A mapping $B=(\mu_B^P,\mu_B^N,\eta_B^P,\eta_B^N)$: $V\times V\to([0,1]\times[-1,0]\times[0,1]\times[-1,0])$ a bipolar intuitionistic fuzzy relation (BIFR) such that $\mu_B^P(v,v')\in[0,1]$, $\mu_B^N(v,v')\in[-1,0]$, $\eta_B^P(v,v')\in[0,1]$, $\eta_B^N(v,v')\in[-1,0]$ with $0\le\mu_B^P(v,v')+\eta_B^P(v,v')\le1$ and $-1\le\mu_B^N(v,v')+\eta_B^N(v,v')\le0$ for any $v,v'\in V$.

A BIFG $G=(V,A,B)$ with $A=(\mu_A^P,\mu_A^N,\eta_A^P,\eta_A^N)$ and $B=(\mu_B^P,\mu_B^N,\eta_B^P,\eta_B^N)$ is a BIFR such that

$\mu_B^P(v,v')\le\mu_A^P(v)\wedge\mu_A^P(v')$,

$\mu_B^N(v,v')\ge\mu_A^N(v)\vee\mu_A^N(v')$,

$\eta_B^P(v,v')\ge\eta_A^P(v)\vee\eta_A^P(v')$,

$\eta_B^N(v,v')\le\eta_A^N(v)\wedge\eta_A^N(v')$,

for any $(v,v')\in V\times V$, and $\mu_B^P(v,v')=\mu_B^N(v,v')=\eta_B^P(v,v')=\eta_B^N(v,v')=0$

for any $(v,v')\in V\times V-E$.

3. Connectivity Indices of BIFGs

Now, we present our new concepts and theoretical results on connectivity indices of BIFGs which can be regarded as an extension of connectivity indices of intuitionistic fuzzy graph raised by Naeem *et al.,* [16]. We begin with the new definitions of BIFGs.

Definition 1: A BIFG is complete if

$\mu_B^P(v,v')=\mu_A^P(v)\wedge\mu_A^P(v')$,

$\mu_B^N(v,v')=\mu_A^N(v)\vee\mu_A^N(v')$,

$\eta_B^P(v,v')=\eta_A^P(v)\vee\eta_A^P(v')$,

$\eta_B^N(v,v')=\eta_A^N(v)\wedge\eta_A^N(v')$,
for any $(v,v')\in V\times V$.

Definition 2: A sequence of $v_1,v_2,\cdots,v_n$ distinct vertices is a path *P* in a bipolar intuitionistic fuzzy graph if for some *i* and *j*, one of the positive relation conditions (i)-(iii) holds and one of the negative relation conditions (iv)-(vi) holds:
(i) $\mu_B^P(v_i,v_j)>0$ and $\eta_B^P(v_i,v_j)=0$;
(ii) $\mu_B^P(v_i,v_j)=0$ and $\eta_B^P(v_i,v_j)>0$;
(iii) $\mu_B^P(v_i,v_j)>0$ and $\eta_B^P(v_i,v_j)>0$;
(iv) $\mu_B^N(v_i,v_j)<0$ and $\eta_B^N(v_i,v_j)=0$;
(v) $\mu_B^N(v_i,v_j)=0$ and $\eta_B^N(v_i,v_j)<0$;
(vi) $\mu_B^N(v_i,v_j)<0$ and $\eta_B^N(v_i,v_j)<0$.

Definition 3: For a path $P=v_1,v_2,\cdots,v_n$ and the μ–positive strength (μ–PS), η–positive strength (η–PS), μ–negative strength (μ–NS), η–negative strength (η–NS) are denoted by
$S_\mu^P=\min_{i,j}\{\mu_B^P(v_i,v_j)\}$,
$S_\eta^P=\max_{i,j}\{\eta_B^P(v_i,v_j)\}$,
$S_\mu^N=\max_{i,j}\{\mu_B^N(v_i,v_j)\}$,
$S_\eta^N=\min_{i,j}\{\eta_B^N(v_i,v_j)\}$, respectively.

Definition 4: If both S_μ^P and S_η^P exist for the same edge, then (S_μ^P,S_η^P) is called the positive strength (PS) of path *P*. If both S_μ^N and S_η^N exist for the same edge, then (S_μ^N,S_η^N) is called the negative strength (NS) of path *P*. If all S_μ^P, S_η^P, S_μ^N, and S_η^N exist for the same edge, then $(S_\mu^P,S_\eta^P,S_\mu^N,S_\eta^N)$ is called the strength of path *P*.

Definition 5: The μ–PS of connectedness between v_i and v_j, η–PS of connectedness between v_i and v_j, μ–NS of connectedness between v_i and v_j, η–negative strength of connectedness between v_i and v_j are denoted respectively by
$\mathrm{CONN}_{\mu(G)}^P(v_i,v_j)=\max\{S_\mu^P\}$,
$\mathrm{CONN}_{\eta(G)}^P(v_i,v_j)=\min\{S_\eta^P\}$,
$\mathrm{CONN}_{\mu(G)}^N(v_i,v_j)=\min\{S_\mu^N\}$,
$\mathrm{CONN}_{\eta(G)}^N(v_i,v_j)=\max\{S_\eta^N\}$,
Where the max and min operators traverse all possible paths between v_i and v_j.

Furthermore, we denote $\mathrm{CONN}_{\mu(G)-(v_i,v_j)}^P(v_i,v_j)$, $\mathrm{CONN}_{\eta(G)-(v_i,v_j)}^P(v_i,v_j)$, $\mathrm{CONN}_{\mu(G)-(v_i,v_j)}^N(v_i,v_j)$, and $\mathrm{CONN}_{\eta(G)-(v_i,v_j)}^N(v_i,v_j)$ by the μ–PS of connectedness, η–PS of connectedness, μ–NSh of connectedness, η–NS of connectedness between v_i and v_j attained by removing the edge (v_i,v_j) from *G*, respectively.

Definition 6: An edge (v_i,v_j) is a positive bridge (PB) in *G* if one of the positive conditions (i) and (ii) is hold:
(i) $\mathrm{CONN}_{\mu(G)-(v_i,v_j)}^P(v_i,v_j)<\mathrm{CONN}_{\mu(G)}^P(v_i,v_j)$, $\mathrm{CONN}_{\eta(G)-(v_i,v_j)}^P(v_i,v_j)\geq\mathrm{CONN}_{\eta(G)}^P(v_i,v_j)$;

(ii) $\mathrm{CONN}^P_{\mu(G)-(v_i,v_j)}(v_i,v_j) \le \mathrm{CONN}^P_{\mu(G)}(v_i,v_j)$, $\mathrm{CONN}^P_{\eta(G)-(v_i,v_j)}(v_i,v_j) > \mathrm{CONN}^P_{\eta(G)}(v_i,v_j)$.

That is to say, removing (v_i,v_j) reduces the PS of connectedness between any pair of vertices.

Correspondingly, an edge (v_i,v_j) is a negative bridge (NB) in *G* if one of the negative conditions (iii) and (iv) is hold:

(iii) $\mathrm{CONN}^N_{\mu(G)-(v_i,v_j)}(v_i,v_j) > \mathrm{CONN}^N_{\mu(G)}(v_i,v_j)$, $\mathrm{CONN}^N_{\eta(G)-(v_i,v_j)}(v_i,v_j) \le \mathrm{CONN}^N_{\eta(G)}(v_i,v_j)$;

(iv) $\mathrm{CONN}^N_{\mu(G)-(v_i,v_j)}(v_i,v_j) \ge \mathrm{CONN}^N_{\mu(G)}(v_i,v_j)$, $\mathrm{CONN}^N_{\eta(G)-(v_i,v_j)}(v_i,v_j) < \mathrm{CONN}^N_{\eta(G)}(v_i,v_j)$.

That is to say, removing (v_i,v_j) increases the NS of connectedness between any pair of vertices.

An edge (v_i,v_j) is a bridge in *G* if (v_i,v_j) is both a PG and a NG.

Definition 7: An edge (v_i,v_j) is strong if

$\mu^P_B(v_i,v_j) \ge \mathrm{CONN}^P_{\mu(G)}(v_i,v_j)$,

$\eta^P_B(v_i,v_j) \le \mathrm{CONN}^P_{\eta(G)}(v_i,v_j)$,

$\mu^N_B(v_i,v_j) \le \mathrm{CONN}^N_{\mu(G)}(v_i,v_j)$,

and

$\eta^N_B(v_i,v_j) \ge \mathrm{CONN}^N_{\eta(G)}(v_i,v_j)$.

An edge (v_i,v_j) is weakest if

$\mu^P_B(v_i,v_j) < \mathrm{CONN}^P_{\mu(G)}(v_i,v_j)$,

$\eta^P_B(v_i,v_j) > \mathrm{CONN}^P_{\eta(G)}(v_i,v_j)$,

$\mu^N_B(v_i,v_j) > \mathrm{CONN}^N_{\mu(G)}(v_i,v_j)$,

and

$\eta^N_B(v_i,v_j) < \mathrm{CONN}^N_{\eta(G)}(v_i,v_j)$.

Definition 8: The strongest path (SP) between two vertices in a BIFG is a path *P* having its PS and NS equal to $\mathrm{CONN}^P_{\mu(G)}(v_i,v_j)$, $\mathrm{CONN}^P_{\eta(G)}(v_i,v_j)$, $\mathrm{CONN}^N_{\mu(G)}(v_i,v_j)$ and $\mathrm{CONN}^N_{\eta(G)}(v_i,v_j)$ lying in the same edge.

Definition 9: A path $P: v_i - v_j$ in a BIFG *G* is called strong path if each edge in *P* is strong.

Definition 10: An edge (v_i,v_j) in a BIFG *G* is called α –strong if;

$\mu^P_B(v_i,v_j) > \mathrm{CONN}^P_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,

$\eta^P_B(v_i,v_j) < \mathrm{CONN}^P_{\eta(G)-(v_i,v_j)}(v_i,v_j)$,

$\mu^N_B(v_i,v_j) < \mathrm{CONN}^N_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,

and

$\eta^N_B(v_i,v_j) > \mathrm{CONN}^N_{\eta(G)-(v_i,v_j)}(v_i,v_j)$.

An edge (v_i,v_j) in a BIFG *G* is called β –strong if

$\mu^P_B(v_i,v_j) = \mathrm{CONN}^P_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,

$\eta^P_B(v_i,v_j) = \mathrm{CONN}^P_{\eta(G)-(v_i,v_j)}(v_i,v_j)$,

$\mu_B^N(v_i,v_j)=\mathrm{CONN}^N_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,
and
$\eta_B^N(v_i,v_j)=\mathrm{CONN}^N_{\eta(G)-(v_i,v_j)}(v_i,v_j)$.
An edge (v_i,v_j) in a BIFG G is called δ –weak if
$\mu_B^P(v_i,v_j)<\mathrm{CONN}^P_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,
$\eta_B^P(v_i,v_j)>\mathrm{CONN}^P_{\eta(G)-(v_i,v_j)}(v_i,v_j)$,
$\mu_B^N(v_i,v_j)>\mathrm{CONN}^N_{\mu(G)-(v_i,v_j)}(v_i,v_j)$,
and
$\eta_B^N(v_i,v_j)<\mathrm{CONN}^N_{\eta(G)-(v_i,v_j)}(v_i,v_j)$.

Moreover, a path in a BIFG containing only α –strong edges is called α –strong, and a path having only β –strong edges is called β –strong.

Definition 11: A BIFG *G* is called a cycle if its crisp graph is a cycle. A BIFG *G* is called a bipolar intuitionistic fuzzy cycle (BIFC) if its crisp graph is a cycle and there is no unique $(v,v')\in E^*$ satisfying
$\mu_B^P(v,v')=\min\{\mu_B^P(x,y):(x,y)\in E^*\}$,
$\mu_B^N(v,v')=\max\{\mu_B^N(x,y):(x,y)\in E^*\}$,
$\eta_B^P(v,v')=\max\{\eta_B^P(x,y):(x,y)\in E^*\}$,
and
$\eta_B^N(v,v')=\min\{\eta_B^N(x,y):(x,y)\in E^*\}$,
Where E^* is denoted by the edge set of crisp graph of *G*.

Below we formally introduce the concept of connectivity index (CI) for the BIFG.

Definition 12: The CI of BIFG $G=(V,A,B)$ with $A=(\mu_A^P,\mu_A^N,\eta_A^P,\eta_A^N)$ and $B=(\mu_B^P,\mu_B^N,\eta_B^P,\eta_B^N)$ is defined by

$$CI(G)=\sum_{u,v\in V(G)}\{(\mu_A^P(u),\mu_A^N(u),\eta_A^P(u),\eta_A^N(u))(\mu_A^P(v),\mu_A^N(v),\eta_A^P(v),\eta_A^N(v))\,\mathrm{CONN}_G(u,v)\}$$
$$=\sum_{u,v\in V(G)}\{(\mu_A^P(u),\mu_A^N(u),\eta_A^P(u),\eta_A^N(u))(\mu_A^P(v),\mu_A^N(v),\eta_A^P(v),\eta_A^N(v))(\mathrm{CONN}_G^P(u,v),\mathrm{CONN}_G^N(u,v))\}$$
$$=\sum_{u,v\in V(G)}\{(\mu_A^P(u),\mu_A^N(u),\eta_A^P(u),\eta_A^N(u))(\mu_A^P(v),\mu_A^N(v),\eta_A^P(v),\eta_A^N(v))(\mathrm{CONN}^P_{\mu(G)}(u,v)+$$
$$\mathrm{CONN}^P_{\eta(G)}(u,v),\mathrm{CONN}^N_{\mu(G)}(u,v)+\mathrm{CONN}^N_{\eta(G)}(u,v))\}$$
$$=\sum_{u,v\in V(G)}(\mu_A^P(u)\mu_A^P(v)\,\mathrm{CONN}^P_{\mu(G)}(u,v)+\eta_A^P(u)\eta_A^P(v)\,\mathrm{CONN}^P_{\eta(G)}(u,v),$$
$$\mu_A^N(u)\mu_A^N(v)\,\mathrm{CONN}^N_{\mu(G)}(u,v)+\eta_A^N(u)\eta_A^N(v)\,\mathrm{CONN}^N_{\eta(G)}(u,v))$$
$$=(\sum_{u,v\in V(G)}\mu_A^P(u)\mu_A^P(v)\,\mathrm{CONN}^P_{\mu(G)}(u,v)+\sum_{u,v\in V(G)}\eta_A^P(u)\eta_A^P(v)\,\mathrm{CONN}^P_{\eta(G)}(u,v),$$
$$\sum_{u,v\in V(G)}\mu_A^N(u)\mu_A^N(v)\,\mathrm{CONN}^N_{\mu(G)}(u,v)+\sum_{u,v\in V(G)}\eta_A^N(u)\eta_A^N(v)\,\mathrm{CONN}^N_{\eta(G)}(u,v))$$
$$=(CI_\mu^P(G)+CI_\eta^P(G),CI_\mu^N(G)+CI_\eta^N(G))$$
$$=(CI^P(G),CI^N(G)),$$

Where

$$CI_{\mu}^{P}(G)=\sum_{u,v\in V(G)}\mu_A^P(u)\mu_A^P(v)\mathrm{CONN}_{\mu(G)}^{P}(u,v),$$

$$CI_{\eta}^{P}(G)=\sum_{u,v\in V(G)}\eta_A^P(u)\eta_A^P(v)\mathrm{CONN}_{\eta(G)}^{P}(u,v),$$

$$CI_{\mu}^{N}(G)=\sum_{u,v\in V(G)}\mu_A^N(u)\mu_A^N(v)\mathrm{CONN}_{\mu(G)}^{N}(u,v),$$

$$CI_{\eta}^{N}(G)=\sum_{u,v\in V(G)}\eta_A^N(u)\eta_A^N(v)\mathrm{CONN}_{\eta(G)}^{N}(u,v)$$

Are positive μ -connectivity index, positive η -connectivity index, negative μ -connectivity index, negative η -connectivity index of *G*, respectively. Moreover,

$$CI^P(G)=CI_{\mu}^{P}(G)+CI_{\eta}^{P}(G),$$

$$CI^N(G)=CI_{\mu}^{N}(G)+CI_{\eta}^{N}(G)$$

Are positive CI and negative CI of *G*, respectively.

Theorem 1: Let $G=(V,A,B)$ be a BIFG with vertex set $V^*=\{v_1,v_2,\cdots,v_n\}$. Assume that $0\le t_1^P\le t_2^P\le\cdots\le t_n^P\le 1$, $1\ge s_1^P\ge s_2^P\ge\cdots\ge s_n^P\ge 0$, $-1\le t_1^N\le t_2^N\le\cdots\le t_n^N\le 0$, $0\ge s_1^P\ge s_2^P\ge\cdots\ge s_n^P\ge -1$, where $t_i^P=\mu_A^P(v_i)$, $s_i^P=\eta_A^P(v_i)$, $t_i^N=\mu_A^N(v_i)$, $s_i^N=\eta_A^N(v_i)$.

Then, we get

$$CI(G)=(CI^P(G),CI^N(G))=(\sum_{i=1}^{n-1}(t_i^P)^2\sum_{j=i+1}^{n}t_j^P+\sum_{i=1}^{n-1}(s_i^P)^2\sum_{j=i+1}^{n}s_j^P,\sum_{i=n}^{2}(t_i^N)^2\sum_{j=i-1}^{1}t_j^N+\sum_{i=n}^{2}(s_i^N)^2\sum_{j=i-1}^{1}s_j^N).$$

Proof. Since Naeem *et al.*, [16] have proven the positive part

$$CI^P(G)=\sum_{i=1}^{n-1}(t_i^P)^2\sum_{j=i+1}^{n}t_j^P+\sum_{i=1}^{n-1}(s_i^P)^2\sum_{j=i+1}^{n}s_j^P,$$

We only need to verify the negative part:

$$CI^N(G)=\sum_{i=n}^{2}(t_i^N)^2\sum_{j=i-1}^{1}t_j^N+\sum_{i=n}^{2}(s_i^N)^2\sum_{j=i-1}^{1}s_j^N.$$

Let v_n be the vertex with the largest negative truth membership value t_n^N. For a complete BIFG, $\mu_B^N(v,v')=\mathrm{CONN}_{\mu(G)}^{N}(v,v')$ for any $v,v'\in V^*$. Thus, $\mu_B^N(v_n,v_i)=t_n^N$ for $1\le i\le n-1$ and $\mu_A^N(v_n)\mu_A^N(v_i)\mathrm{CONN}_{\mu(G)}^{N}(v_n,v_i)=t_n^N\cdot t_i^N\cdot t_n^N=(t_n^N)^2t_i^N$ for $1\le i\le n-1$. Taking summation over *i*, we get

$$\sum_{i=n-1}^{1}\mu_A^N(v_n)\mu_A^N(v_i)\mathrm{CONN}_{\mu(G)}^{N}(v_n,v_i)=\sum_{i=n-1}^{1}(t_n^N)^2t_i^N.$$

Similarly, for vertex v_{n-1}, we yield $\sum_{i=n-2}^{1}\mu_A^N(v_{n-1})\mu_A^N(v_i)\mathrm{CONN}_{\mu(G)}^{N}(v_{n-1},v_i)=\sum_{i=n-2}^{1}(t_{n-1}^N)^2t_i^N$. For v_{n-2}, $\sum_{i=n-3}^{1}\mu_A^N(v_{n-2})\mu_A^N(v_i)\mathrm{CONN}_{\mu(G)}^{N}(v_{n-2},v_i)=\sum_{i=n-3}^{1}(t_{n-2}^N)^2t_i^N$ and so on. Finally for v_2, we obtain

$$\sum_{i=1}^{1}\mu_A^N(v_2)\mu_A^N(v_i)\mathrm{CONN}_{\mu(G)}^{N}(v_2,v_i)=\sum_{i=1}^{1}(t_2^N)^2t_i^N.$$

In terms of adding all the above equations, we infer

$$CI_\mu^N(G)=\sum_{i=n-1}^{1}(t_n^N)^2 t_i^N+\sum_{i=n-2}^{1}(t_{n-1}^N)^2 t_i^N+\cdots+\sum_{i=1}^{1}(t_2^N)^2 t_i^N=\sum_{i=n}^{2}(t_i^N)^2\sum_{j=i-1}^{1}t_j^N .$$

Let v_n be the vertex with the smallest negative falsity membership value s_n^N. For a complete BIFG, $\eta_B^N(v,v')=\mathrm{CONN}_{\eta(G)}^N(v,v')$ for any $v,v'\in V^*$. Thus, $\eta_B^N(v_n,v_i)=s_n^N$ for $1\le i\le n-1$ and $\eta_A^N(v_n)\eta_A^N(v_i)\mathrm{CONN}_{\eta(G)}^N(v_n,v_i)=s_n^N\cdot s_i^N\cdot s_n^N=(s_n^N)^2 s_i^N$ for $1\le i\le n-1$. Taking summation over *i*, we get

$$\sum_{i=n-1}^{1}\eta_A^N(v_n)\eta_A^N(v_i)\mathrm{CONN}_{\eta(G)}^N(v_n,v_i)=\sum_{i=n-1}^{1}(s_n^N)^2 s_i^N .$$

Similarly, for vertex v_{n-1}, we yield $\sum_{i=n-2}^{1}\eta_A^N(v_{n-1})\eta_A^N(v_i)\mathrm{CONN}_{\eta(G)}^N(v_{n-1},v_i)=\sum_{i=n-2}^{1}(s_{n-1}^N)^2 s_i^N$; for v_{n-2}, $\sum_{i=n-3}^{1}\eta_A^N(v_{n-2})\eta_A^N(v_i)\mathrm{CONN}_{\eta(G)}^N(v_{n-2},v_i)=\sum_{i=n-3}^{1}(s_{n-2}^N)^2 s_i^N$ and so on. Finally for v_2, we obtain

$$\sum_{i=1}^{1}\eta_A^N(v_2)\eta_A^N(v_i)\mathrm{CONN}_{\eta(G)}^N(v_2,v_i)=\sum_{i=1}^{1}(s_2^N)^2 s_i^N .$$

In terms of adding all the above equations, we infer

$$CI_\eta^N(G)=\sum_{i=n-1}^{1}(s_n^N)^2 s_i^N+\sum_{i=n-2}^{1}(s_{n-1}^N)^2 s_i^N+\cdots+\sum_{i=1}^{1}(s_2^N)^2 s_i^N=\sum_{i=n}^{2}(s_i^N)^2\sum_{j=i-1}^{1}s_j^N .$$

Hence, we prove the desired result. □

The following conclusion reveals that the value of $CI(G)$ is affected by deleting a bridge of BIFG *G*.

Theorem 2: Let *H* be the bipolar intuitionistic fuzzy subgraph (BIFSG) of a BIFG *G* by removing an edge $uv\in E(G)$. Then $CI^P(G)>CI^P(H)$ and $CI^N(G)<CI^N(H)$ if and only if uv is a bridge of *G*.

Proof. Since Naeem *et al.*, [16] determined that $CI^P(G)>CI^P(H)$ if and only if uv is a PB. Here, we only confirm that $CI^N(G)<CI^N(H)$ if and only if uv is a NB.

If uv is a NB, then using its definition, there exist *v* and *u* such that their negative strength of connectedness will increase. Hence, we have $CI^N(G)<CI^N(H)$.

On the contrary, assume that $CI^N(G)<CI^N(H)$, and then we consider the following three cases.

Case 1: uv is a δ-edge.
In this case, we have;

$$\mathrm{CONN}_{\mu(G)}^N(u,v)=\mathrm{CONN}_{\mu(G)-(u,v)}^N(u,v),$$

$$\mathrm{CONN}_{\eta(G)}^N(u,v)=\mathrm{CONN}_{\eta(G)-(u,v)}^N(u,v).$$

Thus, $CI_\mu^N(G)=CI_\mu^N(H)$ and $CI_\eta^N(G)=CI_\eta^N(H)$, which implies $CI^N(G)=CI^N(H)$, a contradiction.

Case 2: uv is a β-strong edge.
In this case, we get

$$\mu_B^N(u,v)=\mathrm{CONN}_{\mu(G)-(u,v)}^N(u,v),$$

$$\eta_B^N(u,v)=\mathrm{CONN}_{\eta(G)-(u,v)}^N(u,v).$$

Thus there is another $u-v$ path different from edge uv and removing uv doesn't change the negative strength of connectedness between *u* and *v*, which also implies $CI^N(G)=CI^N(H)$, a contradiction.

Case 3: uv is a α -strong edge.
In this case, we have
$\mu_B^N(u,v)<\mathrm{CONN}^N_{\mu(G)-(u,v)}(u,v)$,
$\eta_B^N(u,v)>\mathrm{CONN}^N_{\eta(G)-(u,v)}(u,v)$.

Thus the only SP is uv edge having NS equal to $(\mu_B^N(u,v),\eta_B^N(u,v))$, which also implies $CI^N(G)<CI^N(H)$.
We get the result of Theorem 2. □

According to the proof of Theorem 2, we directly get the following two corollaries.
Corollary 1; Let *H* be the BIFSG of a BIFG *G* by removing an edge $uv\in E(G)$. Then $CI(G)=CI(H)$ if and only if uv is a δ -edge or β -strong edge.

Corollary 2: Let *H* be the BIFSG of a BIFG *G* by removing an edge $uv\in E(G)$. Then $CI(G)\neq CI(G-uv)$ if and only if uv is a unique bipolar intuitionistic fuzzy bridge of *G*.

When two BIFGs are isomorphic, there is a bijection between them. We believe that they have the same properties, so we have the following conclusion and the detailed proof is skipped here.

Theorem 3: Let $G_1=(V_1,A_1,B_1)$ and $G_2=(V_2,A_2,B_2)$ be two isomorphic BIFGs. We have $CI(G_1)=CI(G_2)$.

Definition 13: The positive truth and positive falsity values of the weakest edge in a cycle *C* are defined to be the positive strength of *C*, and negative truth and negative falsity values of the weakest edge in a cycle *C* are defined to be the negative strength of *C* in a BIFG *G*.

Definition 14: Let *C* be a cycle in BIFG *G*. Then *C* is called intuitionistic fuzzy strongest strong cycle if it is the union of two strongest strong $u-v$ paths for each *u* and *v* in *C* unless uv is a bipolar intuitionistic bridge of *G*. Cycle *C* is a strong cycle (in short, SC) if and only if each edge is strong.

Definition 14: Let *G* be a BIFG. Then the positive θ_μ –evaluation of two vertices *u* and *v* in *G* is denoted by
$\theta_\mu^P(u,v)=\{\alpha^P:\alpha^P\in(0,1]\}$,
where α^P represents μ -PS of a strong cycle passing through both *u* and *v*. The positive θ_η –evaluation of two vertices *u* and *v* in *G* is denoted by
$\theta_\eta^P(u,v)=\{\beta^P:\beta^P\in(0,1]\}$,
where β^P represents η -PS of a strong cycle passing through both *u* and *v*. The negative θ_μ –evaluation of two vertices *u* and *v* in *G* is denoted by
$\theta_\mu^N(u,v)=\{\alpha^N:\alpha^N\in[-1,0)\}$,
where α^N represents μ NS of a strong cycle passing through both *u* and *v*. The negative θ_η –evaluation of two vertices *u* and *v* in *G* is denoted by
$\theta_\eta^N(u,v)=\{\beta^N:\beta^N\in[-1,0)\}$,
where β^N represents η -NS of a strong cycle passing through both *u* and *v*.

If strong cycles through *u* and *v* don't exist, then
$\theta_\mu^P(u,v)=\theta_\eta^P(u,v)=\varnothing$,
$\theta_\mu^N(u,v)=\theta_\eta^N(u,v)=\varnothing$.

Definition 15: Let *G* be a BIFG. The positive cycle μ -connectivity between *u* and *v* in *G* is denoted by
$C_{u,v}^{\mu,P}=\vee\{\alpha^P:\alpha^P\in\theta_\mu^P(u,v);u,v\in V^*\}$.

The positive cycle η -connectivity between *u* and *v* in *G* is denoted by
$C_{u,v}^{\eta,P}=\wedge\{\beta^P:\beta^P\in\theta_\eta^P(u,v);u,v\in V^*\}$.

The negative cycle μ -connectivity between *u* and *v* in *G* is denoted by
$C_{u,v}^{\mu,N}=\wedge\{\alpha^N:\alpha^N\in\theta_\mu^N(u,v);u,v\in V^*\}$.

The negative cycle η -connectivity between *u* and *v* in *G* is denoted by
$C_{u,v}^{\eta,N}=\vee\{\beta^N:\beta^N\in\theta_\eta^N(u,v);u,v\in V^*\}$.

Furthermore, we get
$\theta_\mu^P(u,v)=\varnothing\Rightarrow C_{u,v}^{\mu,P}=0$,
$\theta_\eta^P(u,v)=\varnothing\Rightarrow C_{u,v}^{\eta,P}=0$,
$\theta_\mu^N(u,v)=\varnothing\Rightarrow C_{u,v}^{\mu,N}=0$,
$\theta_\eta^N(u,v)=\varnothing\Rightarrow C_{u,v}^{\eta,N}=0$.

Theorem 4: Suppose that *G* is a BIFG and for any $u, v \in V(G)$, there is a strong cycle containing both *u* and *v*. Then, we obtain

$$CI(G) = (CI^P(G), CI^N(G))$$
$$= (\sum_{u,v \in V(G)} \mu_A^P(u)\mu_A^P(v)C_{u,v}^{\mu,P} + \sum_{u,v \in V(G)} \eta_A^P(u)\eta_A^P(v)C_{u,v}^{\eta,P}, \sum_{u,v \in V(G)} \mu_A^N(u)\mu_A^N(v)C_{u,v}^{\mu,N} + \sum_{u,v \in V(G)} \eta_A^N(u)\eta_A^N(v)C_{u,v}^{\eta,N})$$

Proof. Since Naeem *et al.*, [16] have proved the positive part, and we only proof the negative part, i.e.,

$$CI^N(G) = \sum_{u,v \in V(G)} (\mu_A^N(u)\mu_A^N(v)C_{u,v}^{\mu,N} + \eta_A^N(u)\eta_A^N(v)C_{u,v}^{\eta,N}).$$

Assume that $u, v \in V^*$ lie on a common bipolar intuitionistic fuzzy strongest strong cycle. Then $C_{u,v}^{\mu,N} = \min\{\alpha^N : \alpha^N \in \theta_\mu^N(u,v); u, v \in V^*\}$, where $\theta_\mu^N(u,v) = \{\alpha^N \in [-1,0) : \alpha^N$ is μ-NS of a strong cycle through *u* and *v*}. Hence $\mathrm{CONN}_{\mu(G)}^N(u,v) = C_{u,v}^{\mu,N}$ and

$$CI_\mu^N(G) = \sum_{u,v \in V(G)} \mu_A^N(u)\mu_A^N(v)C_{u,v}^{\mu,N}.$$

Similar, $C_{u,v}^{\eta,N} = \max\{\beta^N : \beta^N \in \theta_\eta^N(u,v); u, v \in V^*\}$, where $\theta_\eta^N(u,v) = \{\beta^N \in [-1,0) : \beta^N$ is η-NS of a strong cycle through *u* and *v*}. Hence $\mathrm{CONN}_{\eta(G)}^N(u,v) = C_{u,v}^{\eta,N}$ and

$$CI_\eta^N(G) = \sum_{u,v \in V(G)} \eta_A^N(u)\mu_A^N(v)C_{u,v}^{\eta,N}.$$

Therefore, we finish the proof of Theorem 4. □

Definition 16: Let *G* be a BIFG. Then the positive average μ-connectivity index of *G* is denoted by

$$ACI_\mu^P(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} \mu_A^P(u)\mu_A^P(v)\mathrm{CONN}_{\mu(G)}^P(u,v).$$

The positive average η-connectivity index of *G* is denoted by

$$ACI_\eta^P(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} \eta_A^P(u)\eta_A^P(v)\mathrm{CONN}_{\eta(G)}^P(u,v).$$

The negative average μ-connectivity index of *G* is denoted by

$$ACI_\mu^N(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} \mu_A^N(u)\mu_A^N(v)\mathrm{CONN}_{\mu(G)}^N(u,v).$$

The negative average η-connectivity index of *G* is denoted by

$$ACI_\eta^N(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} \eta_A^N(u)\eta_A^N(v)\mathrm{CONN}_{\eta(G)}^N(u,v).$$

Definition 17: Let *G* be a BIFG. The average connectivity index of *G* is defined by

$$ACI(G) = (ACI^P(G), ACI^N(G)) = (ACI_\mu^P(G) + ACI_\eta^P(G), ACI_\mu^N(G) + ACI_\eta^N(G)),$$

where $ACI^P(G) = ACI^P_\mu(G) + ACI^P_\eta(G)$ is called the positive average connectivity index of *G*, and $ACI^N(G) = ACI^N_\mu(G) + ACI^N_\eta(G)$ is called the negative average connectivity index of *G*.

Obviously, $ACI^P(G)$ as well as $CI^P(G)$ will not increase by removing an edge, and $ACI^N(G)$ as well as $CI^N(G)$ will not decrease by deleting an edge. Furthermore, $0 \le ACI^P(G) \le 1$ and $-1 \le ACI^N(G) \le 0$.

Definition 18: Let *G* be a BIFG and $v \in V^*$. If $ACI^P(G-\{v\}) < ACI^P(G)$ and $ACI^N(G-\{v\}) > ACI^N(G)$, then *v* is called a bipolar intuitionistic fuzzy connectivity reducing vertex (BIFCRV); if $ACI^P(G-\{v\}) > ACI^P(G)$ and $ACI^N(G-\{v\}) < ACI^N(G)$, then *v* is called a bipolar intuitionistic fuzzy connectivity enhancing vertex (BIFCEV); if $ACI^P(G-\{v\}) = ACI^P(G)$ and $ACI^N(G-\{v\}) = ACI^N(G)$, then *v* is called a bipolar intuitionistic fuzzy connectivity neutral vertex (BIFCNV).

Theorem 5: Let *G* be a BIFG with at least three vertices and $v \in V^*$. Set $a = \frac{CI^P(G)}{CI^P(G-\{v\})}$ and $b = \frac{CI^N(G)}{CI^N(G-\{v\})}$. Then

- *v* is a BIFCEV $\Leftrightarrow$ $a < \frac{n}{n-2}$ and $b > \frac{n}{n-2}$;
- *v* is a BIFCRV $\Leftrightarrow$ $a > \frac{n}{n-2}$ and $b < \frac{n}{n-2}$;
- *v* is a BIFCNV $\Leftrightarrow$ $a = \frac{n}{n-2}$ and $b = \frac{n}{n-2}$.

The proof of Theorem 5 is similar to what we presented in Gong and Hua [17] as similar definition and theorem in BIFG setting. From Theorem 5, we directly yield the following corollary.

Corollary 3: Let *G* be a BIFG with at least three vertices and $v \in V^*$. Set $a = \frac{CI^P(G)}{CI^P(G-\{v\})}$ and $b = \frac{CI^N(G)}{CI^N(G-\{v\})}$. Then

- If *v* is a BIFCEV, then $a < b$;
- If *v* is a BIFCRV, then $a > b$;
- If *v* is a BIFCNV, then $a = b$.

The following conclusion is the extension of Theorem 6 in Naeem *et al.*, [16].

Theorem 6: Let *G* be a BIFG with at least three vertices and $v \in V^*$ is an end vertex of *G*. Set

$$c = \sum_{u \in V(G)-\{v\}} (\mathrm{CONN}^P_{\mu(G)}(u,v) + \mathrm{CONN}^P_{\eta(G)}(u,v)),$$

$$d = \sum_{u \in V(G)-\{v\}} (\mathrm{CONN}^N_{\mu(G)}(u,v) + \mathrm{CONN}^N_{\eta(G)}(u,v)).$$

Then,

- *v* is a BIFCEV if $c < \frac{2}{n-2} CI^P(G-\{v\})$ and $d > \frac{2}{n-2} CI^N(G-\{v\})$;
- *v* is a BIFCRV if $c > \frac{2}{n-2} CI^P(G-\{v\})$ and $d < \frac{2}{n-2} CI^N(G-\{v\})$;
- *v* is a BIFCNV if $c = \frac{2}{n-2} CI^P(G-\{v\})$ and $d = \frac{2}{n-2} CI^N(G-\{v\})$.

Finally, we classify the BIFGs by means of the above definitions.

Definition 13: Let *G* be a bipolar intuitionistic fuzzy graph with at least three vertices. Suppose *G* has at least one BIFCEV, in that case, *G* is called bipolar intuitionistic fuzzy connectivity enhancing graph. If *G* has no BIFCRV, then *G* is called bipolar intuitionistic fuzzy connectivity reducing graph. Suppose all the vertices in *G* are BIFCNV, in that case, *G* is called bipolar intuitionistic fuzzy connectivity neutral graph.

4. CONCLUSION

After a graph model represents the network structure, the MF is used to describe the uncertainty of the stations and channels in the network, and thus the entire network is modeled with a FG. To discuss the stability and connectivity of fuzzy network graphs, we define the connectivity index of BIFGs. The main contribution of this paper is to describe the characteristics of the connectivity index from a theoretical point of view. However, we have not made any experimental verification, which requires the definition of the membership function according to the network attack's actual situation and application background. We will discuss these issues in future articles.

For the bipolar Pythagorean fuzzy graph (BPFG), the difference between it and the BIFG lies in the restriction condition of the membership function, that is $(\mu_A^P(v))^2 + (\eta_A^P(v))^2 \le 1$ and $(\mu_A^N(v))^2 + (\eta_A^N(v))^2 \le 1$ hold for any vertex *v*, $(\mu_B^P(v,v'))^2 + (\eta_B^P(v,v'))^2 \le 1$ and $(\mu_B^N(v,v'))^2 + (\eta_B^N(v,v'))^2 \le 1$ hold for any pair of vertices $(v,v') \in V \times V$. Therefore, the related definitions and properties of BIFGs given in this article can be directly extended to BPFG s. For the sake of space, we will no longer make a specific narrative.

The following aspects can be considered as topics for future research:

(1) The bipolar connectivity index reflects the topology of the positive and negative aspects of the network graph and is closely related to the transmission of data, services and resources. Therefore, it is necessary to consider practical applications in network attacks in the future, such as the connectivity of the remaining subgraphs after a specific number of vertices are attacked.

(2) More definitions and theoretical characteristics of connected indices under more FG settings must be further studied. Furthermore, the connectivity index needs to be compared with other types of topology indices, and some measures for evaluating the quality of indicators need to be defined.

REFERENCES

1. Gao, W., Wang, W., & Dimitrov, D. (2019). Toughness condition for a graph to be all fractional (*g*,*f*,*n*)-critical deleted. *Filomat*, 33, 2735-2746.
2. Gao, W., Guirao, J. L. G., & Chen, Y. (2019). A toughness condition for fractional (*k*,*m*)-deleted graphs revisited. *Acta Math. Sin. (Engl. Ser.)*, 35, 1227-1237.
3. Gao, W., Liang, L., & Chen, Y. (2017). An isolated toughness condition for graphs to be fractional (*k*,*m*)-deleted graphs. *Util. Math*., 105, 303-316.
4. Gao, W., & Wang, W. (2017). New isolated toughness condition for fractional (*g*,*f*,*n*)-critical graphs. *Colloq. Math.*, 147, 55-66.
5. Gao, W., Wang, W., & Chen, Y. (2021). Tight bounds for the existence of path factors in network vulnerability parameter settings. *Int. J. Intell. Syst*., 36, 1133-1158.
6. Gao, W., Senel, M., Yel, G., Baskonus, H. M., & Senel, B. (2020). New complex wave patterns to the electrical transmission line model arising in network system. *AIMS Math.*, 5, 1881-1892.
7. Ali, H., Binyamin, M. A., Shafiq, M. K., & Gao, W. (2019). On the degree-based topological indices of some derived networks. *Mathematics*, 7(7), 612. https://doi.org/10.3390/math7070612.
8. Aslam, A., Ahmad, S., Binyamin, M. A., & Gao, W. (2019). Calculating topological indices of certain OTIS Interconnection networks. *Open Chemistry*, 17, 220-228.
9. Gao, W., Zhang, Y., & Chen, Y. (2018). A note on transmission feasibility problem in networks. *Open Phys.*, 16, 889–895.
10. Gao, W., Wu, H., Siddiqui, M. K., & Baig, A. Q. (2018). Study of biological networks using graph theory. *Saudi J. Biol. Sci.*, 25, 1212-1219.
11. Poulik, S., & Ghorai, G. (2020). Note on ‘‘Bipolar fuzzy graphs with applications’’. *Knowl. Based Syst.*, 192, 105315.
12. Akram, M. (2013). Bipolar fuzzy graphs with applications. *Knowl. Based Syst.*, 39, 1-8.
13. Shannon, A., & Atanassov, K. T. (1994). A first step to a theory of the intuitionistic fuzzy graphs, in: D. Lakov (Ed.), *Proceeding of the FUBEST, Bulgarian Academy of Sciences, Sofia, Bulgaria*, 59-61.
14. Shannon, A., & Atanassov, K. T. (1995). Intuitionistic fuzzy graphs from α-, and (α, β)-levels, Notes Intuit. *Fuzzy Sets*, 1, 32-35.
15. Ezhilmaran, D., & Sankar, K. (2015). Morphism of bipolar intuitionistic fuzzy graphs. *J. Discrete Math. Sci. C.*, 18, 605-621.
16. Naeem, T., Gumaei, A., Jamil, M. K., Alsanad, A., & Ullah, K. (2021). Connectivity indices of intuitionistic fuzzy graphs and their applications in internet routing and transport network flow. *Math. Probl. Eng.*, 2021, 4156879.
17. Gong, S., & Hua, G. (2021). Topological indices of bipolar fuzzy incidence graph. *Open Chem.*, 19, 894-903.