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Exact Squaring the Circle with Straightedge and Compass Only

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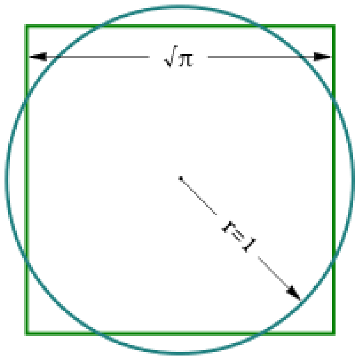
Abstract **Physics**

No great theory lasts forever in science, but only specific research and discoveries continuously complement each other. There have been 3 classical problems remaining from ancient Greek mathematics, which are extremely influential in the development of geometry. They are Trisecting an Angle, Squaring the Circle, and Doubling the Cube problems. The problem of Squaring the Circle is stated: Using only a straightedge and a compass, is it possible to construct a square with an area equal to a given circle? From the oldest mathematical documents known up to today's mathematics, the "Squaring the Circle" problem and related problems concerning π have interested professional & non-professional mathematicians for millenniums. I adopted the technique "ANALYSIS" to solve accurately the "Squaring the Circle" problem with only a straightedge & compass by Euclidean Geometry, and did not use the value π/Pi anywhere in this exact solution. Upstream from this method of exact "squaring the circle", we can deduce, inversely, to get a new Mathematical challenge "Circling the Square" with a straightedge & a compass.

Keywords: Squaring the circle, Quadrature of the circle, Make a circle squared, Find a square area same as the circle, Make a circle squared.

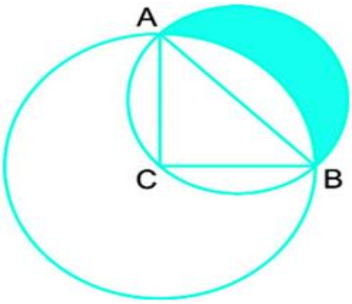
Part I

INTRODUCTION



In the past, knowledge was often called scientific if it could be confirmed by specific evidence or experiments. But Karl Popper, in his book "Logik der Forschung" (The Law of Scientific Discovery), published in 1934, showed that an essential characteristic of scientific hypotheses is that they can be proven wrong (falsifiability). Anything that cannot be refuted by evidence is temporarily considered true until new evidence is found. For example, in astronomy, now everyone believes in the Big Bang theory, but in the future, whoever finds a loophole in that theory will be rewarded by the whole physics community! Also, no great theory lasts forever in science, but only specific research and discoveries continuously complement each other [1].

About three thousand years ago, there were three well-known ancient Greek problems. Among them, the problem of squaring the circle was meticulously studied by Hippocrates. Hippocrates was the first to use a plane construction to find a square with an area equal to a figure with circular sides. He squared certain lunes, and also the sum of a lune and a circle. The problem of squaring a circle is stated: Using only a straightedge and a compass, is it possible to construct a square with an area equal to a given circle? In 1882, mathematician Ferdinand von Lindemann proved that pi is an irrational number, which means that it is impossible to construct a square in the problem of squaring a circle mentioned by Hippocrates. In his attempts to square the circle, Hippocrates was able to find the areas of certain moons, or crescent-shaped figures contained between two intersecting circles. In the figure below, Hippocrates asserted that the shaded green part (called the crescent moon) has an area equal to the area of triangle ABC. But no one understands how he calculated it.



A major step forward in proving that the circle could not be squared using a straightedge and a compass occurred in 1761 when Lambert proved that π was irrational. This was not enough to prove the impossibility of squaring the circle with a straightedge and a compass since certain algebraic numbers can be constructed with these tools. In 1775, the Paris Académie des Sciences passed a resolution that stated that no further attempted solutions submitted to them would be examined! The exact question posed by Anaxagoras was answered in 1882 when the German mathematician Ferdinand von Lindemann proved that squaring the circle is impossible with classical tools! A few years later, the Royal Society in London also banned consideration of any further 'proofs' of squaring the circle as large numbers of amateur mathematicians tried to achieve fame by presenting the Society with a solution! This decision of the Royal Society was described by De Morgan about 100 years later as the official blow to circle-squarers! [2]. Despite the proof of the impossibility of "squaring the circle," the problem has continued to capture the imagination of mathematicians and the general public alike, and it remains an important topic in the history and philosophy of mathematics.

It is difficult to give an accurate date when the problem of Squaring The Circle first appeared. The present article studies what has become the most famous for these problems, namely the problem of squaring the circle or the quadrature of the circle as it is sometimes called. One of the fascinations of this problem is that it has been of interest throughout the history of mathematics. From the oldest mathematical documents known to today's mathematics, the problem and related problems concerning π have interested professional & non-professional mathematicians for millenniums. The problem of Squaring The Circle is stated: Using only a straightedge and a compass, is it possible to construct a square with an area equal to a given circle?

First of all, it is not saying that a square of equal area with a circle does not exist. If the circle has area A, then a square with a side "square root" of A has the same area. Secondly, it is not saying that it is impossible, since it is possible, under the restriction of using only a straightedge and a compass [3].

In seeking the solutions to problems, geometers developed a special technique, which they called "ANALYSIS". They assumed the problem to have been solved and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved based on the givens. To obtain the formally correct solution to the original problem, then, geometers reversed the procedure: first, the data were used to solve the equivalent problem derived in the analysis, and, from the solution obtained, the original problem was then solved. In contrast to analysis, this reversed procedure is called "SYNTHESIS". I adopted the technique "ANALYSIS" to solve exactly the "Squaring The Circle" problem using only a straightedge, compass, and a newly defined shape (Conical-Arc) in Euclidean Geometry at nearly A-Level in the UK, without touching the irrational number π/Pi.

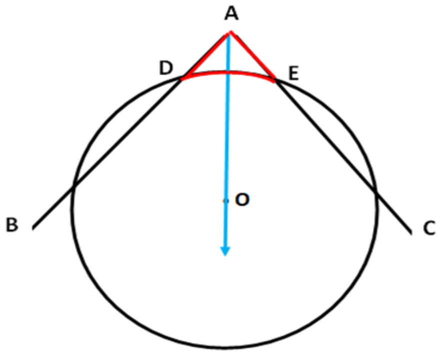
These problems were proven unsolvable using those tools back in the 19th century. In 1837, the French mathematician, Wantzel, L. proved that these 3 ancient challenged problems are IMPOSSIBLE to solve with only a straightedge and compass. Therefore, the three classical Greek problems, mentioned, above have been studied by mathematicians for centuries and, until 2022, have been shown to be unsolvable with a compass and straightedge. As of my December 2022 cutoff, no mathematician has found exact solutions to classical problems such as "Doubling the Cube," "Squaring the Circle," or "Trisecting an Angle" using only a compass and straightedge. However, my article's research result, herein is a clear demonstration as a counter-proof for the impossibility of Wantzel, L. in 1837 [4].

Starting from accepted premises, without proof, people use deductive reasoning to go step by step firmly to theorems and corollaries. With different premises, we have different mathematical systems. For example, with the premise "from a point outside a straight line we can draw only one line parallel to that line" then we have what is called Euclidean geometry. But if we assume that from that point we cannot draw any parallel lines, that is the premise of Riemannian Geometry. But Lobatchewsky Geometry assumes that we have not just one but an infinite number of parallel lines. Despite these facts, I exactly *solved herein the "Squaring The Circle" problem without making the premises (Euclidean Geometry & Straightedge & Compass) of this problem change, in the following Part II.*

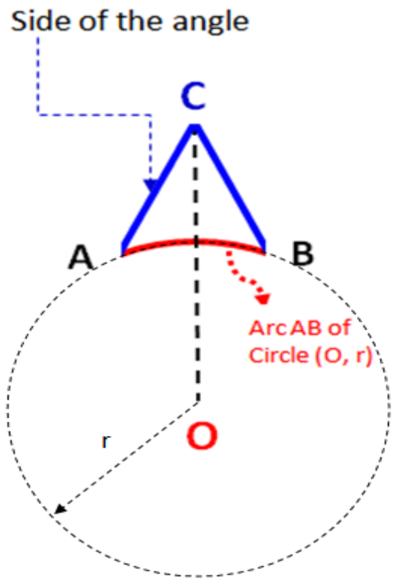
Part II
PROOFS FOR NEW GENERAL PROPOSITIONS

Definition 1: "Conical-Arc" shape

Given a circle (O,r) and an angle $\widehat{BAC}$ *with its vertex outside the circle such that the bisector of the angle passes through the centre O of the circle, then the* *special shape formed by the 2 sides of the angle and arc $\widehat{DE}$ can be called a Conical-Arc (in Figure 1 below, the red shape ADE is a <u>Conical-Arc</u>). If $\widehat{BAC}$ is a right angle then the shape ADE is called a Right-Conical-Arc.*



<u>Figure 1</u>: The **Conical-Arc** ADE



Theorem 1: *If there is a square ABCD of area πr^2 (assumed, yellow colour) of which centre coincides with the centre of a given circle (O, r), then ABCD occupies a stretch in between the inscribed square A"B"C"D" (red) of the circle and the circumscribed square A'B'C'D' of the circle (with 4 sides of A'B'C'D' (blue) being tangents to the circle, in Figure 2 below);*

OR,

Given {
α.) a circle (O, r) – in black colour of the Figure 2 below,
β.) a circumscribing square A'B'C'D' (blue), area $4r^2$,
γ.) and an inscribed square A"B"C"D" (red), area $2r^2$,

then {
α.) a square <u>ABCD with area πr^2</u> occupies a stretched area in between A'B'C'D' & A"B"C"D",
β.) the given circle (O, r) contains the inscribed circle (O, $\frac{r\sqrt{\pi}}{2}$) – yellow dashes - of ABCD (yellow),
γ.) and 4 sides of the square ABCD intersect the circle (O, r) to make 4 equal "small segments" of the circle.

PROOF:

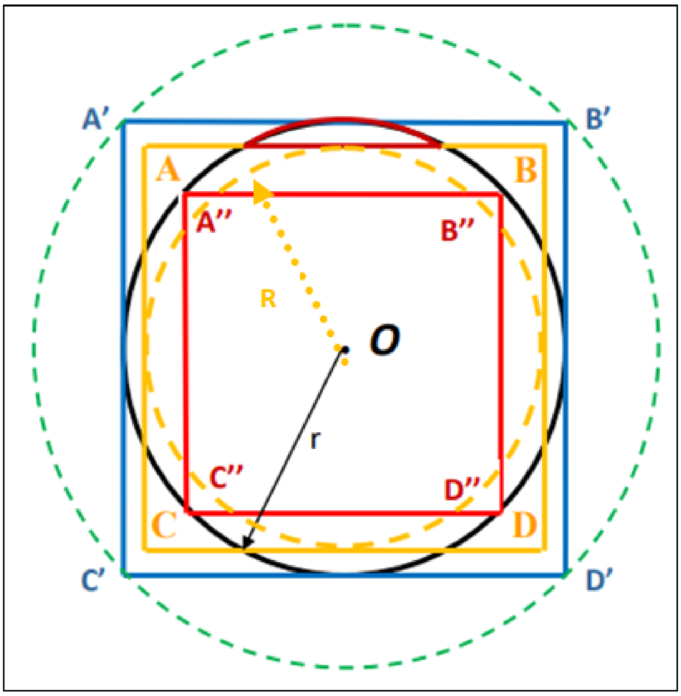


Figure 2: ***A small segment (red) located above side AB of square ABCD***

Assume there exist an unknown square of area πr^2 (assumed), having the same centre O with the given circle (O, r) and having the sides parallel to sides of the circumscribed square A'B'C'D' (blue) of (O, r) - black colour in the Figure 2 above. That means A'B'C'D' is a circumscribing square (blue colour) of the given circle (O, r) and A''B''C''D'' is an inscribing square (red colour) of the circle. Let these squares have the same centre O as the circle (O, r) and let them have parallel sides.

The square A'B'C'D' (blue colour) has properties:
A'B' = C'D' = A'C' = B'C' = 2r

and

A'B'C'D' area = $4r^2$……………………… (1)

Similarly, A''B''C''D'' (red colour) has properties:

A''B'' = C''D'' = A''C' = B''C'' = $r\sqrt{2}$
and

Area of A''B''C''D'' = $2r^2$ ……………… (2)

Then from (1), (2) and $2 < \pi < 4$, we get:
$2r^2$ < assumed area πr^2 of the square ABCD < $4r^2$ ………………………………………………. (3)

α.) From (3), the square A'B'C'D' of area $4r^2$ contains the square ABCD of area πr^2 *(assumed, yellow colour)*, which in turns contains the inscribed A"B"C"D" (with area $2r^2$, *red colour in Figure 2 above*) of the given circle (O, r).

Therefore, square ABCD occupies the stretched area *(yellow colour in the above Figure 2)* in between squares A'B'C'D' & A''B''C''D'', as required.

β.) Let R be a radius of the inscribed circle (yellow dashes in the above Figure 2) of the square ABCD, then $R = \frac{r\sqrt{\pi}}{2}$, which is a half of the side $r\sqrt{\pi}$ of the square ABCD. And then $R = \frac{r\sqrt{\pi}}{2} < r$ to result that the circle (O, r) contains the inscribed circle (O, $R=\frac{r\sqrt{\pi}}{2}$) of ABCD, as $\frac{\sqrt{\pi}}{2} < 1$.

Therefore the given circle (O, r) contains the circle (O, $\frac{r\sqrt{\pi}}{2}$), as required.

γ.) By (3), the area πr^2 of ABCD is less than the area $4r^2$ of A'B'C'D'. The circle (O, r) is the inscribed circle of square A'B'C'D'. Therefore, the 4 sides of square ABCD intersect the circle (O, r) to make 4 equal segments (*which are formed by arc chords and 4 small arcs of the circle (O, r), as described in the above Figure 2, - the* ***red shape*** *near the top of Figure 2 – attached to side AB - illustrates one of the mentioned circle segments*), as required.

Theorem 2: *"ANALYSIS METHOD"*

Given a circle (O,r). If there exists a square ABCD with area πr^2 (assumed), of which the centre is located at centre O of the circle (O,r), then 4 sides of the square ABCD are overlapped 4 non-consecutive sides of *a regular octagon abcdefgh, which is inscribed in the circle (O, r).*

PROOF: **"*ANALYSIS METHOD*"**

Assume there exists a square ABCD (yellow colour) of area πr^2 *(assumed)*, of which centre is located at the centre of the given circle (O, r) then by section γ of the above Theorem 1, the 4 circle segments, formed by the circle (O, r) and 4 sides of ABCD, are equal.

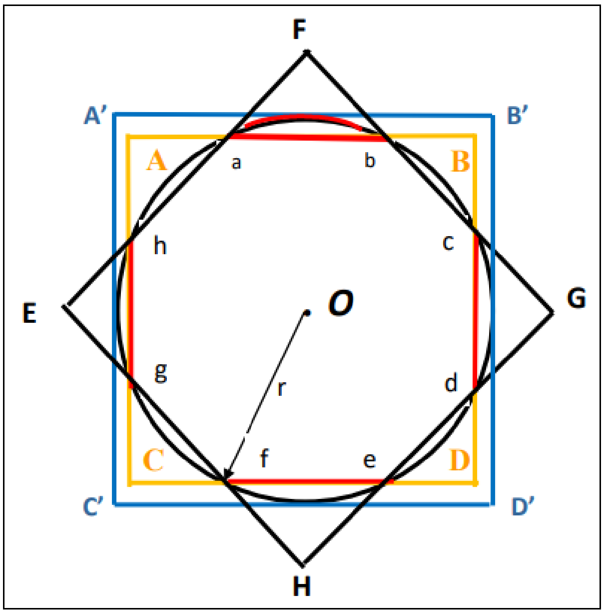


Figure 3: ***abcdefgh (red and black colours) is the regular Octagon inscribed in Circle (O, r)***

Consider the area of the Conical-Arc Aah in Figure 3 above (as defined in Definition 1 above) of the corner A of the square ABCD. From the expression {area πr^2 of ABCD = area πr^2 of the circle (O, r)}, we get the following expressions:

{the area of the Conical-Arc Aha = the area of the circle segment ab (red colour)**}** ………….. (4)

Similarly to (4),

{the area of the Conical-Arc Bbc = the area of the circle segment cd} ………….. (5)

{the area of the Conical-Arc Dde = the area of the circle segment ef}………….. (6)

{the area of the Conical-Arc Cfg = the area of the circle segment gh} ………….. (7)

Note that all expressions (4), (5), (6) & (7) above are illustrated in Figure 3 above.

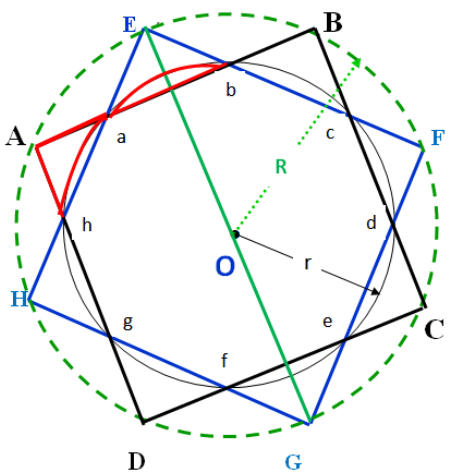


Figure 4: ***2 inscribed squares ABCD (red and black colours) & EFGH (blue colour) of the circle (O, R).***

Let centre O of the circumscribed circle (O, R) – green colour - of square ABCD be the same centre O as the given circle (O, r) – black colour. Then, lengthen the arc chord ah of (O, r) in Figure 4 above that meets the circumscribed circle (O, R) – marked green dashes in Figure 4 above - at E and H. And then, connect the diameter of (O, R) which gets through E & O. From E, draw a symmetric chord to EH that meets the green dashes Circle (O, R) at F. The special octagon abcdefgh inscribed in the given circle (O, r) with 4 equal & parallel side pairs, and Section γ of Theorem 1 shows that EF is the symmetric chord of EH through the symmetric EG-axis (green colour). From Section γ of Theorem 1 above, the distances between O and the 2 chords ha & bc are the same and this equality shows chord EF (Figure 4) in the green dash Circle (O, R) overlaps chord bc of the given circle (O, r). Similarly, chord FG in the green dash Circle (O, R) also overlaps chord de (Figure 4 above) of the given black circle (O, r). By Section γ of Theorem 1, FG // EH, then chords EF & GH of the green dash Circle (O, R) are equal and parallel. This implies

EF = FG = GH = HE …………….. (8)

and

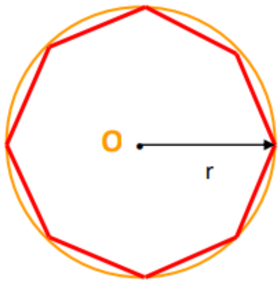
EFGH (blue) is the inscribed square of the circle (O, R) ……………………………..…….. (9)

Then (8) and (9) show that the areas of the two squares ABCD (black) & EFGH (blue) are the same, and equal to πr^2.

Locations of 8 sides of the equal squares ABCD & EFGH above show 8 chords ab, bc, cd, de, ef, fg, gh & ha of the given circle (O, r) are equal. Therefore, these 8 equal chords show the shape abcdefgh is a regular octagon that inscribes in the given circle (O, r), as required.

<u>Definition 2</u>: **2D-SQUARING RULER**

Given a circle (O, r) then a regular octagon, inscribed in the circle, is defined as a 2D-Squaring Ruler – coloured red in Figure 5 below.



<u>Figure 5</u>: a 2D-Squaring Ruler (red) of the circle (O,r) (orange)

Theorem 3:

Given a circle (O, r) with area πr^2, then the square ABCD of an area πr^2 is formed by lengthening the 4 non-consecutive sides of a regular octagon abcdefgh which is the inscribed regular octagon of the circle (this circle O is the circumscribed circle of the octagon abcdefgh, which is defined 2D-Squaring Ruler in Definition 2 above).

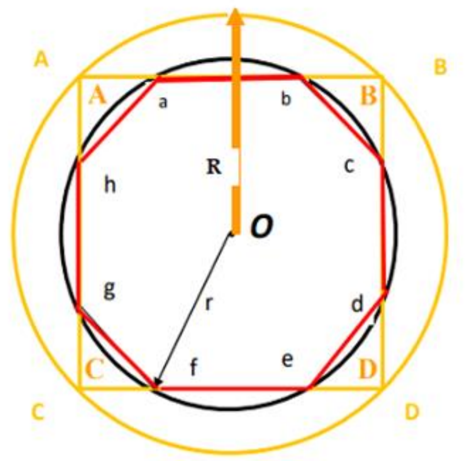
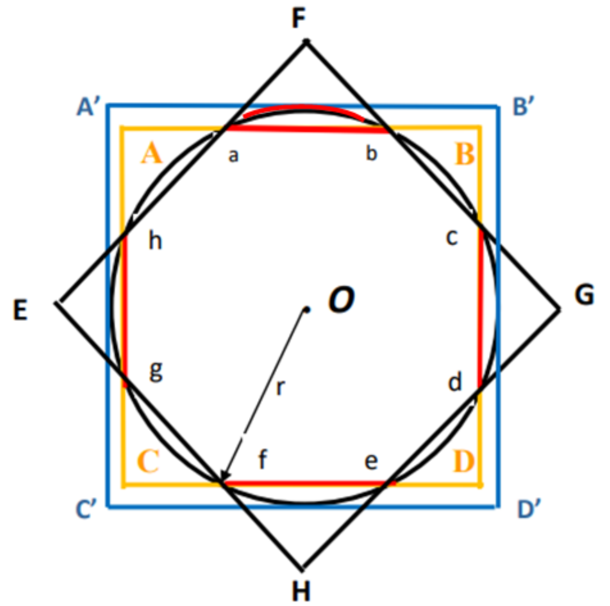


Figure 6: ***The given circle (O, r) and the regular octagon abcdefgh.***

PROOF:

Given a circle (O, r) then by Definition 2 above, we can construct a 2D-Squaring Ruler abcdefgh (red colour in Figure 6 above) inscribed in the circle. And then, by Theorem 2 & Theorem 3 above, the square ABCD (coloured orange in Figure 6 above) is the square with area πr^2, which is equal to the area πr^2 of the given circle (black). Thus the given black circle (O, r) is squared exactly into the yellow square ABCD, as required (Figure 6 above).



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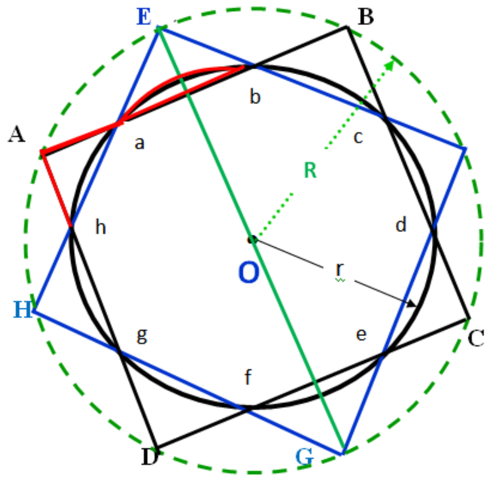

Figure 4: *2 inscribed squares ABCD (red and black colours) & EFGH (blue colour) of the circle (O, R).*

Let the centre of the circumscribed circle (O, R) – green colour - of ABCD be the same centre O of the given circle (O, r) – black colour. Then lengthen the arc chord ah (Figure 4 above) to meet the circumscribed circle (O, R) – marked green dashes in Figure 4 above - at E and H. And then, connect the diameter of (O,R) which gets through E & O. From E, draw a symmetric chord to **EH** that meets the green dash Circle (O, R) at F. The special octagon abcdefgh inscribed in the given circle (O, r) with 4 equal & parallel side pairs, and Section γ of **Theorem 1** shows that **EF** is the symmetric chord of **EH** through the symmetric **EG-axis** (green colour). From Section γ of Theorem 1 above, the distances between O and the 2 chords **ha & bc** are the same and this equality shows chord **EF** (Figure 4) in the green dash Circle (O, R) overlaps chord **bc** in the given circle (O, r). Similarly, chord **FG** in the green dashes Circle (O, R) also overlaps chord **de** (Figure 4 above) in the given black circle (O, r). By Section γ of Theorem 1, FG // EH, then chords EF & GH of the green dash Circle (O, R) are equal and parallel. This implies

EF = FG = GH = HE ………………. (8)

and

EFGH (blue) is the inscribed square of the circle (O, R) …………….. (9)

Then (8) and (9) show that the areas of the two squares ABCD **(black) & EFGH (blue)** are the same, and equal to πr^2.

Locations of 8 sides of the equal squares ABCD & EFGH above show 8 chords ab, bc, cd, de, ef, fg, gh & ha of the given circle (O, r) are equal. Therefore, these 8 equal chords show the shape abcdefgh is the regular octagon that inscribes in the given circle (O, r), as required.

Part III

EXACT "SQUARING THE CIRCLE" METHOD WITH STRAIGHTEDGE AND COMPASS

Given a circle (O, r) with area πr^2, a straightedge, and a compass.

Step 1: *Use the given straightedge & compass to construct a 2D-Squaring Ruler (red colour in Figure 7 below), which is exactly a regular octagon, inscribed in the given circle (O, r).*

Step 2: *Lengthen the 2 pairs of the parallel sides of the 2D-Squaring Ruler, which meet at 4 points A, B, C, and D. Then ABCD is the square with area πr^2 (yellow colour in Figure 7 below), as required.*

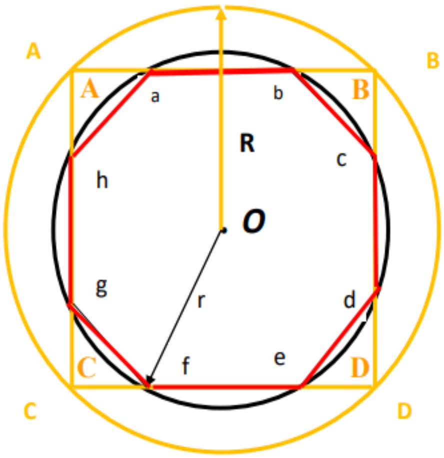

Figure 7: Square ABCD is a result of the exact "Squaring the Circle".

Results of this research show that the square ABCD has the exact area πr^2, therefore if the given circle is a unit circle r = 1 then in terms of Geometry, π can exactly be constructive / expressed by a square with area π. This square comes from the *2D-Squaring Ruler* of the unit circle (O, 1), by *Step 1 & Step 2*, above.

Moreover,

- In the above Figure 6, AB = $\sqrt{\pi r^2}$ = $r\sqrt{\pi}$. Then if the given circle to do squaring is a unit circle r = 1, then in terms of Algebraic Geometry, $\sqrt{\pi}$ can be exactly constructive or expressed by a geometric LENGTH. This length is a side of the square resulting from the solution of "squaring the unit circle (O, 1) problem" by *Step 1 & Step 2* above.

- We can find out a geometric length for the radius R of the circumscribed circle (O, R) of square ABCD (coloured yellow in Figure 7 above), in terms of π as follows:

  From the right angle triangle OAB in Figure 7 above we get

  $R^2 + R^2 = \pi r^2$; then $\mathbf{R} = \frac{r\sqrt{2\pi}}{2}$.

Part IV

DISCUSSION AND CONCLUSION

Can mathematicians use a compass and a straightedge to construct a square of equal area to a given circle? Surprisingly, mathematicians are still working on this question. In January 2022, a paper posted online by Andras Máthé and Oleg Pikhurko of the University of Warwick and Jonathan Noel of the University of Victoria is the latest to join in this ancient tradition challenge. These authors show how a circle can be squared by cutting it into pieces that can be visualized and possibly drawn. It's a result that builds on a rich history. Mathematicians named this method *"the equidecomposition",* but it is also a theoretical proof that the problem can be solved (without straightedge & compass) by cutting the circle into pieces and rearranging them into a square and none knows the number of pieces. Nevertheless, no computer existed in that ancient Greek era.

Results of my independent research show that the square ABCD, constructed by compass & straightedge, has the exact area πr^2, therefore if the given circle to square is a unit circle r = 1 then in terms of Geometry, π can exactly be constructive / expressed by a square with area π. This square comes from the *2D-Squaring Ruler* of the unit circle (O, 1), by Section 1 & Section 2, above. Then, the exact geometric length of $\sqrt{\pi}$ was found.

Moreover,

- In the above Figure 6, AB = $\sqrt{\pi r^2}$ = $r\sqrt{\pi}$. Then if the given circle to square is a unit circle r = 1 then in terms of Algebraic Geometry, $\sqrt{\pi}$ can be exactly constructive or expressed by a geometric LENGTH. This length is a side of the square that comes from the solution of "squaring the unit circle (O, 1) problem" by Section 1 & Section 2, above.
- We can find out a geometric length – in terms of π - for the radius R of the circumscribed circle (O, R) of square ABCD (coloured yellow in Figure 6 & Figure 7 above), as follow:

  $\mathbf{R} = \frac{r\sqrt{2\pi}}{2}$

  This research result is also a new method to calculate the arithmetic value π as follows: *with an accurate length 2 metres given by the International Bureau of Weights and Measures (BIPM) or the International System of Units, we* *use a compass to construct a circle, of which area is accurately 4π.* Then use a straightedge & a compass to construct an accurate square, of which area is also accurately 4π, as proved above. Laser measurement can be used to measure as much accurately as possible to have the arithmetic value of 4π, say it is equal to A. And then,

$$\mathbf{4\pi = A \Rightarrow \pi = \frac{A}{4}}$$

The above arithmetic value of $\pi = \frac{A}{4}$ *could be the nearest arithmetical value of π ever seen.*

My construction method is quite different from approximation and based on the use of a straightedge and a compass within A-Level Geometry, so that any secondary student can solve the problem for any given circle. Moreover, the method shows that $\sqrt{\pi}$ can be expressed accurately in 1-D space and π can be expressed accurately in 2-D space in terms of Geometry. This Geometrical expression of the irrational number π could be an interest field for mathematicians in the 21st Century. In the other words, Algebraic Geometry can express exactly any irrational number kπ, $k \subset \mathbf{R}$.

Upstream from this method of exact "squaring the circle", we can deduced, conversely, to get a new mathematical challenge "**Circling the Square**" with a straightedge & a compass. In details, we can describe it as follows:

Given a square with side a, a ⊂ R, then use a straightedge & a compass to construct an accurate circle, which has the exact area a². Then, how do to solve this new geometry problem is still an open research interest, in order to get a circle area without using the traditional constant π.

In addition, this research result can be used for a further research in the "CUBING THE SPHERE" challenge, with only "a straightedge & compass" in Euclidean Geometry.

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REFERENCES

1. Karl Popper, 21 February 2002, The Logic of Scientific Discovery, 2nd edition, page 544, London, Routledge, ISBN9780203994627, Doi: https://doi.org/10.4324/9780203994627, [Publisher link].
2. Rolf, W. (2000). On Lambert's proof of the irrationality of π, Published by De Gruyter, https://www.semanticscholar.org/paper/On-Lambert's-proof-of-the-irrationality-of-%CF%80 Wallisser/43f1fe3182f2233a1a9f313649547d13ea7311c7.
3. Albertini, T. (1991). La quadrature du cercle d'ibn al-Haytham: solution philosophique ou mathématique?, *J. Hist. Arabic Sci*, *9*(1-2) 5-21, 132. https://www.academia.edu/9342794/_La_quadrature_du_cercle_dIbn_al_Haytham_solution_math%C3%A9matique_ou_philosophique_Ibn_al_Haythams_Quadratura_Circuli_a_mathematical_or_a_philosophical_solution_translation_and_commentary_Journal_for_the_History_of_Arabic_Science_9_1991_5_21
4. Wantzel, L. (1837). Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. *Journal de mathématiques pures et appliquées*, *2*, 366-372.
5. Borwein, J. M., Borwein, P. B., & Plouffe, S. (1997), "The quest for pi", *The Mathematical Intelligencer*, *19*(1), 50-57, doi:10.1007/BF03024340, MR 1439159. S2CID 14318695.
6. Laczkovich, M. (1997). "On Lambert's proof of the irrationality of π", *The American Mathematical Monthly*, *104*(5), 439–443, doi:10.1080/00029890.1997.11990661, JSTOR 2974737, MR 1447977.
7. Hobson, E. W. (1913). Squaring the Circle: A History of the Problem, Cambridge University Press. 34–35.
8. Dixon, R. A. (1987). "Squaring the circle", Mathographics, *Blackwell*, 44–47, Reprinted by Dover Publications, 1991.
9. Beatrix, F. (2022). "Squaring the circle like a medieval master mason", Parabola, *UNSW School of Mathematics and Statistics*, *58*(2).
10. Bird, A. (1996). "Squaring the Circle: Hobbes on Philosophy and Geometry", *Journal of the History of Ideas*, *57*(2), 217–231, doi:10.1353/jhi.1996.0012, S2CID 171077338.
11. Steve, N. (2022). An Ancient Geometry Problem Falls to New Mathematical Techniques, Gemometry, Quanta Magazine,https://www.quantamagazine.org/an-ancient-geometry-problem-falls-to-new-mathematical-techniques-20220208/.
12. Tubbs, R. (2020). "Squaring the circle: A literary history", In Tubbs, Robert; Jenkins, Alice; Engelhardt, Nina (eds.), The Palgrave Handbook of Literature and Mathematics, Springer International Publishing, 169–185, doi:10.1007/978-3-030-55478-1_10, MR 4272388. S2CID 234128826.

https://link.springer.com/chapter/10.1007/978-3-030-55478-1_10 .

13. Otero, D. E. (2010). "The Quadrature of the Circle and Hippocrates' Lunes", Convergence, *Mathematical Association of America*.