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New Constructions of Uninorms on Bounded Lattices via Closure Operators and Interior Operators

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Abstract **Original Research Article**

In this paper, based on closure operators and interior operators, we propose some new methods to construct uninorms on bounded lattices. Meanwhile, we discuss the relationships between new uninorms and some uninorms in the literature.

Keywords: Bounded lattices, t-norms, t-conorms, closure operators, interior operators, uninorms.

1. INTRODUCTION

Schweizer and Sklar [1] introduced triangular norms (t-norms, for short) with the neutral 1 and triangular conorms (t-conorms, for short) with the neutral 0 on the unit interval [0,1] which are widely used to various fields, such as fuzzy set theory, fuzzy logic, image processing and so on (see, e.g., [2-6]). Uninorms on the unit interval [0,1] as a generalization of t-norms and t-conorms were introduced by Yager and Rybalor [7] which were also proved useful in many fields, such as fuzzy logic, fuzzy system modeling, expert systems, neural networks, decision-making and so on (see, e.g., [8-14]). Uninorms are particularly useful in the bipolar decision-making described in expert systems (see, e.g., [8, 15-17]). Besides, the fuzzy modeling inference process consists of an aggregation step in which the contributions of different rules of the fuzzy system model are combined, and uninorms provide a general class of operators to implement this step [14]. A great deal of study on uninorms has been done on the unit interval (see, e.g., [18-20]).

Due to the fact that the bounded lattice is more general than [0,1], uninorms on bounded lattice were introduced by Karaçal and Mesiar [21]. Since then, uninorms on bounded lattices have been studied extensively and a great deal of construction methods have been given in the literature (see, e.g., [21-33]).

As we see uninorms in the literature, $U(r,s) = s$ for $(r,s) \in (0,e) \times I_e$ (resp. $U(r,s) = s$ for $(r,s) \in (e,1) \times I_e$), $U(r,s) = 0$ for $(r,s) \in (0,e) \times I_e$ (resp. $U(r,s) = 1$ for $(r,s) \in (e,1) \times I_e$) or $U(r,s) = r$ for $(r,s) \in (0,e) \times I_e$ (resp. $U(r,s) = r$ for $(r,s) \in (e,1) \times I_e$). However, based on the fact that $U(r,s) \le s$ for $(r,s) \in (0,e) \times I_e$ (resp. $s \le U(r,s)$ for $(r,s) \in (e,1) \times I_e$) in Proposition 1 of [21], we may ask a question that whether the values of $U(r,s)$ can be elements different from 0, r and s for $(r,s) \in (0,e) \times I_e$ (resp. 1, r and s for $(r,s) \in (e,1) \times I_e$). In this paper, under some constraints, we construct new uninorms via closure operators and interior operators.

2. Preliminaries

In this section, we recall some basic concepts and results about lattices and aggregation functions.

Definition 2.1 ([34]) *A lattice $(L,\le)$ is bounded if it has top and bottom elements, which are written as* 1 *and* 0, *respectively, that is, there exist two elements* $1,0 \in L$ *such that* $0 \le r \le 1$ *for all* $r \in L$.

Throughout this article, unless stated otherwise, we denote L as a bounded lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.2 ([34]) *Let L be a bounded lattice, $a,b \in L$ with $a \le b$. A subinterval $[a,b]$ of L is defined as $[a,b] = \{r \in L : a \le r \le b\}$. Similarly, we can define $[a,b) = \{r \in L : a \le r < b\}$, $(a,b] = \{r \in L : a < r \le b\}$ and $(a,b) = \{r \in L : a < r < b\}$. If a and b are incomparable, then we use the notation $a \parallel b$.*

In the following, I_a denotes the set of all incomparable elements with a, that is, $I_a = \{r \in L \mid r \parallel a\}$.

Definition 2.3 ([35]) *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $T : L^2 \to L$ *is called a t-norm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element* $1 \in L$*, that is,* $T(1,r) = r$ *for all* $r \in L$.

Definition 2.4 ([36]) *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $S : L^2 \to L$ *is called a t-conorm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element* $0 \in L$*, that is,* $S(0,r) = r$ *for all* $r \in L$.

Definition 2.5 ([37]) *Let* $(L,\leq,0,1)$ *be a bounded lattice. A mapping* $cl : L^2 \to L$ *is said to be a closure operator on L if, for all* $r,s \in L$*, it satisfies the following three conditions:*

(1) $r \leq cl(r)$;
(2) $cl(r \vee s) = cl(r) \vee cl(s)$;
(3) $cl(cl(r)) = cl(r)$.

Definition 2.6 ([29]) *Let* $(L,\leq,0,1)$ *be a bounded lattice. A mapping* $int : L^2 \to L$ *is said to be an interior operator on L if, for all* $r,s \in L$*, it satisfies the following three conditions:*

(1) $int(r) \leq r$;
(2) $int(r \wedge s) = int(r) \wedge int(s)$;
(3) $int(int(r)) = int(r)$.

Definition 2.7 ([21]) *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $U : L^2 \to L$ *is called a uninorm on L (a uninorm if L is fixed) if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element* $e \in L$*, that is,* $U(e,r) = r$ *for all* $r \in L$.

Definition 2.8 ([36]) *Let L be a bounded lattice and U be a uninorm with the neutral element* $e \in L \setminus \{0,1\}$ *on L.*

(1) *An element* $r \in L$ *is called an idempotent element of U if* $U(r,r) = r$.
(2) *U is called an idempotent uninorm whenever* $U(r,r) = r$ *for all* $r \in L$.

Definition 2.9 ([36]) *Let L be a bounded lattice and U be a uninorm with the neutral element* $e \in L \setminus \{0,1\}$ *on L.*

(1) *U is called conjunctive uninorm if* $U(0,1) = 0$.
(2) *U is called disjunctive uninorm if* $U(0,1) = 1$.

Proposition 2.1 ([28]) *Let S be a nonempty set and* $A_1,A_2,\ldots,A_n$ *be subsets of S. Let H be a commutative binary operation on S, then H is associative on* $A_1 \cup A_2 \cup \ldots \cup A_n$ *if and only if all of the following statements hold:*

(i) *for every combination* $\{i,j,k\}$ *of size* 3 *chosen from* $\{1,2,\ldots,n\}$, $H(r,H(s,t)) = H(H(r,s),t) = H(s,H(r,t))$ *for all* $r \in A_i$, $s \in A_j$, $t \in A_k$;
(ii) *for every combination* $\{i,j\}$ *of size 2 chosen from* $\{1,2,\ldots,n\}$, $H(r,H(s,t)) = H(H(r,s),t)$ *for all* $r \in A_i$, $s \in A_i$, $t \in A_j$;
(iii) *for every combination* $\{i,j\}$ *of size 2 chosen from* $\{1,2,\ldots,n\}$, $H(r,H(s,t)) = H(H(r,s),t)$ *for all* $r \in A_i$, $s \in A_j$, $t \in A_j$;
(iv) *for every* $i \in \{1,2,\ldots,n\}$, $H(r,H(s,t)) = H(H(r,s),t)$ *for all* $r,s,t \in A_i$.

Theorem 2.1 ([21]) *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$*. If* T_e *is a t-norm on* $[0,e]$*, then the uninorm* $U_t : L^2 \to L$ *defined as follows:*

$$U_t(r,s) = \begin{cases} T_e(r,\ s) & \text{if } (r,s) \in [0,e]^2, \\ r \vee s & \text{if } (r,s) \in (e,1] \times [0,e] \\ & \quad \bigcup [0,e] \times (e,1], \\ r & \text{if } (r,s) \in I_e \times [0,\ e], \\ s & \text{if } (r,s) \in [0,e] \times I_e, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.2 ([21]) *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$*. If* S_e *is a t-conorm on* $[e,1]$*, then the uninorm* $U_s : L^2 \to L$ *defined as follows:*

$$U_s(r,s) = \begin{cases} S_e(r,\ s) & \text{if } (r,s) \in [e,1]^2, \\ r \wedge s & \text{if } (r,s) \in [0,e) \times [e,1] \\ & \quad \bigcup [e,1] \times [0,e), \\ r & \text{if } (r,s) \in I_e \times [e,1], \\ s & \text{if } (r,s) \in [e,1] \times I_e, \\ 0 & \text{otherwise.} \end{cases}$$

3. New methods to construct uninorms on bounded lattices

In this section, based on closure operators and interior operators, we propose new methods to construct uninorms on bounded lattices.

Theorem 3.1 *Let* $(L,\leq,0,1)$ *be a bounded lattice with* $e \in L \setminus \{0,1\}$*, T be a t-norm on* $[0,e]$ *and int be an interior operator on L. Let* $U_{Ie,1} : L^2 \to L$ *be a function defined as follows:*

$$U_{Ie,1}(r,s) = \begin{cases} T(r,s) & \text{if } (r,s) \in [0,e]^2, \\ int(r) & \text{if } (r,s) \in I_e \times [0,e), \\ int(s) & \text{if } (r,s) \in [0,e) \times I_e, \\ r & \text{if } (r,s) \in (L \setminus [0,e]) \times \{e\} \\ & \quad \bigcup (e,1] \times [0,e), \\ s & \text{if } (r,s) \in \{e\} \times (L \setminus [0,e]) \\ & \quad \bigcup [0,e) \times (e,1], \\ 1 & \text{otherwise.} \end{cases}$$

(1) *If $r \wedge e < int(r)$ for all $r \in I_e$, then $U_{Ie,1}$ is a uninorm on L with the neutral element $e \in L$.*
(2) *If T is an idempotent t-norm, then $U_{Ie,1}$ is a uninorm on L with the neutral element $e \in L$ iff $r \wedge e < int(r)$ for all $r \in I_e$.*

Proof. (1) Obviously, $U_{Ie,1}$ is commutative and e is the neutral element. Hence, we only need to prove the increasingness and the associativity of $U_{Ie,1}$.

I. Increasingness: We prove that if $r \leq s$, then $U_{Ie,1}(r,t) \leq U_{Ie,1}(s,t)$ for all $t \in L$. It is obvious that $U_{Ie,1}(r,t) \leq U_{Ie,1}(s,t)$ if both r and s belong to one of the intervals $[0,e)$,$\{e\}$,I_e or $(e,1]$ for all $t \in L$. The residual proof can be split into all possible cases.

1. $r \in [0,e)$
   1.1. $s \in \{e\}$
      1.1.1. $t \in [0,e]$
      $U_{Ie,1}(r,t) = T(r,t) \leq T(s,t) = U_{Ie,1}(s,t)$
      1.1.2. $t \in I_e$
      $U_{Ie,1}(r,t) = int(t) \leq t = U_{Ie,1}(s,t)$
      1.1.3. $t \in (e,1]$
      $UIe,1(r,t) = t = UIe,1(s,t)$
   1.2. $s \in I_e$
      1.2.1. $t \in [0,e)$
      $U_{Ie,1}(r,t) = T(r,t) \leq r < int(s) = U_{Ie,1}(s,t)$
      1.2.2. $t \in \{e\}$
      $U_{Ie,1}(r,t) = r < s = U_{Ie,1}(s,t)$
      1.2.3. $t \in I_e$
      $U_{Ie,1}(r,t) = int(t) < 1 = U_{Ie,1}(s,t)$
      1.2.4. $t \in (e,1]$
      $U_{Ie,1}(r,t) = t \leq 1 = U_{Ie,1}(s,t)$
   1.3. $s \in (e,1]$
      1.3.1. $t \in [0,e]$
      $U_{Ie,1}(r,t) = T(r,t) \leq r < s = U_{Ie,1}(s,t)$
      1.3.2. $t \in I_e$
      $U_{Ie,1}(r,t) = int(t) < 1 = U_{Ie,1}(s,t)$
      1.3.3. $t \in (e,1]$
      $U_{Ie,1}(r,t) = t \leq 1 = U_{Ie,1}(s,t)$
2. $r \in \{e\}$, $s \in (e,1]$
   2.1. $t \in [0,e]$
   $U_{Ie,1}(r,t) = T(r,t) = t < s = U_{Ie,1}(s,t)$
   2.2. $t \in I_e \cup (e,1]$
   $U_{Ie,1}(r,t) = t \leq 1 = U_{Ie,1}(s,t)$
3. $r \in I_e$, $s \in (e,1]$
   3.1. $t \in [0,e)$
   $U_{Ie,1}(r,t) = int(r) < s = U_{Ie,1}(s,t)$
   3.2. $t \in \{e\}$
   $U_{Ie,1}(r,t) = r < s = U_{Ie,1}(s,t)$
   3.3. $t \in I_e \cup (e,1]$
   $UIe,1(r,t) = 1 = UIe,1(s,t)$

II. Associativity: It can be shown that $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(U_{Ie,1}(r,s),t)$ for all $r,s,t \in L$. It is obvious that if at least one of r,s,t belongs to $\{e\}$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(U_{Ie,1}(r,s),t)$ for all $r,s,t \in L$. By Proposition 2.1, we need to verify the following cases.

1. If $r,s,t \in [0,e)$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,T(s,t)) = T(r,T(s,t)) = T(T(r,s),t) = U_{Ie,1}(T(r,s),t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$.
2. If $r,s,t \in I_e$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(1,t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$.
3. If $r,s,t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(1,t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$.
4. If $r,s \in [0,e)$ and $t \in I_e$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,int(t)) = int(int(t)) = int(t) = U_{Ie,1}(T(r,s),t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$.
5. If $r,s \in [0,e)$ and $t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,t) = t = U_{Ie,1}(T(r,s),t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$.
6. If $r,s \in I_e$ and $t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(1,t) = UIe,1(UIe,1(r,s),t)$.
7. If $r \in [0,e)$ and $s,t \in I_e$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(int(s),t) = UIe,1(UIe,1(r,s),t)$.
8. If $r \in [0,e)$ and $s,t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(s,t) = UIe,1(UIe,1(r,s),t)$.
9. If $r \in I_e$ and $s,t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(1,t) = UIe,1(UIe,1(r,s),t)$.
10. If $r \in [0,e)$, $s \in I_e$ and $t \in (e,1]$, then $U_{Ie,1}(r,U_{Ie,1}(s,t)) = U_{Ie,1}(r,1) = 1 = U_{Ie,1}(int(s), t) = U_{Ie,1}(U_{Ie,1}(r,s),t)$ and $U_{Ie,1}(s,U_{Ie,1}(r,t)) = U_{Ie,1}(s,t) = 1$. Thus $UIe,1(r,UIe,1(s,t)) = UIe,1(UIe,1(r,s),t) = UIe,1(s,UIe,1(r,t))$.

(2) Next we just prove that if T is an idempotent uninorm, then the condition $r \wedge e < int(r)$ for all $r \in I_e$ is necessary.

By the definition of interior operators, we obtain that $r \wedge e < int(r)$, $int(r) \leq r \wedge e$ or $int(r) \parallel r \wedge e$ for $r \in I_e$. First, we prove that $r \wedge e < int(r)$ or $int(r) \parallel r \wedge e$ for all $r \in I_e$. Assume that there exists $r \in I_e$ such that $int(r) \leq r \wedge e$. Then $U_{Ie,1}(U_{Ie,1}(r \wedge e,r),r) = U_{Ie,1}(int(r),r) = int(r)$ and $U_{Ie,1}(r \wedge e,U_{Ie,1}(r,r)) = U_{Ie,1}(r \wedge e,1) = 1$. Since $int(r) \neq 1$, the associativity of $U_{Ie,1}$ is violated. Next, we prove that $int(r)$ is comparable with $r \wedge e$ for all $r \in I_e$. Assume that there exists $r \in I_e$ such that $int(r) \parallel r \wedge e$. Then $U_{Ie,1}(r \wedge e,r) = int(r)$ and $U_{Ie,1}(r \wedge e,r \wedge e) = T(r \wedge e,r \wedge e) = r \wedge e$. Since $int(r) \parallel r \wedge e$, the increasingness of $U_{Ie,1}$ is violated.

Therefore, if T is an idempotent uninorm, then the condition $r \wedge e < int(r)$ for all $r \in I_e$ is necessary.

Remark 3.1 *Let $(L,\leq,0,1)$ be a bounded lattice. If we put $int(r) = r$ for all $r \in I_e$ in Theorem 3.1, then $r \wedge e < r = int(r)$ for all $r \in I_e$ and Theorem 3.1(1) is exactly Theorem 2.1.*

Remark 3.2 *Let $U_{Ie,1}$ be a uninorm defined by Theorem 3.1.*
(1) *$U_{Ie,1}$ is not idempotent, in general. More precisely, if there exists $r \in I_e$, then $U_{Ie,1}(r,r) = 1 \neq r$*
(2) *$U_{Ie,1}$ is disjunctive, i.e., $U_{Ie,1}(0,1) = 0 \vee 1 = 1$.*

Theorem 3.2 *Let $(L,\leq,0,1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$, S be a t-conorm on $[e,1]$ and cl be a closure operator on L. Let $U_{Ie,2} : L^2 \to L$ be a function defined as follows:*

$$U_{I_e,2}(r,s)=\begin{cases} S(r,\ s) & if\ \ (r,s)\in[e,1]^2, \\ cl(r) & if\ \ (r,s)\in I_e\times(e,1], \\ cl(s) & if\ \ (r,s)\in(e,1]\times I_e, \\ r & if\ \ (r,s)\in\big(L\setminus[e,1]\big)\times\{e\} \\ & \qquad\qquad \cup[0,e)\times(e,1], \\ s & if\ \ (r,s)\in\{e\}\times\big(L\setminus[e,1]\big) \\ & \qquad\qquad \cup(e,1]\times[0,e), \\ 0 & otherwise. \end{cases}$$

(1) *If* $cl(r) < r\vee e$ *for all* $r\in I_e$, *then* $U_{Ie,2}$ *is a uninorm on L with the neutral element* $e\in L$.

(2) *If S is an idempotent t-conorm, then* $U_{Ie,2}$ *is a uninorm on L with the neutral element* $e\in L$ *iff* $cl(r) < r\vee e$ *for all* $r\in I_e$.

Proof. It can be proved with the proof of Theorem 3.1 in a similar way.

Remark 3.3 *Let* $(L,\le,0,1)$ *be a bounded lattice. If we put* $cl(r) = r$ *for all* $r\in I_e$ *in Theorem 3.2, then* $cl(r) = r < r\vee e$ *for all* $r\in I_e$ *and Theorem 3.2(1) is exactly Theorem 2.2.*

Remark 3.4 *Let* $U_{Ie,2}$ *be a uninorm defined by Theorem 3.2.*

(1) $U_{Ie,2}$ *is not idempotent, in general. More precisely, if there exists* $r\in I_e$, *then* $U_{Ie,2}(r,r) = 0\neq r$.

(2) $U_{Ie,2}$ *is conjunctive, i.e.,* $U_{Ie,2}(0,1) = 0\wedge 1 = 0$.

In Theorem 3.1, we know that $U_{Ie,1}(r,s) = int(s)$ for $(r,s)\in(0,e)\times I_e$ and the values of $U_{Ie,1}(r,s)$ for $(r,s)\in(0,e)\times I_e$ can be elements which are different from 0, r and s. Moreover, the values of $U_{Ie,1}(r,s)$ for $(r,s)\in(0,e)\times I_e$ can differ for different interior operators int on L.

Similarly, in Theorem 3.2, $U_{Ie,2}(r,s) = cl(s)$ for $(r,s)\in(e,1)\times I_e$ and then this construction method differs from those in the literature. The values of $U_{Ie,2}(r,s)$ for $(r,s)\in(e,1)\times I_e$ can be elements different from 1, r and s. Moreover, the values of $U_{Ie,2}(r,s)$ for $(r,s)\in(e,1)\times I_e$ can differ for different closure operators on L.

4. CONCLUSION

In this paper, we give new methods to construct uninorms on bounded lattices via closure operators and interior operators, by expending the values of $U(r,s)$ for all $(r,s)\in(0,e)\times I_e$ or $U(r,s)$ for all $(r,s)\in(e,1)\times I_e$. Then we obtain some new uninorms on bounded lattices, which generalized the methods presented in the literature.

REFERENCES

1. Schweizer, B., & Sklar, A. (1960). Statistical metric spaces. *Pacific J Math*, 10, 313-334.
2. Höhle, U. (1995). Commutative, residuated l-monoids, in: Höhle, U., Klement, E. P. (Eds.)., Non-Classical Logics and Their Applications to Fuzzy Subsets: A Handbook on the Mathematical Foundations of Fuzzy Set Theory, Kluwer, Dordrecht.
3. Liang, X., & Pedrycz, W. (2009). Logic-based fuzzy networks: a study in system modeling with triangular norms and uninorms. *Fuzzy Sets and Systems*, 160, 3475- 3502.
4. Schweizer, B., & Sklar, A. (1961). Associative functions and statistical triangular inequalities. *Publ Math*, 8, 169-186.
5. Wang, Z. (2007). TL-filters of integral residuated l-monoids. *Information Science*, 177, 887-896.
6. Zimmermann, H. J. (2001). Fuzzy Set Theory and Its Applications, fourth ed., Kluwer, Aachen.
7. Yager, R. R., & Rybalov, A. (1996). Uninorm aggregation operators. *Fuzzy Sets and Systems*, 80, 111-120.
8. De Baets, B., & Fodor, J. (1999). Van Melle's combining function in MYCIN is a representable uninorm: An alternative proof. *Fuzzy Sets and Systems*, 104, 133-136.
9. Grabisch, M., Marichal, J. L., Mesiar, R., & Pap, E. (2009). Aggregation functions. Cambridge University Press.
10. Grabisch, M., Marichal, J. L., Mesiar, R., & Pap, E. (2011). Aggregation functions: construction methods, conjunctive, disjunctive and mixed classes. *Information Science*, 181, 23-43.
11. Gonzalez-Hidalgo, M., Massanet, S., Mir, A., & Ruiz-Aguilera, D. (2015). On the choice of the pair conjunction–implication into the fuzzy morphological edge detector. *IEEE Transactions on Fuzzy Systems*, 23, 872-884.
12. Pedrycz, W., & Hirota, K. (2007). Uninorm-based logic neurons as adaptive and interpretable processing constructs. Soft Computing, 11, 41-52.
13. Yager, R. R. (2003). Defending against strategic manipulation in uninorm-based multi-agent decision making. Fuzzy Sets and Systems, 140, 331-339.
14. Yager, R. R. (2001). Uninorms in fuzzy system modeling. Fuzzy Sets and Systems, 122, 167-175.
15. Hájek, P. (1985). Combining functions for certainty degrees in consulting systems. *International Journal Man-Machine Studies*, 22, 59-76.
16. Hájek, P., Havránek, T., & Jirousek, R. (1992). Uncertain information processing in expert systems, CRC Press.
17. Shortliffe, E.H. (1976). Computer-Based medical consultations: MYCIN, New York: Elsevier-North Holland.
18. De Baets, B. (1999). Idempotent uninorms. European Journal of Operational Research, 188, 631-642.
19. Drewniak, J., & Drygás, P. (2002). On a class of uninorms. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 10, 5-10.
20. Su, Y., Zong, W., & Drygaá, P. (2019). Properties of uninorms with the underlying operations given as ordinal sums. Fuzzy Sets and Systems, 357, 47-57.

21. Karaçal, F., & Mesiar, R. (2015). Uninorms on bounded lattices. Fuzzy Sets and Systems, 261, 33-43.
22. Çaylı, G. D., Ertugrul, U., & Karaçal, F. (2022). Some further construction methods¨ for uninorms on bounded lattices. International Journal of General Systems, DOI: 10.1080/03081079.2022.2132492.
23. Çaylı, G. D. (2023). An alternative construction of uninorms on bounded lattices. International Journal of General Systems, DOI: 10.1080/03081079.2023.2196421.
24. Dan, Y. X., Hu, B. Q., & Qiao, J. S. (2019). New constructions of uninorms on bounded lattices. International Journal of Approximate Reasoning, 110, 185-209.
25. Dan, Y. X., & Hu, B. Q. (2020). A new structure for uninorms on bounded lattices, Fuzzy Sets and Systems, 386, 77-94.
26. He, P., & Wang, X. P. (2021). Constructing uninorms on bounded lattices by using additive generators, International Journal of Approximate Reasoning, 136, 1-13.
27. Hua, X. J., & Ji, W. (2022). Uninorms on bounded lattices constructed by t-norms and t-subconorms, Fuzzy Sets and Systems, 427, 109-131.
28. Ji, W. (2021). Constructions of uninorms on bounded lattices by means of t-subnorms and t-subconorms, Fuzzy Sets and Systems, 403, 38-55.
29. Ouyang, Y., & Zhang, H.P. (2020). Constructing uninorms via closure operators on a bounded lattice, Fuzzy Sets and Systems, 395, 93-106.
30. Xiu, Z. Y., & Zheng, X. (2023). New construction methods of uninorms on bounded lattices via uninorms. Fuzzy Sets and Systems, doi.org/10.1016/j.fss.2023.108535.
31. Xiu, Z. Y., & Jiang, Y. X. (2023). New structures for uninorms on bounded lattices, Journal of Intelligent and Fuzzy Systems, DOI: 10.3233/JIFS-224537.
32. Zhang, H.P., Wu, M., Wang, Z., Ouyang, Y., & De Baets, B. (2021). A characterization of the classes Umin and Umax of uninorms on a bounded lattice, Fuzzy Sets and Systems, 423, 107-121.
33. Asıcı, E., & Mesiar, R. (2022). On generating uninorms on some special classes of bounded lattices, Fuzzy Sets and Systems, 439, 102-125.
34. Birkhoff, G. (1967). Lattice theory. 3rd Edition, Amer. Math. Soc., Rhode Island.
35. Saminger, S. (2006). On ordinal sums of triangular norms on bounded lattices, Fuzzy Sets and Systems, 157, 1403-1416.
36. Çaylı, G. D., Karaçal, F., & Mesiar, R. (2016). On a new class of uninorms on bounded lattices, Information Sciences, 367, 221-231.
37. Everett, C. J. (1944). Closure operators and Galois theory in lattices, *Transactions of the American Mathematical Society*, 55, 514-525.