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A Sensitivity of the White–Noise Distribution on the Performance of the Bootstrap Method in Uncovering the True Sampling Distribution of Autoregressive Processes

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Abstract **Original Research Article**

Bootstrap is a resampling method of estimating parameters or sampling distributions based on observed data. In order to apply the bootstrap approach when evaluating the parameters of time-series models, we need to consider the lack of independence between the observations. This study addresses the sensitivity of the white-noise distribution to the performance of the bootstrap method in uncovering the true sampling distribution of parameter estimates of autoregressive models. In order to study the performance, we use three white–noise (normal, exponential, and uniform) distributions for three (first and higher-order) models.

Keywords: Bootstrap method, white-noise distributions, percentile method, autoregressive models

1. INTRODUCTION

Efron (1979) introduced the bootstrap as a resampling method and its Monte-Carlo approximate form (Chernick & LaBudde, 2014). He mainly addressed the bootstrap to evaluate the sampling distribution of parameter estimates when the probability distribution of the given data is unknown (Efron & Gong, 1983; Lahiri, 2003; Zoubir & Iskander, 2004). His method is attractive due to the following points: **1)** May consistently estimate the distribution of a statistic under weaker conditions than the traditional approach (derivation of the asymptotic distribution) does, **2)** Makes the derivation of the asymptotic distribution superfluous, **3)** Applicable in cases where the asymptotic distribution is unattractive, **4)** Has some optimal properties (Efron, 1982; Shao, & Tu, 1995; Hall, 2013). In addition, bootstrapping has many applications and has been used to solve problems that would be too difficult to deal with analytically (Efron & Tibshirani, 1994).

This study addresses the sensitivity of the white-noise distribution to the performance of the bootstrap method in uncovering the true sampling distribution of parameter estimates of autoregressive models. In order to study the performance, we simulate data from three (first and higher-order) models with three white-noise (normal, exponential, and uniform) distributions by using the R language and then estimate the true distribution of the simulated data, which is used to assess the distribution of bootstrap estimates and their standard errors. Furthermore, one of the models has coefficients that are very close to the unit circle. The consistency of each bootstrap estimate is also considered, i.e., the effect of sample size on the bootstrap estimates and their standard errors. As a result, we chose two samples (50 and 500) to ensure the consistency of the bootstrap estimates. The percentile method based on the bootstrap distribution is invoked in order to construct $(1-\alpha)100\%$ confidence intervals for the model parameters and compare them to the classical confidence intervals based on the large sample theory (approximately normal). In addition, the normality (Shapiro-Wilk) test is also carried out for parameter estimates of models. Finally, the EasyFit statistical package is used in order to test the goodness of fit (Kolmogorov-Smirnov) of residuals for a normal distribution.

2. METHODOLOGY

2.1 The Bootstrap Method

In the past decades, many broad new approaches to statistical methods have emerged in the form of non- parametric approaches to modelling uncertainty. Thus, not only the individual parameters from the probability distribution but also the entire distributions are searched, based on the available empirical data (Efron & Gong, 1983; Zoubir & Iskander, 2004). The bootstrap method was introduced by Efron (1979), with further developments in Efron (1981), and fully described in Efron and Tibshirani (1994). Generally, the bootstrap technique could be divided into two branches. The parametric bootstrap is the branch of Monte-Carlo simulations that involves resampling from a fully defined probability distribution (Robert *et al*., 1999; Chernick & LaBudde, 2014). The non-parametric bootstrap branch is defined as sampling with a replacement technique from a completely unspecified distribution F. The bootstrap method could be described as follows:

Suppose $X = (X_1, X_2, \dots, X_n)$ is a collection of n numbers drawn at random from a completely unspecified distribution F (X_i's are independent and identically distributed (iid) random variables, each having a distribution F). Let θ denotes an unknown parameter of interest of F, and $\hat{\theta}$ is an estimator of θ. Hence, we are interested in finding the sampling distribution of $\hat{\theta}$. In order to estimate the sampling distribution of $\hat{\theta}\,(X, F)$, we invoke the bootstrap technique, as explained in the following procedure:

1. Construct the empirical probability distribution function $\hat{F}$ by putting mass (weight) n^{-1} at each point of the sample X.
2. Obtain the bootstrap sample $Z = \{Z_1, Z_2, \dots, Z_n\}$ (the same size as the original data) from F̂ with replacement.
3. Approximating the sampling distribution of $\hat{\theta}\ (X, F)$ by the bootstrap distribution of $\hat{\theta}\ (X, \hat{F})$.

In applying the bootstrap, the most difficult part is the actual calculation of the bootstrap distribution. Efron (1979) proposed the following three methods:

Method 1: The theoretical calculation of the bootstrap distribution, which is not easy to obtain in most cases,

Method 2: The Monte-Carlo approximation to the bootstrap distribution,

Method 3: Taylor's series expansion or delta method, which turns out to be the same as the usual Jackknife theory.

However, due to the difficulty in getting and applying theoretical results, we adopt Method 2 in order to obtain the sampling distribution of parameter estimates of autoregressive models. The Monte-Carlo procedure is as follows:

1. Draw a sufficient number (B) of independent bootstrap samples, say $Z_1, Z_2, \dots, Z_n$, where each Z_i represents a sample from $\hat{F}$.
2. The statistic of interest $\hat{\theta}$, is calculated for each bootstrap sample. This bootstrap sample of B estimates is used to obtain the Monte-Carlo estimate of the bootstrap distribution (Robert *et al.*, 1999). For instance, the bootstrap standard error is estimated by $\hat{\sigma}_B = [\sum_{k=1}^{B}(\hat{\theta}_k - \bar{\theta})^2/(B-1)\,]^{1/2}$, where $\bar{\theta} = \sum_{k=1}^{B}\hat{\theta}_k/B$ is the sample mean of the B estimates.

It is obvious that as $B \to \infty$, $\hat{\sigma}_B$, it will almost surely converge to the true bootstrap estimate and the true bootstrap estimated standard error, respectively. Note that the previous Monte-Carlo algorithm is valid only if the bootstrap sample size is equal to the size of the actual data; otherwise, it will not converge to the above limits (Efron, 1982).

2.2 Bootstrapping Autoregressive Models

The applications of bootstrap methods to time-series data are not as straight-forward as in the case of random samples due to the lack of independence between observations. Therefore, in bootstrapping, the assumption of independence among the observations is crucial, and the bootstrap could give incorrect answers if dependence is neglected (Singh, 1981).

The application of bootstrapping in the case of $ARMA$ models with independent and identically distributed errors has been studied by Freedman (1981), Stoffer and Wall (1991), Efron and Tibshirani (1994), and Lahiri (2003).

In this paper, we adopt the method by Efron and Tibshirani (1986) for bootstrapping autoregressive models, which consists of the following steps:

Step 1: Simulate n observations from an autoregressive (AR) model of order p.

Step 2: Fit the AR model in the same order as the realisation of size n simulated according to Step 1.

Step 3: Calculate the residuals for $t = p+1, p+2, \dots, n$ by $\hat{\varepsilon} = X_t - \sum_{i=1}^{p}\hat{a}_i\, X_{t-1}$.

Step 4: Obtain a bootstrap sample $Z1, Z2, \dots, Zn$ of size n by setting $Z1, Z2, \dots, Zp$ to the first p observations in the data, and for $t = p+1, p+2, \dots, n$, we have $Zt = \sum_{i=1}^{p}\hat{a}_i\, Z_{t-1} + \delta$, where δ is randomly chosen from the residuals $\hat{\varepsilon}$.

Step 5: Fit the AR model for the same order in Step 2 to the bootstrap sample and record the values of the estimates.

Step 6: Repeat Steps 4 and 5 a sufficient number of times, say $B = 250$ in order to obtain the bootstrap sampling distribution of each estimate (Efron & Tibshirani, 1994; Kotz & Johnson, 2012; Young & Smith, 2005).

2.3 The Percentile Method

Standard errors are frequently used to give approximate confidence intervals based on normal theory. Thus, an approximate $(1-\alpha)100\%$ confidence interval is

$$\hat{\theta} \pm (standard\ error\ of\ \hat{\theta}).Z\alpha, \qquad (1)$$

Where $Z\alpha$ is the 100α percentile of the standard normal distribution.

Efron and Tibshirani (1986) discussed using the bootstrap to give better approximate confidence intervals than Formula (1). We used one of their proposed methods of obtaining confidence intervals, which is called the percentile method. This consists merely of using appropriate percentiles of the estimated bootstrap distribution to give an evaluated confidence interval. Simply, we ascend the values of the bootstrap estimate and take the $100.\alpha th$ ordered value as the lower interval point and the $100\ (1 - \alpha)th$ ordered value as the upper interval point. Finally, the percentile method takes the following form $(\hat{\theta}_{lo}, \hat{\theta}_{up}) = (\hat{\theta}_{\alpha}, \hat{\theta}_{1-\alpha})$. For further details, see, e.g., Efron and Tibshirani (1994); DiCiccio and Efron (1996).

3. THE SIMULATION STUDY

3.1 Using a normal distribution of white noise

In this section, we apply the bootstrap method to three stationary autoregressive processes through simulation using the R language. However, due to the tremendous difficulty in obtaining the actual (theoretical) bootstrap distribution, we have to appeal to the Monte-Carlo approximation of the bootstrap distribution (Efron & Tibshirani, 1986). Our study consists merely of the following steps:

Step 1: Simulate n observations from the AR scheme with normal errors using the R language. In particular, the following three models have been used in the simulation process:

$X_t = 0.2\, X_{t-1} + \varepsilon_t$ …………………….. (2)
$X_t = 0.9\, X_{t-1} + \varepsilon_t$ …………………….. (3)
$X_t = 0.4\, X_{t-1} - 0.7\, X_{t-2} + \varepsilon_t$ ………… (4)

Where ε_t is a normal white–noise process with a mean equal to 0 and a variance equal to 1.

Step 2: A model with the same order is fitted to the generated data obtained in Step 1.

Step 3: Steps 1 and 2 are repeated 250 times in order to obtain the true sampling distribution of each parameter estimate of Models (2), (3), and (4).

Step 4: The refitted model in Step 2 is bootstrapped as explained in Subsection 2.2 to obtain a bootstrap sample with the same size as the generated data used in Step 1.

Step5: Step 4 is done for 250 replications in order to obtain the bootstrap sampling distribution of each parameter estimate of Models (2), (3), and (4).

Step 6: Finally, the true and bootstrap sampling distributions of each estimate are compared to see if the bootstrap distributions resemble the true one.

3.2 Using a non-normal distribution of white noise

We repeat all Steps in Subsection 3.1, but using non-normal (exponential, uniform) white-noise processes. Hence, the white–noise process has an exponential distribution with a parameter ($\lambda = 1$), and a uniform distribution with parameters ($a = 0, b = 1$).

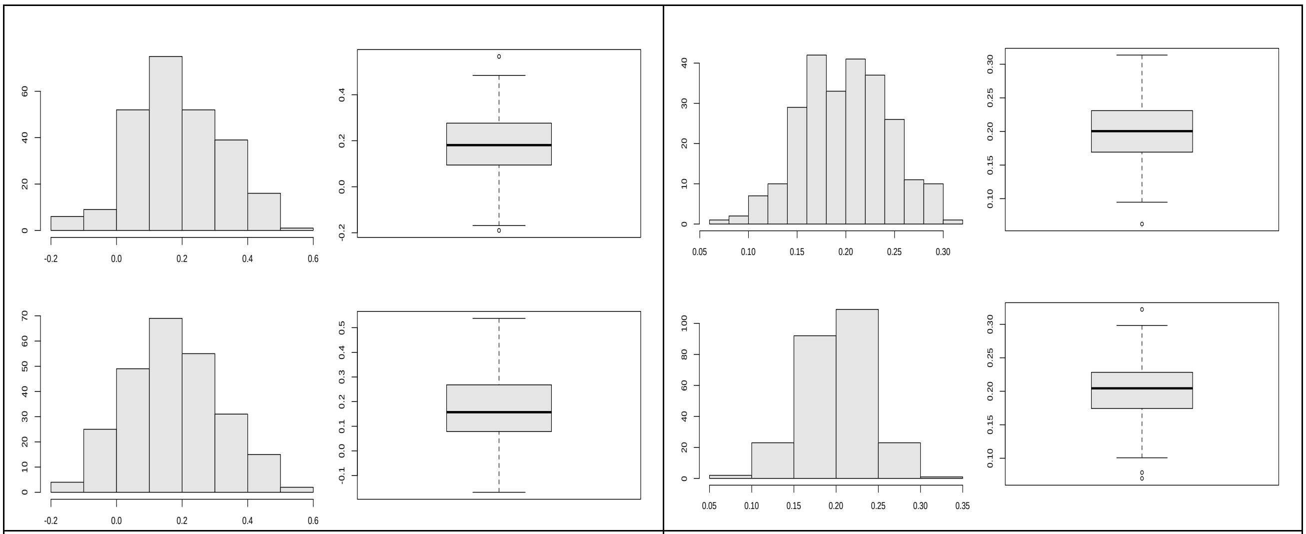
4. RESULTS

In this section, Due to the similarity in the results for Models (2, 3, and 4) and the lack of space, we give only the results for Model (2). The other results can be obtained from the authors. The results of Model (2) are arranged in Table 1, for the sake of comparison. summary statistics, including mean value, standard error, skewness, kurtosis, and variance of residuals, for white–noise (normal, exponential, and uniform) distributions.

Table 1: Summary Statistics of Parameter Estimates of Model (2)

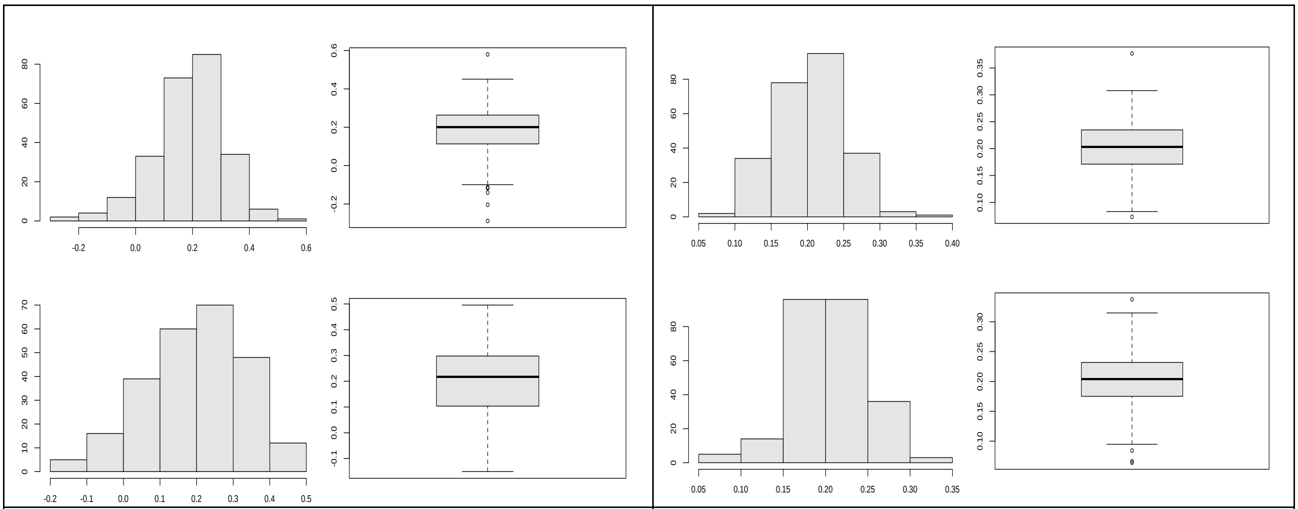
			Sample Size	Mean Value	Standard Error	Variance of Residuals	Skewness	Kurtosis	-95%	+95%
Normal White Noise	True	$\hat{a}$	50	0.1819	0.1376	0.9820	0.0157	2.9239	0.1509	0.2271
	Boot.	$\hat{a}^*$		0.1712	0.1425	0.9995	0.1776	2.5336	-0.0893	0.4454
	True	$\hat{a}$	500	0.1996	0.0438	0.9990	-0.0248	2.8057	0.1957	0.2034
	Boot.	$\hat{a}^*$		0.2007	0.0410	0.9376	-0.2428	3.2331	0.1145	0.2746
Exponential White Noise	True	$\hat{a}$	50	0.1860	0.1382	1.9847	-0.4745	3.9323	0.1477	0.2243
	Boot.	$\hat{a}^*$		0.2002	0.1334	0.8663	-0.2799	2.5952	-0.0997	0.4214
	True	$\hat{a}$	500	0.2025	0.0437	1.9903	0.0753	3.0723	0.1986	0.2063
	Boot.	$\hat{a}^*$		0.2045	0.0436	2.4435	-0.0464	3.5681	0.1169	0.2873
Uniform White Noise	True	$\hat{a}$	50	0.1755	0.1376	0.3317	0.0517	2.9205	0.1374	0.2137
	Boot.	$\hat{a}^*$		0.1885	0.1348	o.5406	-0.0240	2.7844	-0.0765	0.4454
	True	$\hat{a}$	500	0.1968	0.0438	0.3338	-0.0736	2.9928	0.1929	0.2006
	Boot.	$\hat{a}^*$		0.1893	0.0435	0.3341	-0.1656	2.9018	0.1022	0.2701

A variety of graphs (histogram and boxplot) of parameter estimates of the three distributions are shown in Figures 1, 2, and 3, respectively, for the true (top) and bootstrap (bottom) sampling distributions. All results take into account the size of the samples 50 (left), and 500 (right).



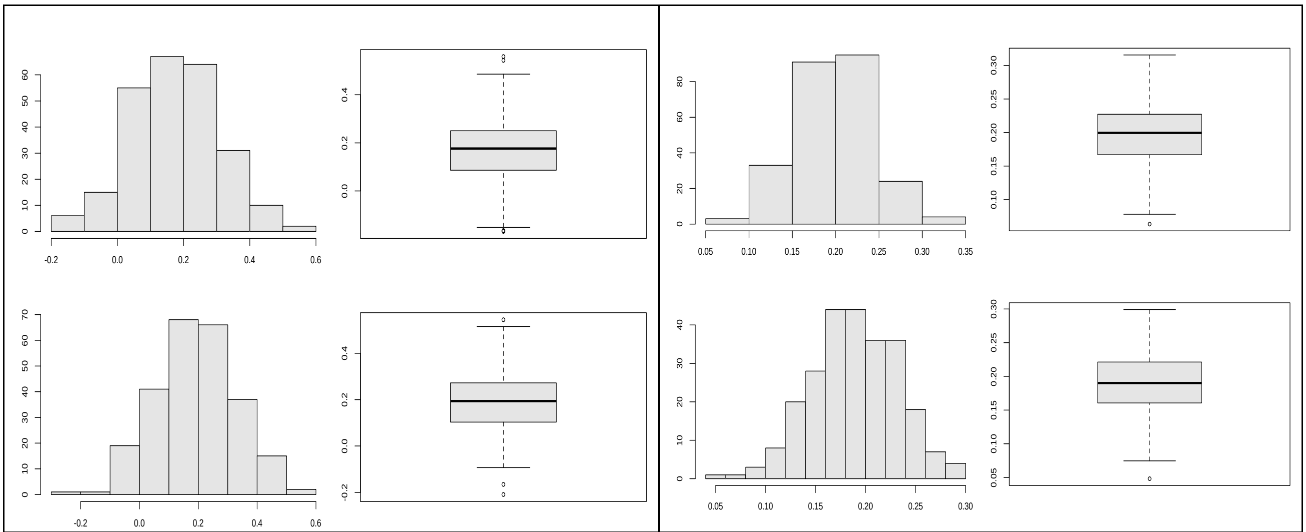
(1) n=50 (2) n=500

Figure 1: Histogram and boxplot of the parameter estimate of Model (2); The Normal Case



(1) n=50 (2) n=500

Figure 2: Histogram and boxplot of the parameter estimate of Model (2); The Exponential Case



(1) n=50 (2) n=500

Figure 3: Histogram and boxplot of the parameter estimate of Model (2); The Uniform Case

The results of the normality (Shapiro-Wilk) test of the coefficients of Model (2) are given in Table 2.

Table 2: The Results of the Normality Test of Coefficients of Model (2)

Sample Size	Sampling Distribution	Shapiro-Wilk					
		W	p-value	W	p-value	W	p-value
		Normal		Exponential		Uniform	
n=50	True $\hat{a}$	0.9936	0.3587	0.9839	0.0063	0.9958	0.7398
	Boot. $\hat{a}^*$	0.9898	0.0766	0.9876	0.0299	0.9970	0.9206
n=500	True $\hat{a}$	0.9970	0.9174	0.9949	0.5779	0.9975	0.9645
	Boot. $\hat{a}^*$	0.9950	0.5901	0.9917	0.1719	0.9966	0.8738

4.1 Testing for Goodness of Fit of Residuals

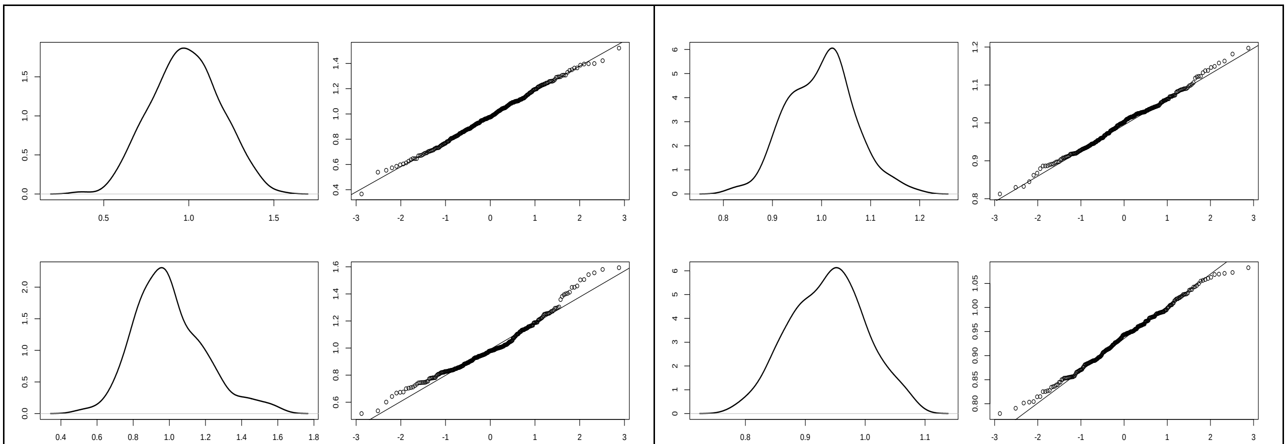
In this section, we apply tests that consider the goodness of fit between a hypothesised distribution function and an empirical distribution function of residuals. Thus, we are interested in testing the hypothesis that a normal distribution will be satisfactory as a probability distribution of residual values, i.e., $H0: F(x) = F0(x)$, against all alternative $H1: F(x) \neq F0(x)$, where $F0(x)$ is a normal distribution function (Hogg *et al*., 2013; D'Agostino, 2017).

The EasyFit statistical package is used to determine the goodness of fit. The Kolmogorov-Smirnov is used, which includes the critical value at the $\alpha = 0.05$ significance level.

Numerical results of goodness of fit for residual values of Model (2) with samples 50 and 500 are given in Tables 3 (normal), 4 (exponential), and 5 (uniform), respectively. In addition, the probability density function and quantile-quantile (Q-Q) plots of true (top) and bootstrap (bottom) distributions from the fitting results with samples 50 (left) and 500 (right) of Tables 3, 4, and 5 are shown in Figures 4, 5, and 6, respectively.

Table 3: Numerical Results of Fit for Residuals of Model (2); Normal

Goodness of Fit	Sample Size		True Sampling Distribution (Normal) $(\boldsymbol{\mu = 0.3317})$ $(\boldsymbol{\sigma = 0.1001})$	Bootstrap Sampling Distribution (Normal) $(\boldsymbol{\mu = 0.5406})$ $(\boldsymbol{\sigma = 0.1969})$
Kolmogorov-Smirnov	n=50	Statistic α Critical Value Reject?	0.0827 0.05 0.0860 No	0.0525 0.05 0.0860 No
Goodness of Fit	**Sample Size**		**True Sampling Distribution (Normal)** $(\boldsymbol{\mu = 0.3338})$ $(\boldsymbol{\sigma = 0.0299})$	**Bootstrap Sampling Distribution (Normal)** $(\boldsymbol{\mu = 0.3341})$ $(\boldsymbol{\sigma = 0.0373})$
Kolmogorov-Smirnov	n=500	Statistic α Critical Value Reject?	0.0420 0.05 0.0860 No	0.0470 0.05 0.0860 No

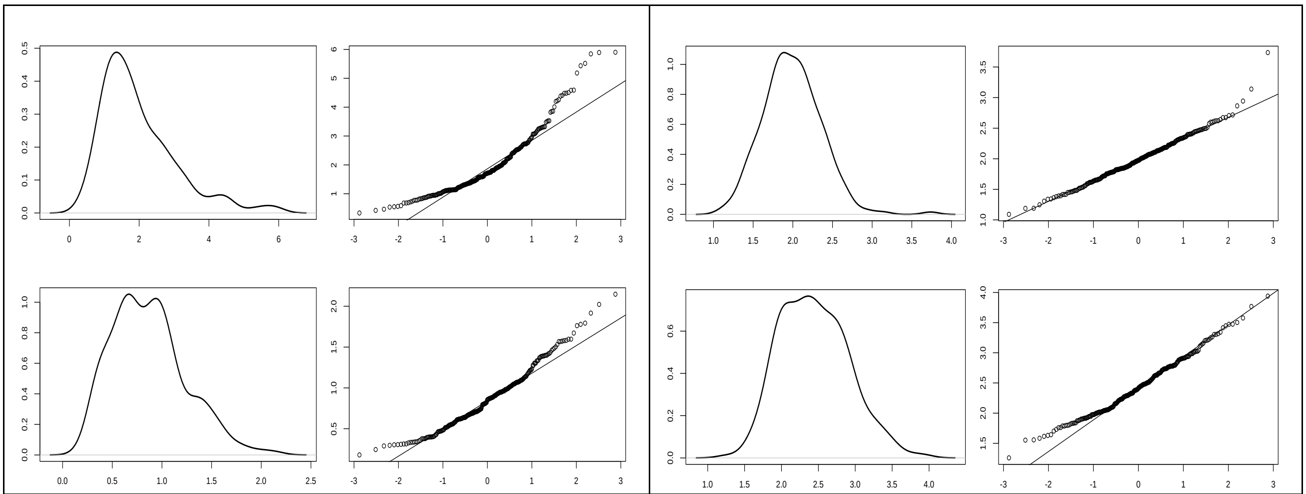


(1) n=50 (2) n=500

Figure 4: The probability density function and q–q plots of Residuals of Model (1); The Normal Case

Table 4: Numerical Results of Fit for Residuals of Model (2); Exponential

Goodness of Fit	Sample Size		True Sampling Distribution (Normal) ($\boldsymbol{\mu = 1.9847}$) ($\boldsymbol{\sigma = 1.0870}$)	Bootstrap Sampling Distribution (Normal) ($\boldsymbol{\mu = 0.8663}$) ($\boldsymbol{\sigma = 0.3710}$)
Kolmogorov-Smirnov	n=50	Statistic α Critical Value Reject?	0.1269 0.05 0.0860 Yes	0.0760 0.05 0.0860 No
Goodness of Fit	**Sample Size**		**True Sampling Distribution (Normal)** **($\boldsymbol{\mu = 1.9903}$)** **($\boldsymbol{\sigma = 0.3668}$)**	**Bootstrap Sampling Distribution (Normal)** **($\boldsymbol{\mu = 0.3341}$)** **($\boldsymbol{\sigma = 0.0373}$)**
Kolmogorov-Smirnov	n=500	Statistic α Critical Value Reject?	0.0333 0.05 0.0860 No	0.0570 0.05 0.0860 No



(1) n=50 (2) n=500

Figure 5: The probability density function and q–q plots of Residuals of Model (1); The Exponential Case

Table 5: Numerical Results of Fit for Residuals of Model (2); Uniform

Goodness of Fit	Sample Size		True Sampling Distribution (Normal) ($\boldsymbol{\mu = 0.3317}$) ($\boldsymbol{\sigma = 0.1001}$)	Bootstrap Sampling Distribution (Normal) ($\boldsymbol{\mu = 0.5406}$) ($\boldsymbol{\sigma = 0.1969}$)
Kolmogorov-Smirnov	n=50	Statistic α Critical Value Reject?	0.0827 0.05 0.0860 No	0.0525 0.05 0.0860 No
Goodness of Fit	**Sample Size**		**True Sampling Distribution (Normal)** **($\boldsymbol{\mu = 0.3338}$)** **($\boldsymbol{\sigma = 0.0299}$)**	**Bootstrap Sampling Distribution (Normal)** **($\boldsymbol{\mu = 0.3341}$)** **($\boldsymbol{\sigma = 0.0373}$)**
Kolmogorov-Smirnov	n=500	Statistic α Critical Value Reject?	0.0420 0.05 0.0860 No	0.0470 0.05 0.0860 No

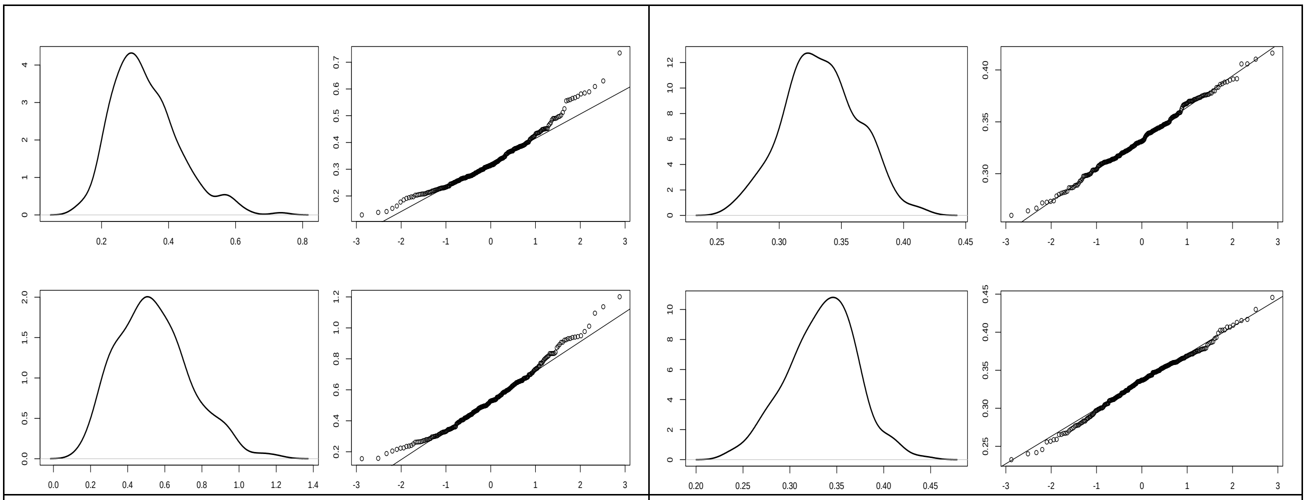


Figure 6: The probability density function and q–q plots of Residuals of Model (1); The Uniform Case

5. DISCUSSION

A close examination of our findings drew the following remarks: The bootstrap method based on Monte-Carlo approximation has a reasonable performance in uncovering the true sampling distribution of each parameter estimate, and this is true for normal, exponential, and uniform white-noise distributions. The previous point can be inferred from the summary statistics of Model (2) (Table 1) when we compare the descriptive statistics (mean, standard error) of the true sampling distribution of each parameter estimate with its counterparts obtained by bootstrapping. Furthermore, the visual comparison between the histogram display, and boxplot of the true sampling distributions with the corresponding ones for bootstrap distributions (Figures 1, 2, and 3) confirmed our finding. In addition, in most cases, bootstrap standard errors are slightly lower than their counterparts obtained from true distributions, perhaps due to sampling variability. The bootstrap estimate of the variance of residuals is near the true value in the Gaussian distribution case. However, in the non-Gaussian cases, there was a clear difference from the true distribution.

Table 1 highlights that the bootstrap confidence interval of each estimate based on the percentile method has a good approximation to its counterpart obtained from true distributions. This outcome means that the bootstrap technique can construct a confidence interval for a parameter without appealing to the approximate normality based on the large sample theory, perhaps at the expense of tedious computation. However, it seems that the performance of the bootstrap method for the sample size of 500 is better than the sample size of 50. It is worth noting that had we used more than 250 replications in our study, the convergence of the Monte-Carlo approximation to the bootstrap distribution would have been faster.

The normality test (Table 2) of coefficients of Model (2) shows that all bootstrap estimates have normal distributions, except in just one case (exponential) with a sample size of 50, we could not accept the null hypothesis, which may be due to the sample size.

Numerical results of goodness of fit for residuals of Model (1) with samples 50 and 500 show that we could not reject $H0$ in all cases except the exponential case with sample 50 (Table 4). In addition, the probability density function and q-q plots of residuals of Model (2) (Figures 4, 5, and 6) show that our decision about H0 might be reasonable.

6. CONCLUSION

It should be noted that these results are specific to Model (2). Moreover, we have used only three distributions (normal, exponential, and uniform) in the simulated data from Model (2), which does not guarantee the sensitivity of the performance of the bootstrap method. For future work, we suggest using more models with different distributions of residuals before these results can be generalised. In addition, it would be interesting to repeat this study using nonlinear time series models, for instance, threshold models. Finally, using the block bootstrap (BBM) method rather than the bootstrap method based on Monte-Carlo approximation in the simulation procedure would be attractive.

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