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On the Global Existence of Solution and Lyapunov Asymptotic Practical Stability for Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle

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Abstract **Review Article**

This paper examines the existence of maximal solution of the comparison differential system and also establishes sufficient conditions for the asymptotic practical stability of the trivial solution of a nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse using the vector Lyapunov functions. First, it was discovered that the vector form of the Lyapunov function was majorized by the maximal solution of the comparison system. From the results obtained, it was also established that the main system is asymptotically practically stable in the sense of Lyapunov.

Keywords: Asymptotic Practical Stability, Caputo Derivative, Impulse, Lyapunov Function
MSC: 34A12; 34A37; 34D05.

1. INTRODUCTION

The study of the concept of fractional calculus which is mainly concerned with the pure mathematical fields is traceable to the 19th century by Liouville, Riemann, Caputo, etc [29, 39].

One of the trends in the stability theory of solutions of differential equations is the so-called practical stability [10-35]. This aspect of stability was introduced by LaSelle in [26], and it is used in estimating the worst-case transient and steady-state responses together with verifying point wise in time constraints imposed on the solution path or the trajectory curve. Fundamental results in this area were established in [10,22,34-36,42] for integer order derivative.

Rapidly revolving alongside the development of the theory of practical stability in recent years is the mathematical theory of impulsive differential equations which have experienced a massive research attention and development. The theory of impulsive differential equation is richer than the corresponding theory of differential equations [18], as they constitutes very important models for describing the true state of several real life processes and phenomena since many evolution processes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly. These processes are assumed to be subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. For instance, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems do exhibit impulsive effects [18].

Again, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [37], and one of the most versatile methods in the study of the stability properties of impulsive systems is the Lyapunov second method (see [6]).

The novelty of the Lyapunov's second method as observed in [6-40] over other methods of examining stability properties of impulsive differential systems like the Razumikhin technique, the use of matrix inequality, the Laplace transform method, variational method, etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential equation - by seeking an appropriate continuously differentiable function (Lyapunov's function) that is positive definite, whose time derivative along the trajectory curve is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [13] and [32], and fundamental results have been obtained for its corresponding fractional order in [3-46]. Atsu in [6] obtained fundamental results on the practical stability of impulsive Caputo fractional differential equations using the vector Lyapunov functions, stressing the importance of the method over the scalar Lyapunov function.

In this paper, the asymptotic practical stability of impulsive Caputo fractional order systems is considered, and by means of the comparison principle, sufficient conditions for the asymptotic practical stability of impulsive fractional order systems is established using a class of piecewise continuous functions. An illustrative example is given to confirm the suitability of the obtained results.

2. Preliminaries and Basic Definitions

The basic concept of calculus such as the derivative and integrals can be generalized to noninteger order using fractional calculus. This allows for more in-depth understanding of behavior of functions, particularly when they have complex or irregular behavior. There are multiple ways to define fractional derivatives and the integrals and the choice of definitions depends on the specific applications (see [16-39]).

There are several definitions of fractional derivatives and fractional integrals.

General Case: Let the number $n-1<\beta<n, \beta>0$ be given, where n is a natural number, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.1

The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}^{RL}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t\geq t_0 $$

Definition 2.2

The Caputo fractional derivative of order β of $\gamma(t)$ is given by (see [36])

$$ {}^{C}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t\geq t_0 $$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3

The Grunwald-Letnikov fractional derivative of order β of $\gamma(t)$ is given by (See [2])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0 $$

and

Definition 2.4: The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (See [2])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0 $$

where ${}^{\beta}C_r$ are the binomial coefficients and $\left[\frac{t-t_0}{h}\right]$ denotes the integer part of $\frac{t-t_0}{h}$.

Particular Case (when n=1). In most applications, the order of β is often less than 1, so that $\beta \in (0,1)$. For simplicity of notation, we will use ${}^{C}D^{\beta}$ instead of ${}^{C}_{t_0}D^{\beta}$ and the Caputo fractional derivative of order β of the function $\gamma(t)$ is

$$ {}^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{-\beta}\gamma'\,ds, \quad t \ge t_0 \tag{2.1} $$

3. Impulses in Fractional Differential Equations

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < \beta < 1$.

$$ \begin{gathered} {}^{C}D^{\beta}\gamma(t) = f(t,\gamma),\ t \ge t_0, \\ \gamma(t_0) = \gamma_0, \end{gathered} \tag{3.1} $$

where $\gamma \in R^N,\ f \in C[R_+ \times R^N, R^N],\ f(t,0) \equiv 0\ and\ (t_0, x_0) \in R_+ \times R^N$.

Some sufficient conditions for the existence of the global solutions to (3.1) are considered in [8-43].
The IVP for FrDE (3.1) is equivalent to the following Volterra integral equation (See [3]),

$$ \gamma(t) = \gamma_0 + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1} f(s,\gamma(s))ds,\ t \ge t_0 \tag{3.2} $$

Consider the IVP for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < \beta < 1$,

$$ \begin{gathered} {}^{C}D^{\beta}\gamma(t) = f(t,\gamma),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,\ldots \\ \Delta\gamma = I_k(\gamma(t_k)),\ t = t_k,\ k \in N, \\ \gamma(t_0^+) = \gamma_0, \end{gathered} \tag{3.3} $$

where $\gamma, \gamma_0 \in R^N,\ f \in C[R_+ \times R^N, R^N],\ and t_0 \in R_+\ I_k : R^N \to R^N,\ k = 1,2,\ldots$ under the following assumptions:

$(i)\ 0 < t_1 < t_2 < \ldots < t_k < \ldots,\ and\ t_k \to \infty\ as k \to \infty$; $(ii)\ f : R_+ \times R^N \to R^N$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $x \in R^N, k = 1,2,\ldots,$ and $\lim\limits_{(t,y)\to(t_k^+,\gamma)} f(t,y) = f(t_k^+,\gamma)$ exists;

$(iii)\ I_k \times R^N \to R^N$

In this paper, we assume that $f(t,0) \equiv 0,\ I_k(0) \equiv 0$ for all k so that we have trivial solution for (3.3), and the points $t_k,\ k = 1,2,\ldots$ are fixed such that $t_1 < t_2 < \ldots.\ and\ \lim\limits_{k\to\infty} t_k = \infty$. The system (3.3) with initial condition $\gamma(t_0) = \gamma_0$ is assumed to have a solution $\gamma(t; t_0, \gamma_0) \in PC^{\beta}([t_0,\infty), R^N)$.

Remark 3.1. The second equation in (3.3) is called the impulsive condition, and the function $I_k(\gamma(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 3.1 Let $\Omega : R_+ \times R^N \to R^N$ Then Ω is said to belong to class χ if,

(i) Ω is continuous in $(t_{k-1}, t_k]$ and for each $\gamma \in R^N$ and $\lim\limits_{(t,y)\to(t_k^+,\gamma)} \Omega(t,y) = \Omega(t_k^+,\gamma)$ exists;

(ii) Ω is locally Lipschitz with respect to its second argument x and $\Omega(t,0) \equiv 0$

Now, for any function $\Omega(t,\gamma) \in PC([t_0,\infty) \times \xi, R_+^N)$. we define the Caputo fractional Dini derivative as:

$$ {}^{C}D_+^{\beta}\Omega(t,\gamma) = \limsup_{h\to 0^+}\frac{1}{h^{\beta}}\{\Omega(t,\gamma) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\,{}^{\beta}C_r[\Omega(t-rh, \gamma - h^{\beta} f(t,\gamma) - \Omega(t_0,\gamma_0)]\} \tag{3.4} $$

$t \ge t_0$ where $t \in [t_0,\infty),\ \gamma, \gamma_0 \in \xi,\ \xi \in R^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T]$.

Definition 3.2 A function $g \in PC[R^n, R^n]$ is said to be quasimonotone nondecreasing in γ, if $\gamma \le y$ and $\gamma_i = y_i$ for $1 \le i \le n$ implies $g_i(\gamma) = g_i(y)$.

Definition 3.3 The zero solution of (3.3) is said to be:
(PS1) practically stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \le \delta$ Implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \ge t_0$;
(PS2) uniformly practically stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$, continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \le \delta$ implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \ge t_0$;
(PS3) asymptotically practically stable if it is practically stable and if for each $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \ge t_0 + T$ and $\|\gamma_0\| \le \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$;
(PS4) uniformly asymptotically practically stable if it is uniformly practically stable and $\delta_0 = \delta_0(\in)$ and $T = T(\varepsilon)$ such that for $t \ge t_0 + T$, the inequality $\|\gamma_0\| \le \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Definition 3.4 A function $a(r)$ is said to belong to the class K if $a \in PC([0, \psi), R_+)$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing in r.
In this paper, we define the following sets:

$$\overline{S}_\psi = \{\gamma \in R^N : \|\gamma\| \le \psi\}$$
$$S_\psi = \{\gamma \in R^N : \|\gamma\| < \psi\}$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.
We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned} {}^{C}_{t_0}D^\beta u &= g(t,u),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,... \\ \Delta u &= \psi_k(u(t_k)),\ t = t_k,\ k \in N, \\ u(t_0^+) &= u_0, \end{aligned} \tag{3.5}$$

existing for $t \ge t_0$,

$u \in R^n$, $R_+ = [t_0, \infty)$, $g : R_+ \times R_+^n \to R^n$, $g(t, 0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (3.5) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC^\beta([t_0, \infty), R^n)$.

Lemma 3.2 [1]Assume $m \in PC([t_0, T] \times \overline{S}_\psi, R^N)$. and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and
$m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \le t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^{C}D_+^\beta m(t^*, \alpha_1) - {}^{C}D_+^\beta m(t^*, \alpha_2) > 0$ holds.

Proof Let $\Omega(t, \gamma) = m(t, \alpha_1) - m(t, \alpha_2)$. Applying (3.4), we have

$${}^{C}D_+^\beta(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h\to 0^+} \frac{1}{h^\beta}\{[m(t^*,\alpha_1) - m(t^*,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\,{}^{\beta}C_r[m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)]\}$$

when $\alpha_1 = \alpha_2$ we have

$${}^{C}D_+^\beta(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h\to 0^+} \frac{1}{h^\beta}\{-[m(t_0,\alpha_1) - m(t_0,\alpha_2)] - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\,{}^{\beta}C_r[m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)]\}$$

$${}^{C}D_+^\beta(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = -\limsup_{h\to 0^+} \frac{1}{h^\beta}[m(t_0,\alpha_1) - m(t_0,\alpha_2)] + \limsup_{h\to 0^+} \frac{1}{h^\beta}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] - \limsup_{h\to 0^+} \frac{1}{h^\beta}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1) - m(t_0,\alpha_2)]$$

$$ {}^{C}D_{+}^{\beta}(m(t^*,\alpha_1)-m(t^*,\alpha_2)) = -\limsup_{h\to 0^+}\frac{1}{h^{\beta}}[m(t_0,\alpha_1)-m(t_0,\alpha_2)] - \limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)] $$

$$ {}^{C}D_{+}^{\beta}(m(t^*,\alpha_1)-m(t^*,\alpha_2)) = -\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)] $$

Applying equation (3.8) in [2], we have

$$ {}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) = -\frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)}\left[m(t^*,\alpha_1)-m(t^*,\alpha_2)\right] $$

By the lemma, we have that

$$ m(t,\alpha_1)-m(t,\alpha_2) < 0 \quad for\ t_0 \le t \le t^* $$

And so, it follows that

$$ {}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) > 0 $$

Note that some existence results for (3.5) are given in [12-15].

Remark 3.3 Lemma 3.2 in [1], extends Lemma 1 in [2], where the vectors $m(t,\alpha_1)\ and\ m(t,\alpha_2)$ are compared component-wise.

In the following, we establish the comparison result for the system (3.3).

4. Fractional Differential Inequalities and Comparison Results for Vector Fractional Differential Equation

In this section, again we assume that $0<\beta<1.$

Theorem 4.1. (Comparison Result) [1].

Assume that:

(i) $g \in PC[R_+ \times R_+^n,\ R^n]$ and is continuous in $(t_{k-1},t_k],\ k=1,2,...\ and\ g(t,u)$ is quasimonotone non-decreasing in u for each $u\in R^n\ and \lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

(ii) $\Omega \in PC[R_+\times R^N, R_+{}^N]\ and\ \Omega\in\chi$ such that ${}^{C}D_{+}^{\beta}\Omega(t,\gamma) \le g(t,\Omega(t,\gamma)),\ (t,\gamma)\in R_+\times R^N$ and and the function $\Omega(t_k,\gamma+I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t=t_k, \gamma\in S_\psi\ \omega_k : R_+^N \longrightarrow R_+^N$ is nondecreasing for $k=1,2,...$

(iii) $\vartheta(t)=\vartheta(t,t_0,u_0)\in PC^{\beta}([t_0,T],R^n)$ be the maximal solution of the IVP for the IFrDE system (3.5)

Then,

$\Omega(t,\gamma(t)) \le \vartheta(t),\ t\ge t_0,$ (4.1)

where $\gamma(t)=\gamma(t,t_0,\gamma_0)\in PC^{\beta}([t_0,T],R^N)$ is any solution of (3.3) existing on $[t_0,\infty),$ provided that

$\Omega(t_0^+,\gamma_0)\le u_0$ (4.2)

Proof.

Let $\eta\in S_\psi$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations,

$$ \begin{aligned} {}^{C}D^{\beta}u &= g(t,u)+\eta, \quad for\ t\in[t_0,\infty) \\ u(t_0^+) &= u_0+\eta, \end{aligned} \tag{4.3} $$

for $t\in[t_0,\infty).$

The function $u_\eta(t,\alpha)$ is a solution of (4.3), where $\alpha > 0$ the fractional differential equation (3.5) if and only if it satisfies the Volterra fractional integral equation,

$$u_\eta(t,\alpha) = u_0 + \eta + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,u_\eta(s,\alpha)) + \alpha)ds, \ \ t \in [t_0,\infty). \tag{4.4}$$

Let the function $m(t,\alpha) \in PC([t_0,T] \times S_\psi, R_+^N)$ be defined as $m(t,\alpha) = \Omega(t,\gamma^*(t))$

We now prove that

$$m(t,\alpha) < u_\eta(t,\alpha) \quad for \ \ t \in [t_0,\infty) \tag{4.5}$$

Observe that the inequality (4.5) holds whenever $t = t_0$ *i.e.*

$$m(t_0,\alpha) = \Omega(t_0,\gamma_0) \le u_0 < u_\eta(t_0,\alpha)$$

Assume that the inequality (4.5) is not true, then there exists a point $t_1 > t_0$ such that

$$m(t_1,\alpha) = u_\eta(t_1,\alpha) \ \ and \ \ m(t,\alpha) < u_\eta(t,\alpha) \text{ for } t \in [t_0,t_1).$$

It follows from Lemma 3.2 that

$${}^C D_+^\beta (m(t_1,\alpha) - u_\eta(t_1,\alpha)) > 0 \ \ i.e.$$

$${}^C D_+^\beta (m(t_1,\alpha)) > {}^C D_+^\beta u_\eta(t_1,\alpha)$$

$${}^C D_+^\beta \Omega(t_1,\gamma(t_1)) > {}^C D_+^\beta u_\eta(t_1,\alpha)$$

and using (4.3) we arrive at

$${}^C D_+^\beta \Omega(t_1,\gamma(t_1)) > g(t_1,u(t_1,\alpha)) + \eta > g(t_1,u(t_1,\alpha)$$

Therefore,

$${}^C D_+^\beta m(t_1,\alpha) > g(t_1,u(t_1,\alpha)) \tag{4.6}$$

From Theorem 4.1, the function $\gamma^*(t) = \gamma(t,t_0,\gamma_0)$ satisfies the IVP (4.3) and the equality

$$\limsup_{h\to 0^+} \tfrac{1}{h^\beta}[\gamma^*(t) - \gamma_0 - S(\gamma^*(t),h)] = f(t,\gamma^*(t)), \ \ holds, \tag{4.7}$$

where $\gamma^*(t) = \gamma(t,t_0,\gamma_0)$ is any other solution of (3.5), and

$$S(\gamma^*(t),h) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,\beta} C_r \left[\gamma^*(t-rh) - \gamma_0\right] \tag{4.8}$$

is the Grunwald Letnikov fractional derivative

Multiply equation (4.7) through by h^β

$$\limsup_{h\to 0^+} [\gamma^*(t) - \gamma_0 - S(\gamma^*(t),h)] = h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t) - \gamma_0 - [S(\gamma^*(t),h) + \rho(h^\beta)] = h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t) - h^\beta f(t,\gamma^*(t)) = [S(\gamma^*(t),h) + \gamma_0 + \rho(h^\beta)] \tag{4.9}$$

Then for $t \in [t_0,\infty)$, we have

$$\begin{aligned}
&m(t,\alpha) - m(t_0,\alpha) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,\beta} C_r \left[m(t-rh,\alpha) - m(t_0,\alpha)\right] = \\
&\Omega(t,\gamma^*(t)) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,\beta} C_r \left[\Omega(t-rh,\gamma^*(t) - h^\beta f(t,\gamma^*(t)) - \Omega(t_0,\gamma_0)\right]\} \\
&= \Omega(t,\gamma^*(t)) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,\beta} C_r \left[\Omega(t-rh) - \gamma^*(t) - h^\beta f(t,\gamma^*(t)) - \Omega(t_0,\gamma_0)\right]\} + \\
&\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,\beta} C_r \{[\Omega(t-rh), S(\gamma^*(t),h) + \gamma_0 + \rho(h^\beta) - \Omega(t_0,\gamma_0)] - [\Omega(t-rh,\gamma^*(t-rh)) - \Omega(t_0,\gamma_0)]\}
\end{aligned} \tag{4.10}$$

Since $\Omega(t,\gamma)$ is locally Lipschitzian with respect to the second variable, we have that,

$$\leq L\left|(-1)^{r+1}\right|\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[S(\gamma^*(t),h)+\gamma_0+\rho(h^{\beta})-\gamma^*(t-rh)]\right\|$$

where $L>0$ is a Lipschitz constant.

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[S(\gamma^*(t),h)+\rho(h^{\beta}))-(\gamma^*(t-rh)-\gamma_0)\right\| \tag{4.11}$$

Using equation (4.8), equation (4.11) becomes,

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)[(\gamma^*(t-rh)-\gamma_0)+\rho(h^{\beta}))-(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)(-1)^{r+1}[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r[(\gamma^*(t-rh)-\gamma_0)]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r\rho(h^{\beta})-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L(-1)^{r+1}\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[(\gamma^*(t-rh)-\gamma_0)]+[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r\rho(h^{\beta})\right\| \tag{4.12}$$

Substituting equation (4.12) into (4.10) we have

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$
$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

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$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$
$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

Dividing through by $h^{\beta}>0$ and taking the $\limsup\ as\ h\to 0^+$ we have,

$${}^{C}D_+^{\beta}m(t,\alpha)=\limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}\left\{\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\right\}+$$
$$\limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

Recall that,

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)=-1\ and\ \limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}\rho(h^{\beta})=0$$

From equations (3.6) and (3.7) in [1], we have that

$${}^{C}D_+^{\beta}m(t,\alpha)={}^{C}D_+^{\beta}\Omega(t,\gamma^*(t))+L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-(-1)-1]+0\right\|$$

Using condition (ii) of the Theorem 4.1, we obtain the estimate

$${}^{C}D_+^{\beta}m(t,\alpha)\leq g(t,\Omega(t,\gamma^*(t))=g(t,m(t,\alpha)), \tag{4.13}$$

Also,

$$m(t_0^+,\alpha)\leq u_0\ and\ m(t_k^+,\alpha)=\Omega(t_k^+,\gamma(t_k)+I_k(\gamma(t_k))\leq\rho_k(m(t_k)) \tag{4.14}$$

Now equation (4.14) with $t=t_1$ contradicts (4.6), hence (4.5) is true. □

For $t\in[t_0,T]$, we now establish that

$$u_{\eta_1}(t,\alpha)<u_{\eta_2}(t,\alpha)\quad \text{whenever } \eta_1<\eta_2 \tag{4.15}$$

Observe that the inequality (4.15) holds for $t = t_0$

Assume that (4.15) is not true. Then there exists a point t_1 such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_2,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \quad for\ t \in [t_0,t_1).$

By Lemma 3.2, we have that

$$^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) > 0$$

However,

$$^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) = {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) \quad = g(t_1,u(t_1,\alpha)) + \eta_1 - [g(t_1,u(t_1,\alpha)) + \eta_2]$$
$$= \eta_1 - \eta_2 < \mathbf{0}$$

which is a contradiction, and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_\eta(t,\alpha)\},\ t \in [t_0,T]$ of (4.3) is uniformly bounded, i.e. there exists $\lambda > 0$ with $|u_\eta(t,\alpha)| \le \lambda$, with bound λ on $[t_0,T].$ We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0,T].$ Let $k = \sup\{|\ g(t,\gamma)\ |: (t,\gamma) \in [t_0,T] \times [-\lambda,\lambda]\},$ where λ is the bound on the family $\{u_\eta(t,\alpha)\}.$ Fix a decreasing sequence $\{\eta_i\}_{i=0}^{\infty}(t)$, such that $\lim_{i\to\infty}\eta_i = 0$ and consider a sequence of functions $\{u_\eta(t,\alpha)\}.$ Again, let $t_1,t_2 \in [t_0,T],\ with\ t_1 < t_2$, then we have the following estimates,

$$\begin{aligned}
\|u_\eta(t_2,\alpha) - u_\eta(t_1,\alpha)\| &\le \|\ u_0 + \eta_i + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}(t_2 - s)^{\beta-1}(g(s,u(s,\eta_i)) + \eta_i)ds \\
&\quad - \left(u_0 + \eta_i + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}(t_1 - s)^{\beta-1}(g(s,u(s,\eta_i)) + \eta_i)ds\right)\| \\
&\le \frac{1}{\Gamma(\beta)}\left\|\left[\int_{t_0}^{t_2}(t_2 - s)^{\beta-1} - \int_{t_0}^{t_1}(t_1 - s)^{\beta-1}\right](g(s,u(s,\eta_i))u(t_2,\eta_i)ds\right\| \\
&\le \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_2}(t_2 - s)^{\beta-1} - \int_{t_0}^{t_1}(t_1 - s)^{\beta-1}\right)ds\right\| \\
&\le \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_1}(t_2 - s)^{\beta-1} + \int_{t_1}^{t_2}(t_2 - s)^{\beta-1} - \int_{t_0}^{t_1}(t_1 - s)^{\beta-1}\right)ds\right\| \\
&\le \frac{k}{\Gamma(\beta)}\left\|\int_{t_0}^{t_1}(t_2 - s)^{\beta-1} - \int_{t_0}^{t_1}(t_1 - s)^{\beta-1} + \int_{t_1}^{t_2}(t_2 - s)^{\beta-1}ds\right\| \\
&\le \frac{k}{\Gamma(\beta)}\left[\left\|\int_{t_0}^{t_1}(t_2 - s)^{\beta-1} - \int_{t_0}^{t_1}(t_1 - s)^{\beta-1}ds\right\| + \left\|\int_{t_1}^{t_2}(t_2 - s)^{\beta-1}ds\right\|\right] \\
&\le \frac{k}{\beta\Gamma(\beta)}\left[\|(t_2 - t_0)^{\beta} - (t_1 - t_0)^{\beta} - (t_2 - t_1)^{\beta}\| + \|(t_2 - t_1)^{\beta}\|\right] \\
&\le \frac{k}{\beta\Gamma(\beta)}\left[(t_2 - t_1)^{\beta} + (t_2 - t_1)^{\beta}\right] = \frac{2k}{\Gamma(\beta+1)}(t_2 - t_1)^{\beta} < \varepsilon,
\end{aligned}$$

provided $\|t_2 - t_1\| < \delta(\varepsilon) = \left(\frac{\Gamma(\beta+1)\varepsilon}{2k}\right)^{\frac{1}{\beta}}$, proving that the family of solutions $\{u_\eta(t,\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t,\alpha)\}$ guarantees the existence of a subsequence $\{u_{\eta_{ij}}(t,\alpha)\}$ which converges uniformly to the function $\vartheta(t)$ on $[t_0,T].$ Then we show that $\vartheta(t)$ is a solution of (4.4). Thus, equation (4.4) becomes

$$u_{\eta_{ij}}(t,\alpha) = u_{0i_j} + \eta_{i_j} + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g_{i_j}(s,u_{\eta_{i_j}}(s,\alpha)) + \eta_{i_j})ds,\ \ t \in [t_0,\infty). \tag{4.16}$$

Taking the $\lim as\ i_j \to \infty$ in (4.16) yields,

$$\vartheta(t) = u_0 + \eta_{i_j} + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,\vartheta(t))ds,\ \ t \in [t_0,\infty). \tag{4.17}$$

Hence, $\vartheta(t)$ is a solution of (3.5) on $[t_0,T]$. We claim that $\vartheta(t)$ is the maximal solution of (3.5). Then, from (4.5), we have that

$$m(t,\alpha) < u_\eta(t,\alpha) \le \vartheta(t)\ \ on\ [t_0,T].$$

Suppose that in Theorem 4.1, $g(t,u) \equiv 0$, then we have the following results

Corollary 4.1.

Assume that Condition (i) of Theorem 4.1 holds and,

(i) $\quad \Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$$^{C}D_+^{\beta}\Omega(t,\gamma) \le 0 \tag{4.14}$$

holds, and

$\Omega(t_k,\gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$
Then for $t \in [t_0,\infty)$, the inequality

$\Omega(t,\gamma(t)) \le \Omega(t_0^+,\gamma_0)$ holds

5. Main Results

In this section, we will obtain sufficient conditions for the practical stability as well as asymptotic practical stability of the system (3.3). Again we assume $0 < \beta < 1.$

Theorem 5.1. Assume that:

(i) $\quad g \in PC[R_+ \times R_+^n,\ R^n]$ is piecewise continuous in $(t_{k-1},t_k]$ and for each $u \in R^n$, $k = 1,2,...,$ and $\lim\limits_{(t,y)\to(t_k^+,u)} g(t,y) = g(t_k^+,u)$ exists, and $g(t,u)$ is quasimonotone nondecreasing in u

(ii) $\quad \Omega \in PC[R_+ \times R^N, R_+{}^N]\ and\ \Omega \in \chi$ such that

$$^{C}D_+^{\beta}\Omega(t,\gamma) \le g(t,\Omega(t,\gamma)), t \ne t_k,\ holds\ \ for\ all\ (t,\gamma) \in R_+ \times S_\psi,$$

There exists $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and $\Omega(t_k,\gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iii) $\quad b(\|\gamma\|) \le \Omega_0(t,\gamma),$ where $b \in K$ and $\Omega_0(t,\gamma), = \sum_{i=1}^{N}\Omega_i(t,\gamma).$

Then the practical stability of the trivial solution $u = 0$ of (3.5) implies the practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (3.5) is stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} < \delta \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), t \geq t_0 \qquad (5.1)$$

where $u(t, t_0, u_0)$ is any solution of (3.5).

Choose $u_0 = \Omega(t_0^+, \gamma_0)$.

Since $\Omega(t, x)$ is continuous, then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \delta) > 0$ that is continuous in t_o for each δ such that the inequalities

$$\|\Omega(t, \gamma) - \Omega(t_0, \gamma_0)\| < \delta \text{ implies } \|\gamma - \gamma_0\| < \delta_1$$

and as $\|\Omega(t, \gamma)\| \to 0, \|\gamma\| \to 0$ then the inequalities

$$\|\gamma_0\| < \delta_1 \quad and \quad \|\Omega(t_0, \gamma_0) < \delta\| \qquad (5.2)$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta_1$ then $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Suppose that this claim is false, then there would exists a point $t_1 \in [t_0, t)$ and the solution $\gamma(t, t_0, \gamma_0)$ with $\|\gamma_0\| < \delta_1$ such that

$$\|\gamma(t_1)\| = \varepsilon \quad and \quad \|\gamma(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \qquad (5.3)$$

So that using equation (5.3); condition (iii) of Theorem 5.1 reduces to the form

$$b(\|\gamma(t_1)\|) \leq \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1)),$$

implying

$$b(\varepsilon) \leq \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1)) \qquad (5.4)$$

for $t \in [t_0, t_1)$ and from Theorem 4.1,

$$\Omega(t, \gamma(t) \leq \vartheta(t) \qquad (5.5)$$

where $\vartheta(t) = \sum_{i=1}^{n} \vartheta_i(t, t_0, u_0)$ is the maximal solution of (3.5).

Then, using equations (5.4), (5.3), (5.5) and condition (iii) of Theorem 5.1 we arrive at the estimate

$$b(\varepsilon) \leq \Omega_0(t_1, \gamma(t_1)) \leq \sum_{i=1}^{N} \vartheta_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the practical stability of the trivial solution $u = 0$ of (3.5) implies the practical stability of the trivial solution $\gamma = 0$ of (3.3).

Theorem 5.2. Assume that:

(i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n, k = 1,2,...,$ and $\lim_{(t,y)\to(t_k^+,u)} g(t, y) = g(t_k^+, u)$ exists, and $g(t, u)$ is quasimonotone nondecreasing in u

(ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$${}^C D_+^\beta \Omega(t, \gamma) \leq g(t, \Omega(t, \gamma)), t \neq t_k, \textit{ holds for all } (t, \gamma) \in R_+ \times S_\psi,$$

There exists $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and $\Omega(t_k, \gamma + I_k(\gamma(t_k))) \leq \omega_k(\Omega(t, \gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iii) $b(\|\gamma\|) \leq \Omega_0(t,\gamma)$, for all $(t,\gamma) \in R_+ \times S_\Psi$, where $b \in K$ and $\Omega_0(t,\gamma),= \sum_{i=1}^{N} \Omega_i(t,\gamma)$.

Then the asymptotic practical stability of the trivial solution $u = 0$ of (3.5) implies the asymptotic practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the trivial solution (3.5) is asymptotically practically stable. Then it is practically stable, and given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} \leq \delta \text{ implies } \sum_{i=1}^{N} u_i(t,t_0,u_0) < b(\varepsilon),\, t \geq t_0 \tag{5.6}$$

Since $\Omega(t,x)$ is continuous, then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o,\delta) > 0$ that is continuous in t_o for each δ such that the inequalities

$$\|\Omega(t,\gamma) - \Omega(t_0,\gamma_0)\| < \delta \text{ implies } \|\gamma - \gamma_0\| < \delta_1$$

and as $\|\Omega(t,\gamma)\| \to 0,\ \|\gamma\| \to 0$ then the inequalities

$$\|\gamma_0\| < \delta_1 \;\; and \;\; \|\Omega(t_0,\gamma_0) < \delta\| \tag{5.7}$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta_1$, then $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon$.

Now, suppose this claim is false, then there exists a sequence $\{t_n\}$ such that $t_n \to \infty \; as \; n \to \infty \;\; and \; t_n \geq t_0 + T$ such that

$\|\gamma(t_k,t_0,\gamma_0)\| \geq \varepsilon$, where $\gamma(t_k,t_0,\gamma_0)$ is some solution of (3.3) starting in $\|\gamma_0\| < \delta_0$.

Now, from condition (iii) of the Theorem 5.2, we have that

$$b(\|\gamma(t_k,t_0,\gamma_0)\|) \leq \Omega_0(t_k,\gamma(t_k,t_0,\gamma_0)),\, t_k \geq t_0 + T \text{ i.e.}$$

$$b(\varepsilon) \leq \Omega_0(t_k,\gamma(t_k,t_0,\gamma_0)),\, t_k \geq t_0 + T \tag{5.8}$$

This implies that $\gamma(t) \in S_\psi \;\; for\, t \in [t_0,t_1)$ and for $t \in [t_0,t_1)$ from Theorem 4.1,

$$\Omega_0(t_k,\gamma(t_k)) \leq \sum_{i=1}^{N} \vartheta_i(t_k,t_0,u_0) \tag{5.9}$$

Combining equations (5.8), (5.9), (5.6) and (5.5) gives the estimate

$$b(\varepsilon) \leq \Omega_0(t_k,\gamma(t_k,t_0,\gamma_0)) \leq \sum_{i=1}^{N} \vartheta_i(t_k,t_0,u_0) < b(\varepsilon)$$

which leads to an absurdity.

Hence, the asymptotic practical stability of the trivial solution $u = 0$ of (3.5) implies the asymptotic practical stability of the trivial solution $\gamma = 0$ of (3.3).

From the results obtained, we can therefore conclude that the trivial solution of the main system (3.3) is asymptotically practically stable.

6. CONCLUSION

In this paper, the existence of the maximal solution of the comparison system for vector Lyapunov function is established, and sufficient condition for the asymptotic practical stability of impulsive Caputo fractional order systems is presented by means of the comparison principle. It was discovered that the maximal solution of the comparison system majorizes the vector form of the Lyapunov function. Again, sufficient conditions for the asymptotic practical stability of the impulsive fractional order systems are established. The results also went t further to emphasize the fact that the solution of the main system is asymptotically practically stable.

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