Scholars Journal of Physics, Mathematics and Statistics OPEN ACCESS
Abbreviated Key Title: Sch J Phys Math Stat
ISSN 2393-8056 (Print) | ISSN 2393-8064 (Online)
Journal homepage: https://saspublishers.com

On Stević-Sharma Operator from Dirichlet-Zygmund Space to Weighted-Type Space

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DOI: 10.36347/sjpms.2023.v10i01.003 | **Received:** 15.12.2022 | **Accepted:** 23.01.2023 | **Published:** 26.01.2023

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Abstract **Review Article**

In this paper, the necessary and sufficient conditions are given for the Stević-Sharma operator to be bounded or compact from Dirichlet-Zygmund space to Weighted-Type space on the unit disk. We also give the estimate of its essential norm.

Keywords: should contain "Stević-Sharma operator, Dirichlet-Zygmund Space, boundedness, essential norm.

1. INTRODUCTION

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$, i.e., $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$, $H(\mathbf{D})$ the space of all analytic functions on $\mathbf{D}$ and $S(\mathbf{D})$ the family of all analytic self-maps of $\mathbf{D}$. Denote by $\mathbf{N}$ the set of positive integers.

Suppose that $1\le p<\infty$, Dirichlet type space D_{p-1}^p consists of all $f\in H(\mathbf{D})$ such that:

$$\|f\|_{D_{p-1}^p}^p=|f(0)|^p+\int_{\mathbf{D}}|f'(z)|^p(1-|z|^2)^{p-1}\mathrm{d}A(z)<\infty,$$

Where $\mathrm{d}A(z)=(1/\pi)\mathrm{d}x\mathrm{d}y$ is the normalized Lebesgue area measure on $\mathbf{D}$. Under the norm $\|\cdot\|_{D_{p-1}^p}$, D_{p-1}^p is a Banach space. If $f'\in D_{p-1}^p$, then f is said to belong to the Dirichlet-Zygmund space, which is denoted by Z_{p-1}^p. Recently, Z_{p-1}^p space was first studied in [1].

A strictly positive continuous function μ on $\mathbf{D}$ is called a weight. If $\mu(z)=\mu(|z|)$ for any $z\in\mathbf{D}$, then μ is called radial. Let μ be a radial weight, the weighted-type space H_μ^∞ consists of all $f\in H(\mathbf{D})$ such that

$$\|f\|_\infty=\sup_{z\in\mathbf{D}}\mu(z)|f(z)|<\infty.$$

In [2, 3], Stević *et al.*, introduced the following Stević-Sharma operator:

$$T_{u,v,\varphi}f(z)=u(z)f(\varphi(z))+v(z)f'(\varphi(z)),\ \ f\in H(\mathbf{D}),$$

Where u, $v\in H(\mathbf{D})$, $\varphi\in S(\mathbf{D})$. By taking some specific choices of the involving symbols, we can easily get the general product-type operators.

Recently, the study of Stević-Sharma operator $T_{u,v,\varphi}$ has aroused the interest of experts. For instance, the author in [4] investigated the metrical boundedness and metrical compactness of the operator $T_{u,v,\varphi}$ from weighted Bergman-Orlicz space to (little) Bloch-type space. Zhang and Liu in [5] characterized the boundedness and compactness of $T_{u,v,\varphi}$ from Hardy space to Zygmund-type space. Zhu *et al.*, in [6] provided the necessary and sufficient conditions for $T_{u,v,\varphi}$ acting from Besov space to Bloch space to be bounded or compact. For more results about Stević-Sharma operator see [7-13] and also related references therein.

Inspired by these, this paper is devoted to investigating the boundedness, compactness and essential norm of $T_{u,v,\varphi}$ from Dirichlet-Zygmund space to weighted-type space.

Throughout the paper we use the abbreviation $X \prec Y$ or $Y \succ X$ for nonnegative quantities X and Y, whenever there is a positive constant C, which is inessential, such that $X \le CY$. If both $X \prec Y$ and $X \succ Y$ hold, we write $X \approx Y$.

2. PRELIMINARIES

In this section, we state several auxiliary results which will be used in the proofs of the main results. The first lemma is the point evaluation estimates of the functions in Z_{p-1}^p, which can be found in [1, Lemma 1].

Lemma 1 Let $1 < p < \infty$, then for each $f \in Z_{p-1}^p$, $\| f \|_\infty \prec \| f \|_{Z_{p-1}^p}$ and

$$| f'(z) | \prec \frac{\| f \|_{Z_{p-1}^p}}{(1-|z|^2)^{1/p}}.$$

By the same method as [1, Lemma 5], we can get the lemma below.

Lemma 2 Let $1 < p < \infty$, u, $v \in H(\mathbf{D})$, $\varphi \in S(\mathbf{D})$ and μ be a radial weight. If operator $T : Z_{p-1}^p \to H_\mu^\infty$ is bounded, then $T : Z_{p-1}^p \to H_\mu^\infty$ is compact if and only if for every sequence $\{f_n\}_{n\in\mathbf{N}}$ which is bounded in Z_{p-1}^p and converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$, we have $\lim_{n\to\infty} \| T f_n \|_{B^\mu} = 0$.

Lemma 3 [1] Let $1 < p < \infty$. Every sequence in Z_{p-1}^p bounded in norm has a subsequence which converges uniformly in $\overline{\mathbf{D}}$ to a function in Z_{p-1}^p.

3. MAIN RESULTS

In this section, we characterize the boundedness, compactness and essential norm of the Stević-Sharma operator $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$.

Theorem 1 Let $1 < p < \infty$, u, $v \in H(\mathbf{D})$, $\varphi \in S(\mathbf{D})$ and μ be a radial weight. Then the operator $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$ is bounded if and only if $u \in H_\mu^\infty$ and

$$M := \sup_{z\in\mathbf{D}} \frac{\mu(z)|v(z)|}{(1-|\varphi(z)|^2)^{1/p}} < \infty$$

Proof Assume that $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$ is bounded. Firstly, consider the function $f(z) = 1 \in Z_{p-1}^p$, by the boundedness of $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$, we have

$$L_1 := \sup_{z\in\mathbf{D}} \mu(z)|u(z)| \le \| T_{u,v,\varphi} 1 \|_\infty < \infty,$$

That is, $u \in H_\mu^\infty$. By using the function $f(z) = z \in Z_{p-1}^p$ yields that

$$\infty > \| T_{u,v,\varphi} z \|_\infty \geq \sup_{z\in\mathbf{D}} \mu(z) | u(z)\varphi(z) + v(z) |$$
$$\geq \sup_{z\in\mathbf{D}} \mu(z) | v(z) | - \sup_{z\in\mathbf{D}} \mu(z) | u(z)\varphi(z) |,$$

Which along with $L_1 < \infty$ and the fact that $|\varphi(z)| < 1$, we get

$$L_2 := \sup_{z\in\mathbf{D}} \mu(z) | v(z) | < \infty.$$

For any fixed $w \in \mathbf{D}$, set

$$f_w(z) = -\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{1/p}} + \frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{1+(1/p)}}.$$

It is easily seen that [1] $f_w(z) \in Z_{p-1}^p$, and by a direct calculation, we have

$$f_w(\varphi(w)) = 0, \quad f_w'(\varphi(w)) = \frac{\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1/p}},$$

Which along with the boundedness of the operator $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$ implies that

$$\infty > \| T_{u,v,\varphi} f_w \|_\infty = \sup_{z\in\mathbf{D}} \mu(z) | T_{u,v,\varphi} f_w(z) | \succ \frac{\mu(w)|v(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^{1/p}}.$$

Therefore

$$\sup_{|\varphi(w)|>1/2} \frac{\mu(w)|v(w)|}{(1-|\varphi(w)|^2)^{1/p}} < \infty.$$

On the other hand, by using $L_2 < \infty$, we have

$$\sup_{|\varphi(w)|\leq 1/2} \frac{\mu(w)|v(w)|}{(1-|\varphi(w)|^2)^{1/p}} \leq (4/3)^{1/p} L_2 < \infty.$$

Hence, $M < \infty$.

Conversely, assume that $M < \infty$ and $u \in H_\mu^\infty$, i.e., $L_1 := \sup_{z\in\mathbf{D}} \mu(z) | u(z) | < \infty$. By Lemma 1, for any $f \in Z_{p-1}^p$, we have

$$\begin{aligned} &\mu(z) | T_{u,v,\varphi} f(z) | \\ &\leq \mu(z)|u(z)||f(\varphi(z))| + \mu(z)|v(z)||f'(\varphi(z))| \\ &\leq \mu(z)|u(z)|\|f\|_{Z_{p-1}^p} + \frac{\mu(z)|v(z)|}{(1-|\varphi(z)|^2)^{1/p}} \|f\|_{Z_{p-1}^p} \\ &\leq (L_1 + M)\|f\|_{Z_{p-1}^p}, \end{aligned}$$

From which we can see that the operator $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$ is bounded.

Now we estimate the essential norm of $T_{u,v,\varphi} : Z_{p-1}^p \to H_\mu^\infty$. Recall that the essential norm of a bounded linear operator $T : X \to Y$ is the distance from T to the compact operators $K : X \to Y$, namely

$$\|T\|_{e,X\to Y} = \inf \left\{ \|T - K\|_{X\to Y} \right\}.$$

Here X and Y are Banach spaces. Notice that $\|T\|_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact.

Theorem 2 Let $1 < p < \infty$, u, $v \in H(\mathbf{D})$, $\varphi \in S(\mathbf{D})$ and μ be a radial weight. Suppose that the operator $T_{u,v,\varphi} : Z_{p-1}^{p} \to H_{\mu}^{\infty}$ is bounded, then

$$\| T_{u,v,\varphi} \|_{e, Z_{p-1}^{p} \to H_{\mu}^{\infty}} \approx P,$$

where

$$P := \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) | v(z) |}{(1 - | \varphi(z) |^2)^{1/p}}.$$

Proof First, we prove that $\| T_{u,v,\varphi} \|_{e, Z_{p-1}^{p} \to H_{\mu}^{\infty}} \succ P$. Let $\{z_n\}_{n \in \mathbf{N}} \subset \mathbf{D}$ such that $| \varphi(z_n) | \mapsto 1$ as $n \to \infty$. Define the sequence of functions:

$$f_n(z) = -\frac{1 - | \varphi(z_n) |^2}{(1 - \overline{\varphi(z_n)} z)^{1/p}} + \frac{(1 - | \varphi(z_n) |^2)^2}{(1 - \overline{\varphi(z_n)} z)^{1 + (1/p)}}.$$

By the proof of Theorem 1, we can see that $\{ f_n \}_{n \in \mathbf{N}}$ belongs to Z_{p-1}^{p}, and

$$f_n(\varphi(z_n)) = 0, \quad f_n'(\varphi(z_n)) = \frac{\overline{\varphi(z_n)}}{(1 - | \varphi(z_n) |^2)^{1/p}}.$$

Moreover, it is easily seen that $\{ f_n \}_{n \in \mathbf{N}}$ converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. By Lemma 2, for any compact operator $K : Z_{p-1}^{p} \to H_{\mu}^{\infty}$, we have

$$\begin{aligned} \| T_{u,v,\varphi} - K \|_{Z_{p-1}^{p} \to H_{\mu}^{\infty}} &\succ \limsup_{n \to \infty} \| T_{u,v,\varphi} f_n \|_{\infty} - \limsup_{n \to \infty} \| K f_n \|_{\infty} \\ &\succ \limsup_{n \to \infty} \frac{\mu(z_n) | v(z_n) | | \varphi(z_n) |}{(1 - | \varphi(z_n) |^2)^{1/p}}. \end{aligned}$$

It follows that

$$\begin{aligned} \| T_{u,v,\varphi} \|_{e, Z_{p-1}^{p} \to H_{\mu}^{\infty}} &= \inf_{K} \| T_{u,v,\varphi} - K \|_{Z_{p-1}^{p} \to H_{\mu}^{\infty}} \\ &\succ \limsup_{n \to \infty} \frac{\mu(z_n) | v(z_n) | | \varphi(z_n) |}{(1 - | \varphi(z_n) |^2)^{1/p}} \\ &= \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) | v(z) |}{(1 - | \varphi(z) |^2)^{1/p}} = P. \end{aligned}$$

Next, we show that $\| T_{u,v,\varphi} \|_{e, Z_{p-1}^{p} \to H_{\mu}^{\infty}} \prec P$. Let $r \in [0,1)$, define the operator K_r by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbf{D}).$$

It is clear that K_r is compact on Z_{p-1}^{p}, and $\| K_r \|_{Z_{p-1}^{p} \to Z_{p-1}^{p}} \le 1$. Moreover, $f - f_r$ converges to zero uniformly on compact subsets of $\mathbf{D}$ as $r \to 1$. Let $\{r_n\}_{n \in \mathbf{N}} \subset (0,1)$ be a sequence such that $r_n \to 1$ as $n \to \infty$. Then for any $n \in \mathbf{N}$, operator $T_{u,v,\varphi} K_{r_n} : Z_{p-1}^{p} \to B^{\mu}$ is compact, and so

$$\| T_{u,v,\varphi} \|_{e, Z_{p-1}^{p} \to H_{\mu}^{\infty}} \le \limsup_{n \to \infty} \| T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_n} \|_{Z_{p-1}^{p} \to H_{\mu}^{\infty}}.$$

Therefore, we only need to show that

$$\limsup_{n \to \infty} \| T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_n} \|_{Z_{p-1}^{p} \to H_{\mu}^{\infty}} \prec P.$$

For any nonnegative integer s, $(f-f_{r_n})^{(s)}$ converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n\to\infty$ by Cauchy's estimation. For each $f\in Z_{p-1}^p$ such that $\|f\|_{Z_{p-1}^p}\le 1$, we have

$$\begin{aligned}
&\|(T_{u,v,\varphi}-T_{u,v,\varphi}K_{r_n})f\|_\infty \\
&=\sup_{z\in\mathbf{D}}\mu(z)|(T_{u,v,\varphi}f-T_{u,v,\varphi}f_{r_n})(z)|\le \underbrace{\sup_{z\in\mathbf{D}}\mu(z)|u(z)||(f-f_{r_n})(\varphi(z))|}_{G_1} \\
&\quad+\underbrace{\sup_{|\varphi(z)|\le r_m}\mu(z)|v(z)||(f-f_{r_n})'(\varphi(z))|}_{G_2}+\underbrace{\sup_{|\varphi(z)|>r_m}\mu(z)|v(z)||(f-f_{r_n})'(\varphi(z))|}_{G_3},
\end{aligned}$$

Where $m\in\mathbf{N}$ such that $r_n\ge 1/2$ for any $n\ge m$. From Theorem 3 it follows that

$$\limsup_{n\to\infty}G_1\prec\limsup_{n\to\infty}\sup_{w\in\mathbf{D}}|(f-f_{r_n})(w)|=0.$$

Since $T_{u,v,\varphi}:Z_{p-1}^p\to H_\mu^\infty$ is bounded, and $\{z:|\varphi(z)|\le r_m\}$ is a compact subset of $\mathbf{D}$, Theorem 1 yields that

$$\limsup_{n\to\infty}G_2=0.$$

Finally, we estimate G_3. By using Lemma 1, we have

$$\limsup_{n\to\infty}G_3\prec\limsup_{n\to\infty}\|f-f_{r_n}\|_{Z_{p-1}^p}\sup_{|\varphi(z)|>r_m}\frac{\mu(z)|v(z)|}{(1-|\varphi(z)|^2)^{1/p}}.$$

Letting $m\to\infty$ in above inequality yields

$$\limsup_{n\to\infty}G_3\prec P.$$

Therefore

$$\limsup_{n\to\infty}\|T_{u,v,\varphi}-T_{u,v,\varphi}K_{r_n}\|_{Z_{p-1}^p\to H_\mu^\infty}\prec P.$$

According to the above discussion, we have $\|T_{u,v,\varphi}\|_{e,Z_{p-1}^p\to H_\mu^\infty}\approx P$.

From Theorem 2, we immediately obtain the following corollary, which characterizes the compactness of $T_{u,v,\varphi}:Z_{p-1}^p\to H_\mu^\infty$.

Corollary 1 Let $1<p<\infty$, u, $v\in H(\mathbf{D})$, $\varphi\in S(\mathbf{D})$ and μ be a radial weight. Suppose that $T_{u,v,\varphi}:Z_{p-1}^p\to H_\mu^\infty$ is bounded, then $T_{u,v,\varphi}:Z_{p-1}^p\to H_\mu^\infty$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|v(z)|}{(1-|\varphi(z)|^2)^{1/p}}=0.$$

4. CONCLUSION

In the present paper, we provide the necessary and sufficient conditions for the Stević-Sharma operator $T_{u,v,\varphi}$ acting from Dirichlet-Zygmund space Z_{p-1}^p to the weighted-type space H_μ^∞ to be bounded. We also estimate the essential norm of the operator $T_{u,v,\varphi}:Z_{p-1}^p\to H_\mu^\infty$. As a corollary, we characterize its compactness.

ACKNOWLEDGEMENTS

This work was supported by the Doctoral Fund of Henan Institute of Technology (No. KQ2003).

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