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The KKT Optimality Conditions and Duality for Constrained Programming Problem with Generalized α- Convex Fuzzy Functions

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Abstract **Review Article**

This paper mainly studies the mixed constraint interval programming problem under the generalized $\alpha-$ convex fuzzy mapping. Firstly, this paper give the concepts of fuzzy mappings, such as $\alpha-$quasiconvex, strictly $\alpha-$ quasiconvex, $\alpha-$pseudoconvex and strictly $\alpha-$pseudoconvex. Then, the relation of generalized $\alpha-$convex fuzzy mapping is studied and some properties are obtained. Finally, the necessary and sufficient KKT conditions are given, and the duality problem is established. The weak duality, strong duality and inverse duality theorem of fuzzy interval programming are proved.

Keywords: Fuzzy interval programming, strictly α- quasiconvex, directional LgH-differentiablity, KKT conditions, dual problem.

I. INTRODUCTION

As we all know, optimization is a problem we have been trying to solve. Convexity plays a very important role in optimization problems. If an optimization problem is convex, it means that the problem has been largely solved. In real life, the optimization problem we often face is not a perfect convex optimization, but we try to find some common methods to solve these optimization problems. And we know that these imperfect convex optimization problems may be caused by many factors, such as weak convexity, differentiability, fuzziness of objective function, weak convexity of constraint conditions and so on.

In recent years, in order to solve these problems caused by different influencing factors, many scholars have explored the weak convex function and studied its application in optimality conditions. For example, O.L. Mangasarian [1] described the properties and applications of pseudoconvex functions in 1965, and obtained sufficient conditions for Kuhn-Tucker differential conditions to be optimality when the objective function is pseudoconvex and the constraint condition is quasiconvex. Jean-Philippe Vial [2] proposed a class of convex functions in 1983, namely, convex functions. According to the sign of constants, this function is called weakly convex or strongly convex. More importantly, he obtained sufficient conditions for the global optimization of non-convex programming problems for such functions. Chanchal, Singh [3] discussed sufficient optimality criteria for quasiconvex functions in continuous time programming. Westerlund, T., and Pörn, R [4] introduced a cutting plane technique to solve the pseudoconvex mixed integer optimization problem. In 2012, Ivanov, V. I. [5] derived the optimality conditions that the objective function is a pseudoconvex function. Mishra, S. K. [6] *et al.*, derived some conditions for the minimization of nonsmooth pseudolinear functions, and obtained that the effective solution can be an appropriate effective solution under these conditions. With the deepening of research, scholars have also found that the differentiability of the objective function plays an extremely important role in the optimization problem. However, in real life, the functions we need to solve are often non- differentiable or subdifferentiable. In order to solve this problem, Craven, B. D. and B. Mond [7] find the minimum value of the function extended to the partial ordered space, and the objective function and constraint function are not always differentiable, and some results are obtained. P., Kanniappan [8] *et al.*, obtained KKT necessary and sufficient conditions and its duality for convex programming problems with sub-differential operator constraints. More importantly, we want to explore a method or a generalized differentiability to solve the non-differentiability of functions. In 2013, Megahed [9] proposed the concept of E-differentiable convex function, which converts the non-differentiable convex function into a differentiable function under the operation symbol, and derives Kuhn-Tucker and Fritz-John conditions to obtain the optimal solution of the optimization problem.

In fuzzy mathematics, since Chang and Zadeh [10] proposed fuzzy mappings, more and more scholars have studied some generalized convex fuzzy mappings. Sudarsan Nanda and Kadambini Kar [11] made pioneering research on mappings and proved that a fuzzy mapping is convex if and only if the above graph is convex set. Yu-Ru Syau [12] derived the relations among convex fuzzy mappings, pre-invariant convex fuzzy mappings and fuzzy mapping classes in 2000. Then Yu-Ru Syau [13] proved in 2001 that the classes of B-vex fuzzy mappings form a subset of quasiconvex fuzzy mappings. In 1983, Puri [14] defined the derivative and H-derivative of fuzzy mapping. Osmo Kaleva [15] studied H-derivative and obtained a necessary and sufficient condition for H-derivative of fuzzy mappings. In 2003, Wang and Wu [16] proposed the concepts of directional derivative, differential and subdifferential of fuzzy mappings from R^n into E^1. With the in-depth study of H-derivative, scholars have found that often the objective function is non-differentiable under the definition of H-differentiability. To solve this problem, B.Bede [17, 18] *et al.*, proposed a generalized Hukuhara differentiable concept to solve the problem of nondifferentiable functions.

With the in-depth study of fuzzy function and the wider application of fuzzy function in real life, fuzzy optimization problem has become an important topic for scholars. Panigrahi [19] extended the concepts of differentiability, convexity and generalized convexity, and derived KKT conditions for constrained fuzzy optimization minimization problems. Wu [20] derived the Karush-Kuhn-Tucker condition of the fuzzy-valued objective function optimization problem and proposed the concept of the solution of the optimization problem. All our academic research is to solve practical problems. Therefore, the objective function of the fuzzy optimization problem we face becomes complicated, such as interval function. In 2012, Zhang [21] extended the concepts of preinvexity and invexity to interval- valued functions, and obtained KKT optimality conditions for Lu-preinvex and invex optimization problems with interval-valued objective functions. Chalco-Cano [22] *et al.*, used the concept of generalized Hukuhara derivative to obtain KKT conditions for interval-valued functions. Finally, the purpose of our study is to solve the optimization problems brought by practical problems. In order to obtain some complex problems, we propose to find the dual problem of the problem. Wanka [23] *et al.*, gave conditions to characterize strong and complete Lagrangian duality for convex optimization problems in separated locally convex spaces. Craven [24] constructed a Wolfe dual problem to solve the continuous weak minimization of the vector objective function.

Inspired by the research in these fields, and so far, few people have studied the weak convexity of optimization problems. Therefore, it is necessary to study the concept of generalized convex fuzzy mapping and related fuzzy optimization problems. Based on the study of $\alpha-$convex fuzzy mapping in [26], the basic concepts and properties of $\alpha-$ quasiconvex, strictly $\alpha-$quasiconvex, $\alpha-$pseudoconvex and strictly $\alpha-$pseudoconvex are given. The optimization problem of fuzzy interval function is studied, and the KKT condition of mixed constraint programming is obtained. Its weak duality, strong duality and inverse duality theorem is given.

This paper is structured as follows. The second part mainly introduces some basic knowledge of fuzzy numbers and fuzzy intervals, and a partial order of fuzzy intervals. The third part discusses the differentiability of fuzzy interval function, gives the definition of $\alpha-$quasiconvex, strictly $\alpha-$quasiconvex, $\alpha-$pseudoconvex, strictly $\alpha-$pseudoconvex, and gives the corresponding examples. We also discuss the relationship between the above fuzzy functions. The fourth part obtains the KKT conditions of the fuzzy optimization problem based on mixed constraints. The fifth part studies its weak duality, strong duality and inverse duality theorems. The sixth part is the conclusion.

II. SYSTEM COORDINATES

We denote by K_C the family of all bounded closed intervals in R , i.e.,

$$K_C=\left\{\left[\underline{c},\overline{c}\right] \middle| \underline{c},\overline{c}\in R \text{ and } \underline{c}\le\overline{c}\right\}.$$

A fuzzy set on R^n is a mapping υ : $R^n\to[0,1]$. We call $[\upsilon]^\alpha=\left\{x\in R^n:\upsilon(x)\ge\alpha\right\}$,

$\alpha-$ cut for any $\alpha\in(0,1]$, and

$$supp(\upsilon)=\left\{x\in R^n:\upsilon(x)>0\right\}$$

is called the support of υ. We defined $[\upsilon]^0$ is the closure of $supp(\upsilon)$. Triangular fuzzy numbers are a special type of fuzzy numbers which are well determined by three real numbers $a \le b \le c$, denoted by $\upsilon = \langle a,b,c \rangle$, with $\alpha - levels$

$$[\upsilon]^\alpha = [a+(b-a)\alpha, c-(c-b)\alpha],$$

for all $\alpha \in (0,1]$.

Definition 2.1. [25] We called a fuzzy set υ on R is a fuzzy interval if:

1. υ is normal, i.e. there exists $x^{(0)} \in R$ such that $\upsilon\left(x^{(0)}\right) = 1$;
2. υ is an upper semi-continuous function;
3. $\upsilon(\lambda x + (1-\lambda)y) \ge \min\{\upsilon(x), \upsilon(y)\}$,
   $x, y \in R^n$, $\lambda \in [0,1]$;
4. $[\upsilon]^0$ is compact.

Let F_C denote the family of all fuzzy intervals. Therefore, for any $\upsilon \in \mathrm{F}_C$ we have that $[\upsilon]^\alpha \in \mathrm{K}_C$ for all $\alpha \in [0,1]$. The $\alpha -$ levels of a fuzzy interval are given by $[\upsilon]^\alpha = [\underline{\upsilon}_\alpha, \overline{\upsilon}_\alpha]$, $\underline{\upsilon}_\alpha$, $\overline{\upsilon}_\alpha \in R$ for all $\alpha \in [0,1]$.

Theorem 2.2.[27] Assume that $I = [0,1]$ and $\upsilon \in \mathrm{F}_C$, then the endpoint functions $\underline{\upsilon}: I \to R$ and $\overline{\upsilon}: I \to R$ satisty the following conditions:

(1) $\overline{\upsilon}$ is a bounded, decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.

(2) $\underline{\upsilon}$ is a bounded, increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.

(3) $\underline{\upsilon}(1) \le \overline{\upsilon}(1)$.

In the following, we consider the fuzzy intervals $\mu, \upsilon \in \mathrm{F}_C$, $\lambda \in R$,for $x \in R$,

$$(\mu+\upsilon)(x) = \sup_{y+z=x} \min\{\mu(y), \upsilon(z)\},$$

$$(\lambda\mu)(x) = \begin{cases} \mu(\lambda^{-1}x), & \lambda \ne 0 \\ 0 \quad , & \lambda = 0 \end{cases}.$$

Also, we know that for any two fuzzy intervals μ, υ represented by $[\underline{\mu}_\alpha, \overline{\mu}_\alpha]$ and $[\underline{\upsilon}_\alpha, \overline{\upsilon}_\alpha]$, for $\alpha \in [0,1]$. For any real number λ, we have

$$[\mu+\upsilon]^\alpha = \left[(\underline{\mu}+\underline{\upsilon})_\alpha, (\overline{\mu}+\overline{\upsilon})_\alpha\right],$$

$$\begin{aligned}[\lambda\upsilon]^\alpha &= \left[(\lambda\underline{\upsilon})_\alpha, (\lambda\overline{\upsilon})_\alpha\right] \\ &= \left[\min\{\lambda\underline{\upsilon}_\alpha, \lambda\overline{\upsilon}_\alpha\}, \max\{\lambda\underline{\upsilon}_\alpha, \lambda\overline{\upsilon}_\alpha\}\right].\end{aligned}$$

Definition 2.3.[26] A function $F: K \to \mathrm{F}_C$ is said to be a fuzzy function. For each $\alpha \in [0,1]$, we associate with F the family of interval-valued functions $F_\alpha : K \to \mathrm{K}_C$ given by

$$F_\alpha(x) = [F(x)]^\alpha.$$

For any $\alpha \in [0,1]$, we denote

$$F_\alpha(x) = \left[\underline{F}_\alpha(x), \overline{F}_\alpha(x)\right].$$

Remark 2.4. Let F is a fuzzy interval represented by

$$\left[\underline{F}_\alpha(x), \overline{F}_\alpha(x)\right],$$

for all $\alpha \in [0,1]$. For any real number λ, we have

$$F+\lambda = \left[\underline{F}_\alpha(x)+\lambda, \overline{F}_\alpha(x)+\lambda\right].$$

Definition 2.5. [28] (H-difference)We denote the set of all fuzzy number of E. For $m,n \in E$, there exsits $w \in E$ such that $m = n + w$, then it is said that the Hukuhara difference between m and n exists. So w is called H-difference between m and n, and is denoted by $m -_H n$.

Definition 2.6.[29] (gH-difference)For $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, and $A, B \in K_C$, we have

$$A \,!_{gH}\, B = C \Leftrightarrow \begin{cases} (i) & A = B + C \\ (ii) & B = A + (-1)C \end{cases}.$$

And here gH-difference C exists, C is equal to

$$C = \left[\min\left\{ \underline{a} - \underline{b}, \overline{a} - \overline{b} \right\}, \max\left\{ \underline{a} - \underline{b}, \overline{a} - \overline{b} \right\} \right].$$

Definition 2.7. [32，33] Given two fuzzy interval μ, υ, the generalized Hukuhara difference (gH-difference for short) is the fuzzy interval ϖ, if it exists, such that

$$\mu \,!_{gH}\, \upsilon = \varpi \Leftrightarrow \begin{cases} (i) & \mu = \upsilon + \varpi \\ (ii) & \upsilon = \mu + (-1)\varpi \end{cases}.$$

It is easy to show that (i) and (ii) are both valid if and only if ϖ is a crisp number.

Definition 2.8. [34, 35] Given two fuzzy interval μ, υ, we define the distance between μ and υ by

$$\begin{aligned} D(\mu, \upsilon) &= \sup_{\alpha \in [0,1]} H\left([\mu]^\alpha, [\upsilon]^\alpha \right) \\ &= \sup_{\alpha \in [0,1]} \max\left\{ \left| \underline{\mu}_\alpha, \underline{\upsilon}_\alpha \right|, \left| \overline{\mu}_\alpha, \overline{\upsilon}_\alpha \right| \right\} \end{aligned},$$

where H is the Pompeiu-Hausdorff distance defined by

$$H(A, B) = \max\left[\max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right]$$

with $d(a, B) = \min_{b \in B} \|a - b\|$.

It is known that (see [35])

$H(A, B) = \|A \,!_{gH}\, B\|$ where,

for $C \in K_C$, $\|C\| = \max\{ |c|; c \in C \}$; then

$$D(\mu, \upsilon) = \sup\left\{ \left\| [\mu]^\alpha \,!_{gH}\, [\upsilon]^\alpha \right\|; \alpha \in [0,1] \right\}.$$

It is well known that (F_C, D) is a complete metric space.

Definition 2.9.[26] For $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, and $A, B \in K_C$, we say that

(i) $A \preceq_{LU} B$ if and only if $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$,

(ii) $A \prec_{LU} B$ if and only if $A \preceq_{LU} B$ and $A \ne B$,

(iii) $A \prec_{LU} B$ if and only if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

We have that, using gH-difference,

$$A \preceq_{LU} B \Leftrightarrow \left(A \,!_{gH}\, B \right) \preceq_{LU} 0$$

$$A \prec_{LU} B \Leftrightarrow \left(A \,!_{gH}\, B \right) \prec_{LU} 0$$

$$A \prec_{LU} B \Leftrightarrow \left(A \,!_{gH}\, B \right) \prec_{LU} 0$$

$$A = B \Leftrightarrow \left(A \,!_{gH}\, B \right) = 0.$$

Definition 2.10.[26, 36, 37] For $\mu, \upsilon \in F_C$ and given $\alpha \in [0,1]$, we say that

(i) $\mu \preceq_{\alpha-LU} \upsilon$ if and only if $\mu_\alpha \preceq_{LU} \upsilon_\alpha$, that is, $\underline{\mu}_\alpha \le \underline{\upsilon}_\alpha$ and $\overline{\mu}_\alpha \le \overline{\upsilon}_\alpha$,

(ii) If $\mu \prec_{\alpha-LU} \upsilon$ if and only if $\mu_\alpha \prec_{LU} \upsilon_\alpha$,

(iii) If $\mu \prec_{\alpha-LU} \upsilon$ if and only if $\mu_\alpha \prec_{LU} \upsilon_\alpha$.

Correspondingly, the analogous LU-fuzzy orders can be obtained by
(i) $\mu \preceq_{LU} \upsilon$, if $\mu \preceq_{\alpha-LU} \upsilon$ for all $\alpha \in [0,1]$.
(ii) $\mu \precneqq_{LU} \upsilon$, if $\mu \precneqq_{\alpha-LU} \upsilon$ for all $\alpha \in [0,1]$.
(iii) $\mu \prec_{LU} \upsilon$, if $\mu \prec_{\alpha-LU} \upsilon$ for all $\alpha \in [0,1]$.

Remark 2.11. Obviously, we can conclude from the similar literature of $\alpha - LU$ partial order that not all of them satisfy the order relation (maybe some values or close to 0). In the comparison of fuzzy numbers, the comparative relationship cannot be fully reflected. In practical applications, if the value does not satisfy the $\alpha - LU$ partial order, some examples may lead to wrong conclusions. On the other hand, if the LU order relation cannot be compared, but for $\alpha \in [\bar{\alpha},1]$, the $\alpha - LU$ order relation can be compared, then this can help us analyze the problem.

In order to avoid above problem, we give the following concept
Definition 2.12. For $\mu, \upsilon \in \mathrm{F}_C$. For all $\alpha \in [0,1]$,
(i) if either $\mu \preceq_{\alpha-LU} \upsilon$ or $\upsilon \preceq_{\alpha-LU} \mu$;
(ii) if either $\mu \precneqq_{\alpha-LU} \upsilon$ or $\upsilon \precneqq_{\alpha-LU} \mu$;
(iii) if either $\mu \prec_{\alpha-LU} \upsilon$ or $\upsilon \prec_{\alpha-LU} \mu$,
Then we say that μ and υ are comparable, otherwise they are non-comparable.

In the following, we discuss the problem that $\alpha - LU$ order relation is comparable and LU and $\alpha - LU$ order relation are equivalent.

III. GENERALIZED α − CONVEX FUZZY MAPPINGS AND PROPERTIES

Differentiability and gradient are two important concepts in generalized convexity study. In recent years, many scholars have conducted extensive research on it. Different differentiability will bring different research results; this paper mainly studies according to gH-differentiable.

Definition 3.1.[17] Let $K \subset R$ wtih $F: K \to \mathrm{F}_C$ be a fuzzy function and $x^{(0)} \in K$ and $h \in R$ be such that $x^{(0)} + h \in K$. Then generalized Hukuhara derivative (gH- derivative, for short) of F at $x^{(0)}$ is defined as

$$F'(x^{(0)}) = \lim_{h \to 0} \frac{F(x^{(0)} + h) \ominus_{gH} F(x^{(0)})}{h}, \qquad (1)$$

If $F'(x^{(0)}) \in \mathrm{F}_C$ satisfying (1) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at $x^{(0)}$.

Definition 3.2.[18] Let $K \subset R$ wtih $F: K \to \mathrm{F}_C$ a fuzzy function, $x^{(0)} \in K$ and $h \in R$ be such that $x^{(0)} + h \in K$. Given $\alpha \in [0,1]$, the level-wise gH-derivative (LgH- derivative, for short) of the corresponding interval-valued function $F_\alpha : K \to \mathrm{K}_C$ at $x^{(0)}$ is defined as

$$F'_{LgH,\alpha}(x^{(0)}) = \lim_{h \to 0} \frac{F_\alpha(x^{(0)} + h) \ominus_{gH} F_\alpha(x^{(0)})}{h}, \qquad (2)$$

if it exists. If $F'_{LgH,\alpha}(x^{(0)}) \in \mathrm{K}_C$ for all $\alpha \in [0,1]$, we say that F is level-wise generalized differentiable (LgH-differentiable, for short) at $x^{(0)}$ of the family of intervals $\{F'_{LgH,\alpha}(x^{(0)}) : \alpha \in [0,1]\}$ is the LgH-derivative of F at $x^{(0)}$, denoted as $F'_{LgH,\alpha}(x^{(0)})$.

As a consequence of the previous definitions, it is derived that LgH-differentiability, and consequently level-wise continuity, is a necessary condition for gH-differentiability, but it is not sufficient (see [18, 38]).

Theorem 3.3.[26] Let $F : K \to \mathrm{F}_C$ be a fuzzy function. If F is gH-differentiable in the form (1), then F_α is LgH-differentiable in the form (2) for each $\alpha \in [0,1]$. Moreover,

$$F'_{LgH,\alpha}(x) = \left[F'(x)\right]^\alpha . \qquad (3)$$

Proof. Verified by the definition of gH-differentiability.

On the other hand, the existence of the gH-derivative for a fuzzy function does not necessarily imply that the corresponding endpoint functions are differentiable, such as the following example shows.

Example 3.4. Let us consider the fuzzy mapping $F : R \to \mathrm{F}_C$ defined by $F(x) = C \cdot x$, where C is a fuzzy interval defined via its $\alpha-$level sets by $[C]^\alpha = [\alpha, 3+\alpha]$. Then

$$F_\alpha = \begin{cases} \left[\alpha x, (3+\alpha)x\right] & ,if \ x \geq 0 \\ \left[(3+\alpha)x, \alpha x\right] & ,if \ x < 0 \end{cases}.$$

So, we have

$$\lim_{h \to 0^+} \frac{\overline{F}_\alpha(0+h) - \overline{F}_\alpha(0)}{h} = \lim_{h \to 0^+} \frac{(3+\alpha)h - 0}{h} = 3+\alpha,$$

$$\lim_{h \to 0^-} \frac{\overline{F}_\alpha(0+h) - \overline{F}_\alpha(0)}{h} = \lim_{h \to 0^-} \frac{\alpha h - 0}{h} = \alpha ,$$

$$\lim_{h \to 0^+} \frac{\underline{F}_\alpha(0+h) - \underline{F}_\alpha(0)}{h} = \lim_{h \to 0^+} \frac{\alpha h - 0}{h} = \alpha ,$$

$$\lim_{h \to 0^-} \frac{\underline{F}_\alpha(0+h) - \underline{F}_\alpha(0)}{h} = \lim_{h \to 0^-} \frac{(3+\alpha)h - 0}{h} = 3+\alpha.$$

If let $\alpha = 0$, now

$$\lim_{h \to 0^+} \overline{F}_\alpha(0) = 3+\alpha \neq \lim_{h \to 0^-} \overline{F}_\alpha(0) = \alpha ,$$

and

$$\lim_{h \to 0^+} \underline{F}_\alpha(0) = \alpha \neq \lim_{h \to 0^-} \underline{F}_\alpha(0) = 3+\alpha .$$

We can see that the endpoint functions $\underline{F}_\alpha(x)$, $\overline{F}_\alpha(x)$ are not differentabile at $x = 0$.

However F is gH-differentiable and $F'(x) = C$ for all $x \in R$. Then it follows relatively easily that gH-derivative exists and it is $F'(x) = C$ but the endpoint functions $\underline{F}_\alpha(x)$ and $\overline{F}_\alpha(x)$ are not necessarily differentiable.

Theorem 3.5. Let $F : K \to \mathrm{F}_C$ be a fuzzy function. F is LgH-differentiable at $x^{(1)}$ if and only if , for each $\alpha \in [0,1]$, following case hold:

$(\underline{F}_\alpha)'_+(x^{(1)})$, $(\overline{F}_\alpha)'_+(x^{(1)})$, $(\underline{F}_\alpha)'_-(x^{(1)})$ and $(\overline{F}_\alpha)'_-(x^{(1)})$ exsit, and

$$\left[F'(x^{(1)})\right]^\alpha = \left[\min\left\{(\underline{F}_\alpha)'_-(x^{(1)}), (\overline{F}_\alpha)'_-(x^{(1)})\right\}, \max\left\{(\underline{F}_\alpha)'_-(x^{(1)}), (\overline{F}_\alpha)'_-(x^{(1)})\right\}\right]$$

or

$$\left[F'(x^{(1)})\right]^\alpha = \left[\min\left\{(\underline{F}_\alpha)'_+(x^{(1)}), (\overline{F}_\alpha)'_+(x^{(1)})\right\}, \max\left\{(\underline{F}_\alpha)'_+(x^{(1)}), (\overline{F}_\alpha)'_+(x^{(1)})\right\}\right].$$

Obviously, the generalized Hukuhara differentiability of interval-valued function is not fully equivalent to the one-sided differentiability of its endpoint functions.(see [39])

Definition 3.6.[26] Consider an interval-valued function $F : K \to \mathrm{K}_C$, where K is any open subset of R^n. If $d \in R^n$ is any admissible direction at $x^{(0)} \in K$, we say that F has the one-sided directional gH-derivative at $x^{(0)}$ in direction d, if the following right limit exists and is an interval:

$$F'_{gH}(x^{(0)};d)=\lim_{t\to 0^+}\frac{F(x^{(0)}+td)\ominus_{gH} F(x^{(0)})}{t}. \qquad (4)$$

If the left limit for $t\to 0^-$ of the function above exists and the two are equal, we say that F has the two-sided directional gH-derivable in direction d at $x^{(0)}$.

Definition 3.7.[26] Consider an interval-valued function $F:K\to \mathrm{F}_C$, where K is any open subset of R^n. If $d\in R^n$ is any admissible direction at $x^{(0)}\in K$, then given $\alpha\in[0,1]$, the directional level-wise deneralized derivative (directional LgH-derivative, for short) of the corresponding interval-valued function $F_\alpha:K\to \mathrm{K}_C$ at $x^{(0)}$ in the direction d is defined as

$$F'_{LgH,\alpha}(x^{(0)};d)=\lim_{t\to 0^+}\frac{F_\alpha(x^{(0)}+hd)\ominus_{gH} F_\alpha(x^{(0)})}{h} \quad (4)$$

If it exists.

(1) If $F'_{LgH,\alpha}(x^{(0)};d)\in \mathrm{K}_C$ exists for all $\alpha\in[0,1]$, then F is said to have the directional LgH-derivative at $x^{(0)}$ in direction d.

(2) We say that F is directionally (or weak) level-wise generalized differentiable (directionally or weak LgH-differentiable) at $x^{(0)}$ if F admits directional LgH-derivatives at $x^{(0)}$ in any direction $d\in R^n$ and for all $\alpha\in[0,1]$; the family of intervals

$$\left\{F'_{LgH,\alpha}(x^{(0)};d):\alpha\in[0,1]\right\}$$

Is the directional LgH-derivative of F at $x^{(0)}$ in direction d, denoted as $F'_{LgH,\alpha}(x^{(0)};d)$.

(3) We say that F is directionally (weak) gH-differentiable at $x^{(0)}$ if it is directionally (weak) LgH-differentiable at $x^{(0)}$ in any direction d and the directional LgH-derivative $F'_{LgH}(x^{(0)};d)$ defines a fuzzy interval (i.e., the intervals $F'_{LgH,\alpha}(x^{(0)};d)$ define the level-cuts of a fuzzy interval);

(4) F is said directionally (weak) LgH-differentiable on K if it is directionally LgH-differentiable at each point $x^{(0)}\in K$ and is said directionally (weak) gH-differentiable on K if it is directionally gH-differentiable at each point $x^{(0)}\in K$.

Definition 3.7. Let $f:W\to R^n$ be defined on a nonempty open convex set $W\subseteq R^n$. We define the i th partial derivative of f at $x^{(0)}$ as the family, if it exists,

$$\frac{\partial f\left(x^{(0)}\right)}{\partial x_i}=\left\{\frac{\partial f\left(x^{(0)}\right)}{\partial x_i}:i=1,2,3\cdots\right\}.$$

And we define the directional derivative of f at $x^{(0)}$ as follows.

$$f'\left(x^{(0)};d\right)=\lim_{\theta\to 0^+}\frac{f\left(x^{(0)}+\theta d\right)-f\left(x^{(0)}\right)}{\theta},$$

For any $d\in R^n$. $\nabla f\left(x^{(0)};d\right)$ denotes the directional gradient of f at $x^{(0)}$ in the direction d.

Definition 3.8.[26]We define the i th partial LgH-derivative of F at $x^{(0)}$ as the family, if it exists,

$$\frac{\partial_{LgH}F\left(x^{(0)}\right)}{\partial x_i}=\left\{\frac{\partial_{LgH,\alpha}F\left(x^{(0)}\right)}{\partial x_i}:\alpha\in[0,1]\right\}.$$

And we define the LgH-gradient of F at $x^{(0)}$ as follows.

$$\tilde{\nabla}_{LgH}F\left(x^{(0)}\right)=\left(\frac{\partial_{LgH}F\left(x^{(0)}\right)}{\partial x_1},\cdots,\frac{\partial_{LgH}F\left(x^{(0)}\right)}{\partial x_n}\right).$$

In the one-dimensional case, we can state a rule to calculate the directional LgH-derivative via the the LgH-derivative as follows.

Theorem 3.9.[26] Let K be a non-empty open subset of R and $F: K \to \mathrm{F}_C$ be an fuzzy function. If F is LgH-differentiable, then F is directionally LgH-differentiable on K, and

$$F'_{LgH,\alpha}(t_0; d_0) = F'_{LgH,\alpha}(t_0) \cdot d_0,$$

for all $t_0 \in K$, $d_0 \in R$ and $\alpha \in [0,1]$.

Definition 3.10.[26]Let F be LgH-directional differentiable and $\alpha \in [0,1]$.We say that F is $\alpha-$convex at $x \in R^n$ on $W \subseteq R^n$ if

$$F'_{LgH,\alpha}(x; \bar{x} - x) \preceq_{\alpha-LU} F_\alpha(\bar{x}) ! _{gH} F_\alpha(x).$$

for all $\bar{x} \in W$. We say that F is $\alpha-$convex on W if it is $\alpha-$convex at every $x \in R^n$ on W.We say that F is $\alpha-$convex at $x \in R^n$ if it is $\alpha-$convex on R^n. And we say that F is $\alpha-$convex on R^n if it is $\alpha-$convex at every $x \in R^n$.

Definition 3.11. Let W be a nonempty open convex set in R^n, let F be LgH-directional differentiable on W. For all $\alpha \in [0,1]$, we say that F is $\alpha-$quasiconvex if and only if either one of the following equivalent statements holds true:

(i) If $x^{(1)}, x^{(2)} \in W$, $F_\alpha(x^{(1)}) \preceq_{\alpha-LU} F_\alpha(x^{(2)})$,we have $F'_{LgH,\alpha}(x^{(2)}; x^{(1)} - x^{(2)}) \preceq_{\alpha-LU} [0,0]$.

(ii) If $x^{(1)}, x^{(2)} \in W$,and $F'_{LgH,\alpha}(x^{(2)}; x^{(1)} - x^{(2)}) \succ_{\alpha-LU} [0,0]$,we have $F_\alpha(x^{(1)}) \succ_{\alpha-LU} F_\alpha(x^{(2)})$.

for all $x^{(1)}$ and $x^{(2)} \in W$.

Example 3.12. Let W be a nonempty open convex set in R^n and $W = (0, +\infty)$. Let us consider the fuzzy mapping $F: R \to \mathrm{F}_C$ defined by $F(x) = C \cdot x^2$, where C is a fuzzy interval defined via its $\alpha-$level sets by $[C]^\alpha = [\alpha, 3\alpha]$. Then,

$$F_\alpha(x) = [\alpha x^2, 3\alpha x^2].$$

Obviously, F is gH-differentiable. According to Theorem 3.3, F is LgH-differentiable. Moreover, F is a $\alpha-$quasiconvex function. Such as, for $x^{(1)} = 1$, $x^{(2)} = 2$, we have

$$F_\alpha(1) = [\alpha, 3\alpha],$$

$$F_\alpha(2) = [4\alpha, 12\alpha],$$

$$F'_{LgH,\alpha}(2; -1) = [-4\alpha, -12\alpha].$$

Obviously,

$$F_\alpha(1) \preceq_{\alpha-LU} F_\alpha(2)$$

and $[-4\alpha, -12\alpha] \preceq_{\alpha-LU} [0,0]$, for any $\alpha \in [0,1]$, satisfies the Definition 3.11. Therefore, F is a $\alpha-$quasiconvex function.

Definition 3.13. Let W be a nonempty open convex set in R^n, let F be LgH-directional differentiable . For all $\alpha \in [0,1]$, the F is said to be strictly $\alpha-$quasiconvex if for each $x^{(1)}, x^{(2)} \in W$ with $F_\alpha(x^{(1)}) \neq F_\alpha(x^{(2)})$, we have

$$F_\alpha(\theta x^{(1)} + (1-\theta) x^{(2)}) \prec_{\alpha-LU} \max\{F_\alpha(x^{(1)}), F_\alpha(x^{(2)})\}$$

for each $\theta \in (0,1)$.

The function F is called strictly $\alpha-$quasiconvex.

Example 3.14. Let W be a nonempty open convex set in R^n and $W = (0, +\infty)$. Let us consider the fuzzy mapping $F: R \to \mathrm{F}_C$ defined by $F(x) = C \cdot x^2$, where C is a fuzzy interval defined via its $\alpha-$level sets by

$$[C]^\alpha = [\alpha, 1+\alpha].$$

Then,
$F_\alpha(x)=\left[\alpha x^2,(1+\alpha)x^2\right]$.

Obviously, F is gH-differentiable. According to Theorem 3.3., F is LgH-differentiable. For any $x^{(1)}, x^{(2)} \in W$ and $W=(0,+\infty)$, we have

$\theta x^{(1)}+(1-\theta)x^{(2)} \in W$.

Moreover, F is a strictly $\alpha-quasiconvex$ function. Such as, for $x^{(1)}=1$, $x^{(2)}=2$, we have

$F_\alpha(1)=[\alpha,1+\alpha]$,

$F_\alpha(2)=[4\alpha,4(1+\alpha)]$,

$F_\alpha(2-\theta)=\left[\alpha(2-\theta)^2,(1+\alpha)(2-\theta)^2\right]$.

Obviously, $F_\alpha(1)\neq F_\alpha(2)$, for any $\alpha\in[0,1]$,

$\left[\alpha(2-\theta)^2,(1+\alpha)(2-\theta)^2\right]\prec_{\alpha-LU} F_\alpha(2)$

Satisfies the Definition 3.12 for $\alpha\in[0,1]$. Therefore, F is a strictly $\alpha-$ quasiconvex function.

Definition 3.15. Let W be a nonempty open convex set in R^n , let F be LgH-directional differentiable on W . For all $\alpha\in[0,1]$, the F is said to be $\alpha-pseudoconvex$ if for each $x^{(1)}$ and $x^{(2)}\in W$ with

$F'_{LgH,\alpha}\left(x^{(2)};x^{(1)}-x^{(2)}\right)\succeqq_{\alpha-LU}[0,0]$,

we have $F_\alpha\left(x^{(1)}\right)\preceqq_{\alpha-LU} F_\alpha\left(x^{(2)}\right)$, for all $x^{(1)}$ and $x^{(2)}\in W$.

Example 3.16. Consider the following fuzzy mapping

$F(x)=\begin{cases}\tilde{1}x, & x\in[-2,0)\\ \tilde{1}x+1, & x\in(0,2]\end{cases}$,

where $\tilde{1}$ is triangular fuzzy number, namely $\tilde{1}=\langle 0,1,0\rangle$. Then,

$F_\alpha(x)=\begin{cases}[\alpha x,\alpha x], & x\in[-2,0)\\ [\alpha x+1,\alpha x+1], & x\in(0,2]\end{cases}$.

Obviously, F is gH-differentiable. According to Theorem 3.3, F is LgH-differentiable. Therefore,

$F'_{LgH,\alpha}(x)=\begin{cases}[\alpha,\alpha], & x\in[-2,0)\\ [\alpha,\alpha], & x\in(0,2]\end{cases}$.

Such as, for $x^{(1)}=2$, $x^{(2)}=1$, we have

$F'_{LgH,\alpha}(1;1)=[\alpha,\alpha]\succeqq[0,0]$.

Moreover,

$F_\alpha(1)=[\alpha,\alpha]$,

$F_\alpha(2)=[2\alpha+1,2\alpha+1]$,

and

$F_\alpha(1)\preceqq F_\alpha(2), \alpha\in[0,1]$.

Definition 3.17. Let W be a nonempty open convex set in R^n, let F be LgH-directional differentiable . For all $\alpha\in[0,1]$, we say that F is strictly $\alpha-$ pseudoconvex if for each $x^{(1)}$ and $x^{(2)}\in W$ satisfies

$F'_{LgH,\alpha}\left(x^{(2)};x^{(1)}-x^{(2)}\right)\succeqq_{\alpha-LU}[0,0]$,

we have $F_\alpha\left(x^{(1)}\right)\prec_{\alpha-LU} F_\alpha\left(x^{(2)}\right)$, for all $x^{(1)}$ and $x^{(2)}\in W$.

Example 3.18. Consider the following fuzzy mapping

$$F(x)=\begin{cases}\tilde{1}x, & x\in(0,2]\\ -\tilde{1}x, & x\in[-2,0)\end{cases},$$

where $\tilde{1}$ is triangular fuzzy number, namely $\tilde{1}=\langle 0,1,0\rangle$. Then,

$$F_\alpha(x)=\begin{cases}[\alpha x,\alpha x], & x\in(0,2]\\ [-\alpha x,-\alpha x], & x\in[-2,0)\end{cases}.$$

Obviously, F is gH-differentiable. According to Theorem 3.3, F is LgH-differentiable. Therefore,

$$F'_{LgH,\alpha}(x)=\begin{cases}[\alpha,\alpha], & x\in(0,2]\\ [-\alpha,-\alpha], & x\in[-2,0)\end{cases}.$$

Such as, for $x^{(1)}=2$, $x^{(2)}=1$, we have

$$F'_{LgH,\alpha}(1;1)=[\alpha,\alpha]\succeq[0,0].$$

Moreover,

$$F_\alpha(1)=[\alpha,\alpha],$$

$$F_\alpha(2)=[2\alpha,2\alpha],$$

Obviously,

$$F_\alpha(1)\prec F_\alpha(2),\ \alpha\in[0,1].$$

Theorem 3.19. Let W be a nonempty open convex set in R^n, let F be LgH-directional differentiable . For all $\alpha\in[0,1]$, if F is $\alpha-$convex on W, We have F is $\alpha-$pseudoconvex on W.

Proof. Let F be $\alpha-$convex, let $x^{(1)}$, $x^{(2)}\in W$. And W is convex aggregation, we have

$$F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)\preceq_{\alpha-LU} F_\alpha\left(x^{(2)}\right)!_{gH} F_\alpha\left(x^{(1)}\right).$$

If $F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)\succeq_{\alpha-LU}[0,0]$, we have

$$F_\alpha\left(x^{(2)}\right)!_{gH} F_\alpha\left(x^{(1)}\right)\succeq_{\alpha-LU}[0,0],$$

that is,

$$\begin{bmatrix}\min\left\{\underline{F}_\alpha\left(x^{(2)}\right)-\underline{F}_\alpha\left(x^{(1)}\right),\overline{F}_\alpha\left(x^{(2)}\right)-\overline{F}_\alpha\left(x^{(1)}\right)\right\}\\ \max\left\{\underline{F}_\alpha\left(x^{(2)}\right)-\underline{F}_\alpha\left(x^{(1)}\right),\overline{F}_\alpha\left(x^{(2)}\right)-\overline{F}_\alpha\left(x^{(1)}\right)\right\}\end{bmatrix}\succeq_{\alpha-LU}[0,0],$$

i.e.,

$$\begin{cases}\underline{F}_\alpha\left(x^{(2)}\right)-\underline{F}_\alpha\left(x^{(1)}\right)\geq 0\\ \overline{F}_\alpha\left(x^{(2)}\right)-\overline{F}_\alpha\left(x^{(1)}\right)\geq 0\end{cases},$$

what is equivalent to $F_\alpha\left(x^{(2)}\right)\succeq_{\alpha-LU} F_\alpha\left(x^{(1)}\right)$,

And F be $\alpha-$pseudoconvex.

Theorem 3.20. Let W be a nonempty open convex set in R^n, let F be LgH-directional differentiable . For all $\alpha\in[0,1]$, if F is $\alpha-$pseudoconvex, we have F both strictly $\alpha-$quasiconvex.

Proof. We first show that F is strictly $\alpha-$quasiconvex. By contradiction, suppose that there exist $x^{(1)}$ and $x^{(2)}\in W$, such that

$$F_\alpha\left(x^{(1)}\right)\neq F_\alpha\left(x^{(2)}\right)$$

and

$$F_\alpha(x')\succeq_{\alpha-LU}\max\left\{F_\alpha\left(x^{(1)}\right),F_\alpha\left(x^{(2)}\right)\right\},$$

where $x' = \theta x^{(1)} + (1-\theta)x^{(2)}$, for some $\theta \in (0,1)$. Without loss of generality, assume that $F_\alpha\left(x^{(2)}\right) \preceq_{\alpha-LU} F_\alpha\left(x^{(1)}\right)$, so that

$F_\alpha\left(x^{(2)}\right) \prec_{\alpha-LU} F_\alpha\left(x^{(1)}\right) \preceq_{\alpha-LU} F_\alpha\left(x'\right)$ (5)

Next, we consider $F_\alpha(x) = \left[\underline{F}_\alpha(x), \overline{F}_\alpha(x)\right]$. Since W is a convex subset, F is differentiable on the closed segment I of $x^{(1)}$ and $x^{(2)}$. Then $\underline{F}_\alpha(x), \overline{F}_\alpha(x)$ must be continuous on $I \subset W$.

From the property of continuous function on closed region, we can see that $\overline{F}_\alpha(x)$ has the maximum value on I, which must be obtained on I.

So there is $\overline{x} \in I$ and $\theta \in (0,1)$,we have

$\overline{F}_\alpha\left(\overline{x}\right) \geq \overline{F}_\alpha\left(\overline{x} + \theta\left(x^{(1)} - \overline{x}\right)\right)$,

$\overline{F}_\alpha\left(\overline{x}\right) \geq \overline{F}_\alpha\left(\overline{x} + \theta\left(x^{(2)} - \overline{x}\right)\right)$. (6)

From the differentiability of F:

$$\lim_{\theta \to 0} \frac{\overline{F}_\alpha\left(\overline{x} + \theta\left(x^{(1)} - \overline{x}\right)\right) - \overline{F}_\alpha\left(\overline{x}\right)}{\theta} = \overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(1)} - \overline{x}\right),$$

$$\lim_{\theta \to 0} \frac{\overline{F}_\alpha\left(\overline{x} + \theta\left(x^{(2)} - \overline{x}\right)\right) - \overline{F}_\alpha\left(\overline{x}\right)}{\theta} = \overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(2)} - \overline{x}\right).$$

From (5):

$\overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(1)} - \overline{x}\right) \preceq_{\alpha-LU} [0,0]$,

$\overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(2)} - \overline{x}\right) \preceq_{\alpha-LU} [0,0]$. (7)

And $\overline{x} \in I$, suppose $\overline{x} = \overline{\theta}x^{(1)} + \left(1-\overline{\theta}\right)x^{(2)}$, $\overline{\theta} \in (0,1)$, we have $x^{(1)} = \dfrac{\overline{x} - x^{(2)} + \overline{\theta}x^{(2)}}{\overline{\theta}}$.

Substitute (7):

$\overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(2)} - \overline{x}\right) \succeq_{\alpha-LU} [0,0]$

$\overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(2)} - \overline{x}\right) \preceq_{\alpha-LU} [0,0]$.

Therefore,

$\overline{F}'_{LgH,\alpha}\left(\overline{x}; x^{(2)} - \overline{x}\right) =_{\alpha-LU} [0,0]$.

Also known that F is $\alpha-$pseudoconvex function on W, so $\overline{F}_\alpha(x)$ is a $\alpha-$pseudoconvex function, then

$\overline{F}_\alpha\left(x^{(2)}\right) =_{\alpha-LU} \overline{F}_\alpha\left(\overline{x}\right)$.

Similarly,

$\underline{F}_\alpha\left(x^{(2)}\right) =_{\alpha-LU} \underline{F}_\alpha\left(\overline{x}\right)$.

Overall, we have $F_\alpha\left(x^{(2)}\right) =_{\alpha-LU} F_\alpha\left(\overline{x}\right)$.

Since $F_\alpha\left(x^{(2)}\right) \prec_{\alpha-LU} F_\alpha\left(x^{(1)}\right)$,

So $F_\alpha\left(\overline{x}\right) \prec_{\alpha-LU} F_\alpha\left(x^{(1)}\right)$.

IV. SUFFICIENT AND NECESSARY OPTIMALITY CONDITIONS OF THE MIXED CONSTRAINED OPTIMIZATION PROBLEM

In this section, we establish some sufficient and necessary Karush-Kuhn-Tucker conditions for a $x^{(1)} \in W$ to be a feasible solution of constrained optimization problem (MFP). In the following, we consider the problem (MFP):

Min $F(x) = \left[\underline{F}(x), \overline{F}(x)\right]$

$s.t.\ \ g_j(x) \leq 0,\ \ j = 1, \cdots, m;$

$h_k(x)=0,\ \ k=1,\cdots,p$;
$x\in W$
where $F:W\to \mathrm{F}_C$, $g_j(x)$ and $h_k(x)$ is a differentiable real-valued function. $W\subseteq R^n$ is an open convex set. We denote the feasible solution set for (MFP) as D :
$D=\left\{x\in W: g_j(x)\le 0, h_k(x)=0\right\}$
we define
$I(x)=\left\{j\in\{1,\cdots,m\}: g_j(x)=0\right\}$,
$\tilde{I}(x)=\left\{j\in\{1,\cdots,m\}: g_j(x)\ne 0\right\}$,
$\hat{I}(x)=\left\{k\in\{1,\cdots,p\}: h_k(x)=0\right\}$.

Definition 4.1.[26] Given $x^{(1)}\in W$ and $\alpha\in[0,1]$, we say that $x^{(1)}$ is a weak α -LU-minimum point of F if there exists no $x^{(2)}\in W$ such that $F\left(x^{(2)}\right)\prec_{\alpha-LU} F\left(x^{(1)}\right)$. Correspondingly, we say that $x^{(1)}$ is a weak LU-minimum of F if there exists no $x^{(2)}\in W$ such that $F\left(x^{(2)}\right)\prec_{LU} F\left(x^{(1)}\right)$; and $x^{(1)}$ is a weak global LU-minimum of F if $x^{(1)}$ is a weak α -LU-minimum of F , for all α .

Definition 4.2.[26] Given $x^{(1)}\in W$ and $\alpha\in[0,1]$, we say that $x^{(1)}$ is a α -LU-minimum point of F if there exists no $x^{(2)}\in W$ such that
$F\left(x^{(2)}\right)\preceq_{\alpha-LU} F\left(x^{(1)}\right)$.
Correspondingly, we say that $x^{(1)}$ is a LU- minimum of F if there exists no $x^{(2)}\in W$ such that $F\left(x^{(2)}\right)\preceq_{LU} F\left(x^{(1)}\right)$; and $x^{(1)}$ is a global LU-minimum of F if $x^{(1)}$ is a α -LU-minimum of F , for all α .

Theorem 4.3. (Sufficient $\alpha-optimality$ condition) Let F be a directional LgH- differentiable fuzzy function. Let $x^{(1)}\in W$ and for all $\alpha\in[0,1]$, if there does not exist $d\in R^n$ such that
$F'_{LgH,\alpha}\left(x^{(1)};d\right)\not\prec_{\alpha-LU}[0,0]$ (8)
and F is strictly $\alpha-$ quasiconvex. Then $x^{(1)}$ is a weak α -LU-solution of F .

Proof. Suppose there exists $x^{(2)}\in W$ such that $F_\alpha\left(x^{(2)}\right)\prec_{\alpha-LU} F_\alpha\left(x^{(1)}\right)$. By the convexity of W , $\theta x^{(2)}+(1-\theta)x^{(1)}\in W$ for each $\theta\in(0,1)$.
But because F is strictly $\alpha-$ quasiconvex, we have
$F_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right)\prec_{\alpha-LU}\max\left\{F_\alpha\left(x^{(1)}\right),F_\alpha\left(x^{(2)}\right)\right\}$,
where $\max\left\{F_\alpha\left(x^{(1)}\right),F_\alpha\left(x^{(2)}\right)\right\}=F_\alpha\left(x^{(1)}\right)$.
It follows that,
$F_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right)!_{\ gH} F_\alpha\left(x^{(1)}\right)\prec_{\alpha-LU}[0,0]$.
According definition 3.6, we have

$$(i)F'_{LgH,\alpha}\left(x^{(1)};d\right)=\lim_{\theta\to 0^+}\frac{F_\alpha\left(x^{(1)}+\theta\left(x^{(2)}-x^{(1)}\right)\right)!_{\ gH} F_\alpha\left(x^{(1)}\right)}{\theta}\prec_{\alpha-LU}[0,0] \text{ or}$$

$$(ii)F'_{LgH,\alpha}\left(x^{(1)};d\right)=\lim_{\theta\to 0^+}\frac{F_\alpha\left(x^{(1)}+\theta\left(x^{(2)}-x^{(1)}\right)\right)!_{\ gH} F_\alpha\left(x^{(1)}\right)}{\theta}\preceq_{\alpha-LU}[0,0] \text{ or}$$

$$(iii)F'_{LgH,\alpha}\left(x^{(1)};d\right)=\lim_{\theta\to 0^+}\frac{F_\alpha\left(x^{(1)}+\theta\left(x^{(2)}-x^{(1)}\right)\right)!_{\ gH} F_\alpha\left(x^{(1)}\right)}{\theta}\underline{\preceq}_{\alpha-LU}[0,0],$$

i.e.,

$F'_{LgH,\alpha}\left(x^{(1)};d\right)\not\prec_{\alpha-LU}[0,0]$.

With $d=x^{(2)}-x^{(1)}$, what is a contradiction to the hypothesis that there exists no d such that $F'_{LgH,\alpha}\left(x^{(1)};d\right)\prec_{\alpha-LU}[0,0]$.

Definition 4.4. Let $x^{(1)}$ be a feasible solution of (MFP) and $\alpha\in[0,1]$, then

(i) $x^{(1)}$ is said to be a weak $\alpha-LU-solution$ of (MFP) if there no $x^{(2)}\in D$ such that $F_{\alpha}\left(x^{(2)}\right)\prec_{\alpha-LU}F_{\alpha}\left(x^{(1)}\right)$.

(ii) $x^{(1)}$ is said to be a $\alpha-LU-solution$ of (MFP) if there no $x^{(2)}\in D$ such that $F_{\alpha}\left(x^{(2)}\right)\prec_{\alpha-LU}F_{\alpha}\left(x^{(1)}\right)$.

Lemma 4.5. Let F is LgH-partial differentiable. If $x^{(1)}$ is a weak α-LU-solution of (MFP), $g_j(x)$ is continuous at $x^{(1)}$ for $j\in\tilde{I}(x^{(1)})$. For all $\alpha\in[0,1]$. Then the system

$$F'_{LgH,\alpha}\left(x^{(1)};x-x^{(1)}\right)\prec_{\alpha-LU}[0,0], \qquad (9)$$

$$\nabla g_j\left(x^{(1)};x-x^{(1)}\right)<0, \qquad (10)$$

$$\nabla h_k\left(x^{(1)};x-x^{(1)}\right)<0, \qquad (11)$$

has no solution $x\in W$, where $W\subseteq R^n$ is an open convex set.

Proof. Assume there exist $\tilde{x}$ such that the inequalities (9), (10) and (11) are ture, i.e.,

$$F'_{LgH,\alpha}\left(x^{(1)};\tilde{x}-x^{(1)}\right)\prec_{\alpha-LU}[0,0],$$

$$\nabla g_j\left(x^{(1)};\tilde{x}-x^{(1)}\right)<0,$$

$$\nabla h_k\left(x^{(1)};\tilde{x}-x^{(1)}\right)<0.$$

Let

$$\psi_F(x^{(1)},\tilde{x},q)=F_{\alpha}(x^{(1)}+q(\tilde{x}-x^{(1)}))\ominus_{gH}F_{\alpha}(x^{(1)}).$$

We observe that this function vanishes at $q=0$,

i.e.,

$$\psi_F(x^{(1)},\tilde{x},0)=[0,0].$$

Given $\alpha\in[0,1]$, by definition 3.7, the right differential of $\psi_F(x^{(1)},\tilde{x},q)$ with respect to q is

$$\lim_{q\to0^+}\frac{\psi_F(x^{(1)},\tilde{x},q)\ominus_{gH}\psi_F(x^{(1)},\tilde{x},0)}{q}$$

$$=\lim_{q\to0^+}\frac{F_{\alpha}(x^{(1)}+q(\tilde{x}-x^{(1)}))\ominus_{gH}F_{\alpha}(x^{(1)})}{q}$$

$$=F'_{LgH,\alpha}\left(x^{(1)};\tilde{x}-x^{(1)}\right)\prec_{\alpha-LU}[0,0]$$

Therefore, $\psi_F(x^{(1)},\tilde{x},q)\prec_{\alpha-\mathrm{LU}}[0,0]$ if q is in some open interval $(0,\delta_1)$.

i.e.,

$$F_{\alpha}(x^{(1)}+q(\tilde{x}-x^{(1)}))\ominus_{gH}F_{\alpha}(x^{(1)})\prec_{\alpha-LU}[0,0].$$

That is

$$\left[\begin{array}{l}\min\left\{\begin{array}{l}\underline{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\underline{F}_\alpha\left(x^{(1)}\right),\\ \bar{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\bar{F}_\alpha\left(x^{(1)}\right)\end{array}\right\},\\ \max\left\{\begin{array}{l}\underline{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\underline{F}_\alpha\left(x^{(1)}\right),\\ \bar{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\bar{F}_\alpha\left(x^{(1)}\right)\end{array}\right\}\end{array}\right],$$

i.e.,

$$\begin{cases}\underline{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\underline{F}_\alpha\left(x^{(1)}\right)<0\\ \bar{F}_\alpha\left(x^{(1)}+q(\tilde{x}-x^{(1)})\right)-\bar{F}_\alpha\left(x^{(1)}\right)<0\end{cases}.$$

It follows that

$F_\alpha(x^{(1)}+q(\tilde{x}-x^{(1)}))\prec_{\alpha-LU} F_\alpha(x^{(1)})$, $q\in(0,\delta^1)$.

Similarly, by defining

$\psi_{g_{I\left(x^{(1)}\right)}}(x^{(1)},\tilde{x},q)$

$=g_{I\left(x^{(1)}\right)}(x^{(1)}+q(\tilde{x}-x^{(1)}))-g_{I\left(x^{(1)}\right)}(x^{(1)}),\ j\in I\left(x^{(1)}\right).$

When $q=0$, we have $\psi_{g_{I\left(x^{(1)}\right)}}(x^{(1)},\tilde{x},0)=0$.

The right differential of $\psi_{g_{I\left(x^{(1)}\right)}}(x^{(1)},\tilde{x},q)$ with respect to q is

$$\lim_{q\to0^+}\frac{\psi_{g_{I\left(x^{(1)}\right)}}(x^{(1)},\tilde{x},q)-\psi_{g_{I\left(x^{(1)}\right)}}(x^{(1)},\tilde{x},0)}{q}$$

$$=\lim_{q\to0^+}\frac{g_{I\left(x^{(1)}\right)}(x^{(1)}+q(\tilde{x}-x^{(1)}))-g_{I\left(x^{(1)}\right)}(x^{(1)})}{q}$$

$$=\nabla g_j\left(x^{(1)};\tilde{x}-x^{(1)}\right)<0.$$

we can prove that

$g_{I\left(x^{(1)}\right)}(x^{(1)}+q(\tilde{x}-x^{(1)}))<g_{I\left(x^{(1)}\right)}(x^{(1)}),q\in(0,\delta^2)\cdot$

By definition of $I\left(x^{(1)}\right)$ and $j\in I\left(x^{(1)}\right)$, we have $g_{I\left(x^{(1)}\right)}(x^{(1)})=0$. We obtain

$g_{I\left(x^{(1)}\right)}(x^{(1)}+q(\tilde{x}-x^{(1)}))<0,q\in(0,\delta^2)$.

Since g_j is continuous at $x^{(1)}$ for $j\in\tilde{I}(x^{(1)})$, Therefore , there exists δ^3 such that

$g_j(x^{(1)}+q(\tilde{x}-x^{(1)}))<0,q\in(0,\delta^3)$.

Similarly, by defining

$\psi_h(x^{(1)},\tilde{x},q)$

$=h_k(x^{(1)}+q(\tilde{x}-x^1))-h_k(x^{(1)}),k\in\hat{I}\left(x^{(1)}\right)$

When $q=0$, we have $\psi_h(x^{(1)},\tilde{x},0)=0$.

The right differential of $\psi_h(x^{(1)},\tilde{x},q)$ with respect to q is

$$\lim_{q\to0^+}\frac{\psi_h(x^{(1)},\tilde{x},q)-\psi_h(x^{(1)},\tilde{x},0)}{q}$$

$$=\lim_{q\to0^+}\frac{h_k(x^{(1)}+q(\tilde{x}-x^1))-h_k(x^{(1)})}{q}$$

$$=\nabla h_k\left(x^{(1)};\tilde{x}-x^{(1)}\right)<0.$$

we can prove that

$h_k(x^{(1)}+q(\tilde{x}-x^{(1)}))<h_k(x^{(1)}), q\in(0,\delta^4)$.

Let $\bar{\delta}=\min\left\{\delta^1,\delta^2,\delta^3,\delta^4\right\}$.

Then , $x^{(1)}+q(\tilde{x}-x^{(1)})\in N_{\bar{\delta}}(x^{(1)}), q\in(0,\bar{\delta})$,

where $N_{\bar{\delta}}(x^{(1)})$ is a neighborhood of $x^{(1)}$.

Now

$$F_\alpha(x^{(1)}+q(\tilde{x}-x^{(1)}))\prec_{\alpha-LU} F_\alpha(x^{(1)}) \qquad (12)$$

$$g_{I(x^{(1)})}(x^{(1)}+q(\tilde{x}-x^{(1)}))<g_{I(x^{(1)})}(x^{(1)}) \qquad (13)$$

$$g_j(x^{(1)}+q(\tilde{x}-x^{(1)}))<g_j(x^{(1)}) \qquad (14)$$

$$h_k(x^{(1)}+q(\tilde{x}-x^{(1)}))<h_k(x^{(1)}) \qquad (15)$$

By (12-15), we get

$x^{(1)}+q(\tilde{x}-x^{(1)})\in N_{\bar{\delta}}(x^{(1)})\bigcap D,\ \ q\in(0,\bar{\delta})$.

Hence (12) is contradiction to the assumption that $x^{(1)}$ is a weak α -LU-solution of (MFP). Thus, there exists no $x\in W$ satifying the system.

Theorem 4.6. (KKT necessary condition)

Let W be a nonempty open convex set of R^n . For all $\alpha\in[0,1]$, and we can find a point $x^{(2)}\in W$ such that $F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)=_{\alpha-LU}[0,0]$.

(i) Assume F is LgH-partial differentiable and LgH-directional differentiable at $x^{(1)}$ such that

$$F'_{LgH,\alpha}\left(x^{(1)};d\right)=\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial x_i}(x^{(1)})\cdot d_i ,$$

for any $d\in R^n$.

(ii) Suppose $x^{(1)}$ be a weak α -LU-solution of (MFP) and assume that g_j is continuous at $x^{(1)}$ for $j\in\tilde{I}(x^{(1)})$.

(iii) g_j and h_k is directionally differentiable at $x^{(1)}$ such that

$$\nabla g_j\left(x^{(1)};d\right)=\sum_{i=1}^{m}\frac{\partial g}{\partial x_i}(x^{(1)})\cdot d_i ,$$

$$\nabla h_k\left(x^{(1)};d\right)=\sum_{i=1}^{m}\frac{\partial h}{\partial x_i}(x^{(1)})\cdot d_i .$$

Then, there exist $\lambda\in R$, $\mu=(\mu_1,\cdots,\mu_m)\in R^m$ and $\upsilon=(\upsilon_1,\cdots,\upsilon_m)\in R^m$ such that

$$[0,0]\in\lambda\tilde{\nabla}_{LgH}F_\alpha(x^{(1)})+\sum_{j\in\tilde{I}(x^1)}\mu_j\nabla g_j(x^{(1)})+\sum_{k=1}^{p}\upsilon_k\nabla h_k(x^{(1)}) \qquad (16)$$

$$\mu_j g_j(x^{(1)})=0 , \qquad (17)$$

$$(\lambda,\mu,\upsilon)\geq 0 . \qquad (18)$$

Proof. Since $x^{(1)}$ be a weak α -LU-solution of (MFP). By the lemma 4.5 , there exist no $x\in W$ statisfying

$F'_{LgH,\alpha}\left(x^{(1)};x-x^{(1)}\right)\prec_{\alpha-LU}[0,0]$,

$\nabla g_j\left(x^{(1)};x-x^{(1)}\right)<0$,

$\nabla h_k\left(x^{(1)};x-x^{(1)}\right)<0$.

Then we take $x^{(2)}$, there exists $x^{(2)}-x^{(1)}$ such that

$$F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)=\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)}) \ . \qquad (19)$$

And according to conditions, the opposite of

$F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)\prec_{\alpha-LU}[0,0]$

has and only has the following case :

$F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)\succeq_{\alpha-LU}[0,0]$, (20)

$\nabla g_j\left(x^{(1)};x^{(2)}-x^{(1)}\right)\geq 0$, (21)

$\nabla h_k\left(x^{(1)};x^{(2)}-x^{(1)}\right)\geq 0$. (22)

We multiply inequalities (20-22) by the multipliers λ, μ_j *and* υ_k, respectively, obtaining

$\lambda F'_{LgH,\alpha}\left(x^{(1)};x^{(2)}-x^{(1)}\right)\succeq_{\alpha-LU}\lambda[0,0]=[0,0]$, (23)

$\mu_j\nabla g_j\left(x^{(1)};x^{(2)}-x^{(1)}\right)\geq 0$, (24)

$\upsilon_k\nabla h_k\left(x^{(1)};x^{(2)}-x^{(1)}\right)\geq 0$. (25)

According to the conditions, we have

$$\lambda\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})\succeq_{\alpha-LU}[0,0], \qquad (26)$$

$$\mu_j\sum_{j\in\tilde{I}(x^{(1)})}\frac{\partial g}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})\geq 0, \qquad (27)$$

$$\upsilon_k\sum_{j\in\hat{I}(x^{(1)})}\frac{\partial h}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})\geq 0. \qquad (28)$$

Now, combining interval inequalities in (26-28), we get

$$\lambda\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})+\mu_j\sum_{i=1}^{m}\frac{\partial g}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})+\upsilon_k\sum_{i=1}^{m}\frac{\partial h}{\partial x_i}(x^{(1)})\cdot(x^{(2)}-x^{(1)})\succeq_{\alpha-LU}[0,0]$$

From this inequality, we have

$$[0,0]\in\lambda\tilde{\nabla}_{LgH}F_\alpha(x^{(1)})+\sum_{j\in\tilde{I}(x^{(1)})}\mu_j\nabla g_j(x^{(1)})+\sum_{k=1}^{p}\upsilon_k\nabla h_k(x^{(1)})$$

The proof is completed.

Theorem 4.7. (KKT sufficient condition)

Let W be a nonempty open convex set of R^n. For all $\alpha\in[0,1]$,

F is LgH-partial differentiable and LgH-directional differentiable at $x^{(1)}$ such that

$$F'_{LgH,\alpha}\left(x^{(1)};d\right)=\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial x_i}(x^{(1)})\cdot d_i,$$

for any $d\in R^n$. g_j and h_k is directionally differentiable at $x^{(1)}$ and there exist $\lambda\in R$, $\mu=(\mu_1,\cdots,\mu_m)\in R^m$ and $\upsilon=(\upsilon_1,\cdots,\upsilon_m)\in R^m$ such that

$$[0,0]\in\lambda\tilde{\nabla}_{LgH}F_\alpha(x^{(1)})+\sum_{j\in\tilde{I}(x^{(1)})}\mu_j\nabla g_j(x^{(1)})+\sum_{k=1}^{p}\upsilon_k\nabla h_k(x^{(1)}) \qquad (29)$$

$\mu_j g_j(x^{(1)})=0$. (30)

$(\lambda,\mu,\upsilon)\geq 0$. (31)

If F is strictly $\alpha-$quasiconvex at $x^{(1)}$, g_j is convex at $x^{(1)}$ in D, for $j\in\tilde{I}(x^{(1)})$, h_k is convex at $x^{(1)}$ in D, for $k=1,\cdots,p$, then $x^{(1)}$ be a α-LU-solution of (MFP).

Proof. Let us suppose the contrary. So, there exists $x^{(2)}\in D$ such that

$F_\alpha(x^{(2)})\prec_{\alpha-LU}F_\alpha(x^{(1)})$.

By the convexity of W, we have

$\theta x^{(2)}+(1-\theta)x^{(1)}\in W,\theta\in(0,1)$

By the definition of strictly $\alpha-$quasiconvex of F, it follows that

$F_\alpha(\theta x^{(2)}+(1-\theta)x^{(1)})\prec_{\alpha-LU}\max\left\{F_\alpha(x^{(1)}),F_\alpha(x^{(2)})\right\}$,

where $\max\left\{F_\alpha(x^{(1)}),F_\alpha(x^{(2)})\right\}=F_\alpha(x^{(1)})$.

Then we have

$$F_\alpha(\theta x^{(2)}+(1-\theta)x^{(1)}) \prec_{\alpha-LU} F_\alpha(x^{(1)}),$$

that is,

$$\begin{cases} \underline{F}_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right) < \underline{F}_\alpha\left(x^{(1)}\right) \\ \bar{F}_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right) < \bar{F}_\alpha\left(x^{(1)}\right) \end{cases},$$

what implies that,

$$\begin{cases} \underline{F}_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right) - \underline{F}_\alpha\left(x^{(1)}\right) < 0 \\ \bar{F}_\alpha\left(\theta x^{(2)}+(1-\theta)x^{(1)}\right) - \bar{F}_\alpha\left(x^{(1)}\right) < 0 \end{cases},$$

Given $\alpha \in [0,1]$, According definition 3.6,

$$\bar{F}'_{LgH,\alpha}\left(x^{(1)}; x^{(2)}-x^{(1)}\right) = \lim_{\theta\to 0^+} \frac{\bar{F}_\alpha(x^{(1)}+\theta(x^{(2)}-x^{(1)})) - \bar{F}_\alpha(x^{(1)})}{\theta} \le 0,$$

$$\underline{F}'_{LgH,\alpha}\left(x^{(1)}; x^{(2)}-x^{(1)}\right) = \lim_{\theta\to 0^+} \frac{\underline{F}_\alpha(x^{(1)}+\theta(x^{(2)}-x^{(1)})) - \underline{F}_\alpha(x^{(1)})}{\theta} \le 0,$$

what is equivalent to

$$F'_{LgH,\alpha}\left(x^{(1)}; x^{(2)}-x^{(1)}\right) = \lim_{\theta\to 0^+} \frac{F_\alpha(x^{(1)}+\theta(x^{(2)}-x^{(1)})) - F_\alpha(x^{(1)})}{\theta} \preceqq_{\alpha-LU} [0,0].$$

Since F satifies condition

$$F'_{LgH,\alpha}\left(x^{(1)}; d\right) = \sum_{i=1}^{n} \frac{\partial_{LgH} F_\alpha}{\partial x_i}(x^{(1)}) \cdot d_i,$$

we have

$$F'_{LgH,\alpha}\left(x^{(1)}; x^{(2)}-x^{(1)}\right) = \sum_{i=1}^{n} \frac{\partial_{LgH} F_\alpha}{\partial x_i}(x^{(1)}) \cdot (x^{(2)}-x^{(1)}) \preceqq_{\alpha-LU} [0,0]. \quad (32)$$

By differentiable and convexity of g_j and h_k, for $j \in \tilde{I}(x)$ and $k = 1,\cdots,p$, we have that

$$\nabla g_j\left(x^{(1)}; x^{(2)}-x^{(1)}\right) \le g_j(x^{(2)}) - g_j(x^{(1)}) = g_j(x^{(2)}) < 0. \quad (33)$$

And

$$\nabla h_k\left(x^{(1)}; x^{(2)}-x^{(1)}\right) \le h_k(x^{(2)}) - h_k(x^{(1)}) = 0. \quad (34)$$

By hypothesis there exist $\lambda \in R$ and $(\lambda,\mu,\upsilon) \ge 0$ such that the conditions (29-31) are satisfied. We multiply inequalities (32-34) by the multipliers λ, μ_j and υ_k, respectively, obtaining

$$\lambda \sum_{i=1}^{n} \frac{\partial_{LgH} F_\alpha}{\partial x_i}(x^{(1)}) \cdot (x^{(2)}-x^{(1)}) \preceqq_{\alpha-LU} [0,0]. \quad (35)$$

$$\mu_j \nabla g_j\left(x^{(1)}; x^{(2)}-x^{(1)}\right) < 0,\ j \in \tilde{I}\left(x^{(1)}\right). \quad (36)$$

and

$$\upsilon_k \nabla h_k\left(x^{(1)}; x^{(2)}-x^{(1)}\right) \le 0, k = 1,\cdots,p. \quad (37)$$

Now, combining interval inequalities in (35-37), we get

$$\lambda \sum_{i=1}^{n} \frac{\partial_{LgH} F_\alpha}{\partial x_i}(x^{(1)}) \cdot (x^{(2)}-x^{(1)}) + \mu_j \nabla g_j\left(x^{(1)}; x^{(2)}-x^{(1)}\right) + \upsilon_k \nabla h_k\left(x^{(1)}; x^{(2)}-x^{(1)}\right) \prec_{\alpha-LU} [0,0]. \quad (38)$$

The interval inequality (38) implies that

$$[0,0] \notin \lambda \tilde{\nabla}_{LgH} F_\alpha(x^{(1)}) + \sum_{j\in\tilde{I}(x^{(1)})} \mu_j \nabla g_j(x^{(1)}) + \sum_{k=1}^{p} \upsilon_k \nabla h_k(x^{(1)}),$$

what is contradiction to (29), and proof is completed.

Example 4.8. Consider the following fuzzy optimization problem:

$$\min \quad F(x) = \left[\underline{F}(x), \bar{F}(x)\right]$$

$s.t.\ x_1 + x_2 - 4 \le 0$
$x_2 = 1$
$x_1 \ge 0$
$x \in R^2$

Where F is fuzzy function via its $\alpha - cut$ as follows:
$F_\alpha(x) = \left[\underline{F}_\alpha(x), \overline{F}_\alpha(x)\right] = [-1, \alpha x_1 - 1], \alpha \in [0,1]$.

Let us verify the hypothesis of Theorem 4.6. It is easy to get that
$(\underline{F}_\alpha)'_+\left(x^{(1)}\right), (\overline{F}_\alpha)'_+\left(x^{(1)}\right), (\underline{F}_\alpha)'_-\left(x^{(1)}\right), (\overline{F}_\alpha)'_-\left(x^{(1)}\right)$, do exist. Moreover,

$$\left[F'\left(x^{(1)}\right)\right]^\alpha = \left[\begin{array}{l}\min\left\{(\underline{F}_\alpha)'_-\left(x^{(1)}\right), (\overline{F}_\alpha)'_-\left(x^{(1)}\right)\right\},\\ \max\left\{(\underline{F}_\alpha)'_-\left(x^{(1)}\right), (\overline{F}_\alpha)'_-\left(x^{(1)}\right)\right\}\end{array}\right].$$

Then, by Theorem 3.5, we have that F is LgH-partial differentiable. Furthermore, and given $\alpha \in [0,1]$, we find

$$\frac{\partial_{LgH} F_\alpha(x)}{\partial x_1} = [0,\ \alpha],$$

$$\frac{\partial_{LgH} F_\alpha(x)}{\partial x_2} = [0,\ 0].$$

Now, given $d = (d_1, d_2) \in R^2$, and by simple calculus, we get that there exists the LgH-derivative at x in the direction d, and it is

$$\begin{aligned} F'_{LgH,\alpha}(x;d) &= \lim_{h\to 0^+} \frac{F_\alpha(x_1 + hd_1, x_2 + hd_2) \ominus_{gH} F_\alpha(x_1, x_2)}{h} \\ &= [0, \alpha d_1] \\ &= \frac{\partial_{LgH} F_\alpha(x)}{\partial x_1}\cdot d_1 + \frac{\partial_{LgH} F_\alpha(x)}{\partial x_2}\cdot d_2 \end{aligned}$$

Therefore, $F'_{LgH,\alpha}(x;d) = \sum_{i=1}^{2} \frac{\partial_{LgH} F_\alpha(x)}{\partial x_i}\cdot d_i$.

On the other hand, the real-valued functions defined as $g_1(x) = x_1 + x_2 - 4$, $g_2(x) = -x_1$ is differentiable.

If we choose $\bar{\alpha} = 0.5$, and by calculus, we obtain that $\bar{x} = (0,1)$ is the unique feasible point such that condition(16)-(18) are fulfilled, although there exsit several different values for the multipliers λ, μ_j and υ_k, $j = 1,2,\ k = 1$. For instance, $\bar{x} = (0,1), \lambda = 0, \mu_1 = 1,\ \mu_2 = 1,\ \upsilon_1 = -1$.

V. DUALITY PROBLEM FOR GENERALIZED α − CONVEX FUZZY MAPPINGS

Now, we consider the dual problem (DMFP) of (MFP)

Max $F(b) = \left[\underline{F}(b), \overline{F}(b)\right]$

s.t.

$$[0,0] \in \bar{\lambda}\tilde{\nabla}_{LgH} F_\alpha(b) + \sum_{j\in \bar{I}(b)} \bar{\mu}_j \nabla g_j(b) + \sum_{k=1}^{p} \bar{\upsilon}_k \nabla h_k(b) \qquad (39)$$

$$\bar{\mu}_j g_j(b) = 0, \qquad (40)$$

$$(\bar{\lambda}, \bar{\mu}, \bar{\upsilon}) \ge 0. \qquad (41)$$

We denote the feasible set of the dual problem (DMFP) by
$\bar{D} = \left\{\left(b, \bar{\lambda}, \bar{\mu}_j, \bar{\upsilon}_k\right) \in R^n \times R^n \times R^m \times R^p\right\}$.

It satifies

$[0,0] \in \bar{\lambda}\tilde{\nabla}_{LgH} F_\alpha(b) + \sum_{j\in \bar{I}(b)} \bar{\mu}_j \nabla g_j(b) + \sum_{k=1}^{p} \bar{\upsilon}_k \nabla h_k(b)$, $\bar{\mu}_j g_j(b) = 0$, $\bar{\lambda} \ne 0$ and $(\bar{\lambda}, \bar{\mu}, \bar{\upsilon}) \ge 0$. According to Theorem 4.7, we have the following weakly duality.

Theorem 5.1.(Weakly duality) Let $x^{(1)}$ be MFP-feasible, $\left(b,\bar{\lambda},\bar{\mu}_j,\bar{\upsilon}_k\right)$ be DMFP-feasible. Let W be a nonempty open convex set of R^n. For all $\alpha\in[0,1]$, $x^{(1)}$ and b, we have $F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)\neq_{\alpha-LU}[0,0]$.

(i) Assume that F is LgH-partial differentiable and LgH-directional differentiable at b such that

$$F'_{LgH,\alpha}\left(b;d\right)=\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial b_i}(b)\cdot d_i \text{ , for any } d\in R^n . \qquad (42)$$

(ii) Assume that g_j and h_k is directionally differentiable at b such that

$$\nabla g_j\left(b;d\right)=\sum_{i=1}^{m}\frac{\partial g}{\partial b_i}(b)\cdot d_i ,\ \ j\in\tilde{I}\left(b\right)$$

$$\nabla h_k\left(b;d\right)=\sum_{i=1}^{m}\frac{\partial h}{\partial b_i}(b)\cdot d_i ,\ \ k=1,\cdots,p.$$

(iii) If F is strictly $\alpha-$quasiconvex, g_j and h_k at b satisfy the following conditions

$$\nabla g_j\left(b;x^{(1)}-b\right)\leq g_j(x^{(1)})-g_j(b) \qquad (43)$$

$$\nabla h_k\left(b;x^{(1)}-b\right)\leq h_k(x^{(1)})-h_k(b) \qquad (44)$$

for all feasible solution $x^{(1)}$.

Then, $F_\alpha\left(x^{(1)}\right)\not\prec_{\alpha-LU} F_\alpha\left(b\right)$.

Proof. We proceed by contradiction. The opposite of this inequality

$$F_\alpha\left(x^{(1)}\right)\not\prec_{\alpha-LU} F_\alpha\left(b\right)$$

has the following three cases :

$$F_\alpha\left(x^{(1)}\right)\prec_{\alpha-LU} F_\alpha\left(b\right),$$

$$F_\alpha\left(x^{(1)}\right)\preceq_{\alpha-LU} F_\alpha\left(b\right),$$

$$F_\alpha\left(x^{(1)}\right)\preceqq_{\alpha-LU} F_\alpha\left(b\right).$$

According to the above inequalities, we have

$$\max\left\{F_\alpha(x^{(1)}),F_\alpha(b)\right\}=F_\alpha(b) .$$

Since F is strictly $\alpha-$quasiconvex, by definition 3.10, we have

$$F_\alpha(\theta x^{(1)}+(1-\theta)b)\prec_{\alpha-LU}\max\left\{F_\alpha(x^{(1)}),F_\alpha(b)\right\},\ \theta x^{(1)}+(1-\theta)b\in W,\theta\in(0,1) .$$

It implies that

$$F_\alpha(\theta x^{(1)}+(1-\theta)b)\, !_{\ gH}\, F_\alpha(b)\prec_{\alpha-LU}[0,0]$$

According definition 3.6,

$$(i)F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)$$

$$=\lim_{\theta\to0^+}\frac{F_\alpha(b+\theta(x^{(1)}-b))\, !_{\ gH}\, F_\alpha(b)}{\theta}\prec_{\alpha-LU}[0,0] ,$$

$$(ii)F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)$$

$$=\lim_{\theta\to0^+}\frac{F_\alpha(b+\theta(x^{(1)}-b))\, !_{\ gH}\, F_\alpha(b)}{\theta}\preceq_{\alpha-LU}[0,0] ,$$

According to the condition in the inequality :

$$F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)\neq_{\alpha-LU}[0,0],$$

we know that

$$F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)\preceqq_{\alpha-LU}[0,0]$$

do not exist.

According to (i) :

$$F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)\prec_{\alpha-LU}[0,0].$$

By assumption (i), it follows that

$$F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)$$
$$=\sum_{i=1}^{n}\frac{\partial_{Lg}F_\alpha}{\partial b_i}(b)\cdot(x^{(1)}-b)\prec_{\alpha-LU}[0,0].$$

Multiplying the inequality by $\bar{\lambda}>0$, we have

$$\bar{\lambda}F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)=\bar{\lambda}\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial b_i}(b)\cdot(x^{(1)}-b)\prec_{\alpha-LU}[0,0]. \quad (45)$$

From (iii),

$$\nabla g_j\left(b;x^{(1)}-b\right)\le g_j(x^{(1)})-g_j(b),$$
$$\nabla h_k\left(b;x^{(1)}-b\right)\le h_k(x^{(1)})-h_k(b).$$

Multiplying the above two inequalities by $\bar{\mu}_j\ge 0$ and $\bar{\upsilon}_k\ge 0$, respectively, we have

$$\bar{\mu}_j\nabla g_j\left(b;x^{(1)}-b\right)\le\bar{\mu}_j g_j(x^{(1)})-\bar{\mu}_j g_j(b). \quad (46)$$
$$\bar{\upsilon}_k\nabla h_k\left(b;x^{(1)}-b\right)\le\bar{\upsilon}_k h_k(x^{(1)})-\bar{\upsilon}_k h_k(b). \quad (47)$$

Now, from feasility of $x^{(1)}$ for (MFP) we have $\mu_j g_j(x^{(1)})\le 0$ and $\bar{\mu}_j g_j(b)=0$, respectively. Since $\left(b,\bar{\lambda},\bar{\mu}_j,\bar{\upsilon}_k\right)$ for (DMFP), $\bar{\upsilon}_k h_k(x^{(1)})=0$ and $\bar{\upsilon}_k h_k(b)=0$, respectively. Hence, by (46) and (47), we obtain

$$\bar{\mu}_j\nabla g_j\left(b;x^{(1)}-b\right)\le 0,$$
$$\bar{\upsilon}_k\nabla h_k\left(b;x^{(1)}-b\right)\le 0.$$

From (ii), we have

$$\bar{\mu}_j\sum_{i=1}^{m}\frac{\partial g}{\partial b_i}(b)\cdot\left(x^{(1)}-b\right)\le 0. \quad (48)$$
$$\bar{\upsilon}_k\sum_{i=1}^{m}\frac{\partial h}{\partial b_i}(b)\cdot\left(x^{(1)}-b\right)\le 0. \quad (49)$$

Now, combining interval inequalities in (45), (48) and (49), we get

$$\bar{\lambda}\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial b_i}(b)\cdot(x^{(1)}-b)+\bar{\mu}_j\sum_{i=1}^{m}\frac{\partial g}{\partial b_i}(b)\cdot\left(x^{(1)}-b\right)+\bar{\upsilon}_k\sum_{i=1}^{m}\frac{\partial h}{\partial b_i}(b)\cdot\left(x^{(1)}-b\right)\prec[0,0]$$

equivalent to

$$\bar{\lambda}\tilde{\nabla}_{LgH}F_\alpha(b)+\sum_{j\in\tilde{I}(b)}\bar{\mu}_j\nabla g_j(b)+\sum_{k=1}^{p}\bar{\upsilon}_k\nabla h_k(b)\prec[0,0] \quad (50)$$

The interval inequality (50) implies that

$$[0,0]\notin\bar{\lambda}\tilde{\nabla}_{LgH}F_\alpha(b)+\sum_{j\in\tilde{I}(b)}\bar{\mu}_j\nabla g_j(b)+\sum_{k=1}^{p}\bar{\upsilon}_k\nabla h_k(b),$$

which is a contradiction to the dual constraint

$$[0,0]\in\bar{\lambda}\tilde{\nabla}_{LgH}F_\alpha(b)+\sum_{j\in\tilde{I}(b)}\bar{\mu}_j\nabla g_j(b)+\sum_{k=1}^{p}\bar{\upsilon}_k\nabla h_k(b).$$

Similarly, According to (ii):

$$F'_{LgH,\alpha}\left(b;x^{(1)}-b\right)\preceq_{\alpha-LU}[0,0],$$

we have

$$\bar{\lambda}\tilde{\nabla}_{LgH}F_\alpha(b)+\sum_{j\in\tilde{I}(b)}\bar{\mu}_j\nabla g_j(b)+\sum_{k=1}^{p}\bar{\upsilon}_k\nabla h_k(b)\prec[0,0].$$

i.e.,

$$[0,0]\notin\bar{\lambda}\tilde{\nabla}_{LgH}F_\alpha(b)+\sum_{j\in\tilde{I}(b)}\bar{\mu}_j\nabla g_j(b)+\sum_{k=1}^{p}\bar{\upsilon}_k\nabla h_k(b).$$

The proof is completed.

Theorem 5.2. (Strong duality) Let $x^{(1)}$ be a weak α-LU-solution for (MFP) which the constraint qualification is satisfied.
(i) Let F is LgH-partial differentiable and LgH-directional differentiable at $x^{(1)}$.
(ii) g_j and h_k is directionally differentiable at $x^{(1)}$, and g_j is continuous for $j \in \bar{I}(x^{(1)})$.
Then there exsits $\bar{\lambda} \in R^n$, $\bar{\mu}, \bar{\upsilon} \in R_+^m$ such that $\left(x^{(1)}, \bar{\lambda}, \bar{\mu}_j, \bar{\upsilon}_k\right)$ is feasible for (DMFP). Moreover, if weak duality (Theorem 5.1) between (MFP) and (DMFP) holds then $\left(x^{(1)}, \bar{\lambda}, \bar{\mu}_j, \bar{\upsilon}_k\right)$ is a weak maximum for (DMFP).

Proof. Since $x^{(1)}$ satisfies all the conditions of theorem 4.7, there exists $\bar{\lambda} \in R^n$, $\bar{\mu}, \bar{\upsilon} \in R_+^m$ such that

$$[0,0] \in \bar{\lambda} \tilde{\nabla}_{LgH} F_\alpha(x^{(1)}) + \sum_{j \in \tilde{I}(x^{(1)})} \bar{\mu}_j \nabla g_j(x^{(1)}) + \sum_{k=1}^{p} \bar{\upsilon}_k \nabla h_k(x^{(1)})$$
$$\bar{\mu}_j g_j(x^{(1)}) = 0,$$
$$(\lambda, \bar{\mu}, \bar{\upsilon}) \geq 0.$$

It implies that $\left(x^{(1)}, \bar{\lambda}, \bar{\mu}_j, \bar{\upsilon}_k\right)$ is feasible for (DMFP). Also, by weak duality (Theorem 5.1), it follows that $\left(x^{(1)}, \bar{\lambda}, \bar{\mu}_j, \bar{\upsilon}_k\right)$ is optimal for (DMFP).

Theorem 5.3. (Converse duality) Let $\left(\bar{b}, \bar{\lambda}, \bar{\mu}, \bar{\upsilon}\right)$ be a weak α-LU-maxinmum for (DMFP). Let W be a nonempty open convex set of R^n. For all $\alpha \in [0,1]$, $\bar{b}$ and $\tilde{b}$, we have $F'_{LgH,\alpha}\left(\bar{b}; \tilde{b} - \bar{b}\right) \neq_{\alpha-LU} [0,0]$.
Moreover,
(i) Assume that F is LgH-partial differentiable and LgH-directional differentiable at $\bar{b}$ such that

$$F'_{LgH,\alpha}\left(\bar{b}; d\right) = \sum_{i=1}^{n} \frac{\partial_{LgH} F_\alpha}{\partial \bar{b}_i}(\bar{b}) \cdot d_i, \text{ for any } d \in R^n. \qquad (42)$$

(ii) Assume that g_j and h_k is directionally differentiable at $\bar{b}$ such that

$$\nabla g_j\left(\bar{b}; d\right) = \sum_{i=1}^{m} \frac{\partial g}{\partial \bar{b}_i}(\bar{b}) \cdot d_i, \quad j \in \tilde{I}\left(\bar{b}\right)$$
$$\nabla h_k\left(\bar{b}; d\right) = \sum_{i=1}^{m} \frac{\partial h}{\partial \bar{b}_i}(\bar{b}) \cdot d_i, \quad k = 1, \cdots, p.$$

(iii) If F is strictly $\alpha-$quasiconvex, g_j and h_k at $\bar{b}$ satisfy the following conditions

$$\nabla g_j\left(\bar{b}; x^{(1)} - \bar{b}\right) \leq g_j(x^{(1)}) - g_j(\bar{b}) \qquad (43)$$
$$\nabla h_k\left(\bar{b}; x^{(1)} - \bar{b}\right) \leq h_k(x^{(1)}) - h_k(\bar{b}) \qquad (44)$$

for all feasible $x^{(1)}$.
Then, $\bar{b}$ is optimal in (MFP).

Proof. We proceed by contradiction. Suppose that $\bar{b}$ is not optimal for (MFP), that is, there exists $\tilde{b} \in D$ such that

$$F_\alpha\left(\tilde{b}\right) \prec_{\alpha-LU} F_\alpha\left(\bar{b}\right),$$
$$F_\alpha\left(\tilde{b}\right) \preceq_{\alpha-LU} F_\alpha\left(\bar{b}\right).$$

It impies that $F_\alpha\left(\tilde{b}\right) \prec_{\alpha-LU} F_\alpha\left(\bar{b}\right)$.
Since F is strictly $\alpha-$quasiconvex, it follows the inequality

$$F_\alpha(\theta\tilde{b} + (1-\theta)\bar{b}) \prec_{LU} \max\left\{F_\alpha(\tilde{b}), F_\alpha(\bar{b})\right\},$$
$$\theta\tilde{b} + (1-\theta)\bar{b} \in W, \theta \in (0,1), \tilde{b}, \bar{b} \in W$$

By $F_\alpha\left(\tilde{b}\right) \prec_{\alpha-LU} F_\alpha\left(\bar{b}\right)$, we have

$$F_\alpha(\theta\tilde{b} + (1-\theta)\bar{b}) \prec_{\alpha-LU} F_\alpha(\bar{b})$$

i.e.,

$$F_\alpha(\theta\tilde{b}+(1-\theta)\bar{b})\ominus_{gH} F_\alpha(\bar{b})\prec_{\alpha-LU}[0,0].$$

Given $\alpha\in[0,1]$, by the definition 3.6 and assumption (i),we have

$$(i)F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)$$

$$=\lim_{\theta\to 0^+}\frac{F_\alpha(\theta\tilde{b}+(1-\theta)\bar{b})\ominus_{gH} F_\alpha(\bar{b})}{\theta}\prec_{\alpha-LU}[0,0],$$

$$(ii)F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)$$

$$=\lim_{\theta\to 0^+}\frac{F_\alpha(\theta\tilde{b}+(1-\theta)\bar{b})\ominus_{gH} F_\alpha(\bar{b})}{\theta}\preceq_{\alpha-LU}[0,0],$$

According to the condition in the inequality :

$$F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)\neq_{\alpha-LU}[0,0],$$

we know that

$$F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)\underline{\preceq}_{\alpha-LU}[0,0]$$

do not exist.

According to (i) :

$$F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)\prec_{\alpha-LU}[0,0].$$

Multiplying the inequality by $\bar{\lambda}$, we have

$$\bar{\lambda}F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)=\bar{\lambda}\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial\bar{b}_i}(\bar{b})\cdot(\tilde{b}-\bar{b})\prec_{\alpha-LU}[0,0].\qquad(51)$$

Similarly, from (iii), the inequality

$$\bar{\mu}_j\triangledown g_j\left(\bar{b};\tilde{b}-\bar{b}\right)\le\bar{\mu}_j g_j(\tilde{b})-\bar{\mu}_j g_j(\bar{b}).\qquad(52)$$

$$\bar{\upsilon}_k\triangledown h_k\left(\bar{b};\tilde{b}-\bar{b}\right)\le\bar{\upsilon}_k h_k(\tilde{b})-\bar{\upsilon}_k h_k(\bar{b}).\qquad(53)$$

Now, from of feasibility of $\tilde{b}$ for (MFP) and $\left(\bar{b},\bar{\lambda},\bar{\mu},\bar{\upsilon}\right)$ for (DMFP), we have

$$\mu_j g_j(\tilde{b})\le 0 \text{ and } \bar{\mu}_j g_j(\bar{b})=0,$$

$$\bar{\upsilon}_k h_k(\tilde{b})=0 \text{ and } \bar{\upsilon}_k h_k(\bar{b})=0,$$

respectively. Hence, by (52), (53) and assumption (ii), we get

$$\bar{\mu}_j\sum_{i=1}^{m}\frac{\partial g}{\partial\bar{b}_i}(\bar{b})\cdot\left(\tilde{b}-\bar{b}\right)\le 0.\qquad(54)$$

$$\bar{\upsilon}_k\sum_{i=1}^{m}\frac{\partial h}{\partial\bar{b}_i}(\bar{b})\cdot\left(\tilde{b}-\bar{b}\right)\le 0.\qquad(55)$$

Now, adding (51), (54) and (55), we obtain

$$\bar{\lambda}\sum_{i=1}^{n}\frac{\partial_{LgH}F_\alpha}{\partial\bar{b}_i}(\bar{b})\cdot(\tilde{b}-\bar{b})+\bar{\mu}_j\sum_{i=1}^{m}\frac{\partial g}{\partial\bar{b}_i}(\bar{b})\cdot\left(\tilde{b}-\bar{b}\right)+\bar{\upsilon}_k\sum_{i=1}^{m}\frac{\partial h}{\partial\bar{b}_i}(\bar{b})\cdot\left(\tilde{b}-\bar{b}\right)<0.$$

equivalent to

$$\bar{\lambda}\tilde{\triangledown}_{LgH}F_\alpha(\bar{b})+\sum_{j\in\tilde{I}(\bar{b})}\bar{\mu}_j\triangledown g_j(\bar{b})+\sum_{k=1}^{p}\bar{\upsilon}_k\triangledown h_k(\bar{b})\prec[0,0].\qquad(50)$$

The interval inequality (50) implies that

$$[0,0]\notin\bar{\lambda}\tilde{\triangledown}_{LgH}F_\alpha(\bar{b})+\sum_{j\in\tilde{I}(\bar{b})}\bar{\mu}_j\triangledown g_j(\bar{b})+\sum_{k=1}^{p}\bar{\upsilon}_k\triangledown h_k(\bar{b})$$

which is a contradiction to the dual constraint

$$[0,0]\in\bar{\lambda}\tilde{\triangledown}_{LgH}F_\alpha(\bar{b})+\sum_{j\in\tilde{I}(\bar{b})}\bar{\mu}_j\triangledown g_j(\bar{b})+\sum_{k=1}^{p}\bar{\upsilon}_k\triangledown h_k(\bar{b}).$$

Similarly, According to (ii) :

$$F'_{LgH,\alpha}\left(\bar{b};\tilde{b}-\bar{b}\right)\preceq_{\alpha-LU}[0,0],$$

we have

$$\bar{\lambda}\tilde{\nabla}_{LgH} F_\alpha(\bar{b}) + \sum_{j\in \tilde{I}(\bar{b})} \bar{\mu}_j \nabla g_j(\bar{b}) + \sum_{k=1}^{p} \bar{\upsilon}_k \nabla h_k(\bar{b}) \prec [0,0]$$

i.e.,

$$[0,0] \notin \bar{\lambda}\tilde{\nabla}_{LgH} F_\alpha(\bar{b}) + \sum_{j\in \tilde{I}(\bar{b})} \bar{\mu}_j \nabla g_j(\bar{b}) + \sum_{k=1}^{p} \bar{\upsilon}_k \nabla h_k(\bar{b}).$$

The proof is completed.

Example 5.4. Consider the following fuzzy optimization problem:

$$max \quad F(b) = \left[\underline{F}(b), \bar{F}(b)\right]$$

$$s.t. \;\; 0 \in \bar{\lambda}\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}\right] + \bar{\mu}_1\begin{pmatrix}1\\1\end{pmatrix} + \bar{\mu}_2\begin{pmatrix}-1\\0\end{pmatrix} + \bar{\upsilon}_1\begin{pmatrix}0\\1\end{pmatrix}$$

$$\bar{\mu}_1\begin{pmatrix}1\\1\end{pmatrix} + \bar{\mu}_2\begin{pmatrix}-1\\0\end{pmatrix} = 0 \,,$$

$$(\bar{\lambda}, \bar{\mu}_1, \bar{\mu}_2, \bar{\upsilon}_1) \geq 0, \quad \alpha \in [0,1],$$

where F is fuzzy function via its $\alpha - cut$ as follows:

$$F_\alpha(b) = \left[\underline{F}_\alpha(b), \bar{F}_\alpha(b)\right] = [0, \alpha b_1 - 1]$$

If we choose $\bar{\alpha} = 0.5$, and by calculus, we obtain that $(0,1,1,-1)$ is the unique feasible point.

VI. CONCLUSION

Convexity and generalized convexity play an important role in optimization theory. The study of generalized convexity is one of the important directions in optimization problems. With the emergence of fuzzy optimization problems, fuzzy generalized convexity has attracted more attention. More and more scholars have studied fuzzy optimization problems, and fuzzy generalized convexity has also been widely studied.

In this paper, we first define quasiconvex, strictly quasiconvex, pseudoconvex, strictly pseudoconvex based on the concept of convexity given in [26]. Then, some relations and properties between them are discussed. Finally, the KKT condition of fuzzy optimization problem and its weak duality, strong duality theory are given, and an example is given to illustrate. These results are useful for solving practical problems. In addition, in life, there are various optimization problems. In the next step, we can reduce the convexity of the constraint conditions for research and discussion. These studies may bring more novel results.

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