Scholars Journal of Physics, Mathematics and Statistics
OPEN ACCESS
Abbreviated Key Title: Sch J Phys Math Stat
ISSN 2393-8056 (Print) | ISSN 2393-8064 (Online)
Journal homepage: https://saspublishers.com

Exploration of Symmetric Groups: Cayley Tables, Subgroup Analysis, and Real-World Applications in Card Tricks

Udoaka Otobong G.[1*], Udoakpan I. U.[2]

[1]Department of Mathematics, Akwa Ibom State University, Nigeria
[2]Department of Mathematics and Statistics, University of Port Harcourt, Nigeria

DOI: 10.36347/sjpms.2024.v11i01.003 | **Received:** 10.12.2023 | **Accepted:** 22.01.2024 | **Published:** 25.01.2024

***Corresponding author:** Udoaka Otobong G.
Department of Mathematics, Akwa Ibom State University, Nigeria

Abstract **Review Article**

This research delves into the extensive analysis of symmetric groups of various orders and degrees. We explore different representations of permutations, construct a Cayley table for the symmetric group of degree four, and systematically identify all its subgroups using Lagrange's theorem. An intriguing discovery unfolds as we demonstrate that the converse of the theorem countered by Sylow's theorem does not hold universally. Furthermore, we apply the concepts of permutations and product of disjoint cycles to address a real-world problem – the Card Trick game. This study amalgamates theoretical exploration with practical applications, showcasing the versatility and depth of symmetric group theory.

Keywords: Symmetric Groups, Cayley Table, Lagrange Theorem, Sylow's Theorem, Permutations, Disjoint Cycles, Card Trick Game.

1. INTRODUCTION

Symmetric groups form a cornerstone of group theory, providing rich insights into the algebraic structures of permutations. Cayley's theorem is fundamental in understanding group representations. It establishes that every group is isomorphic to a subgroup of a symmetric group [1]. The theorem forms the cornerstone for exploring structures within symmetric groups. Cayley tables provide visual insights into group operations. The construction of a Cayley table for the symmetric group of degree four (S_4) enables a comprehensive exploration of the group's multiplication rules and internal structure [2]. Lagrange's theorem plays a crucial role in identifying subgroups systematically. The theorem, proposed by [3], establishes the order relationships between subgroups and the parent group, guiding the exploration of subgroup structures within symmetric groups. The research introduces an intriguing discovery by demonstrating that the converse of Lagrange's theorem, countered by Sylow's theorem, may not hold universally. Sylow's theorem, proposed by [4], addresses the existence of subgroups of prime order within a group. The study applies group theory concepts to a real-world problem, the Card Trick game. Diaconis and Graham [5] contributes to the mathematics of perfect shuffles, showcasing the practical relevance of symmetric group theory in solving problems beyond theoretical exploration. This research embarks on a comprehensive journey, analyzing symmetric groups of various orders and degrees. We delve into diverse representations of permutations, explore the structure of the symmetric group of degree four through a Cayley table, and systematically identify its subgroups using Lagrange's theorem. An intriguing exploration unfolds as we challenge a theorem countered by Sylow's theorem, demonstrating its limitations. Bridging theory and application, we leverage the concepts of permutations and disjoint cycles to solve a real-world problem – the Card Trick game. Also, see [6] – [13] for further studies.

2. PRELIMINARY

Symmetric Groups of Degree *N* (S_n) 2.1. The set of all permutations of X with when endowed with the binary operation "∘" (composition) is called the symmetric group often denoted by S_n.

When $X = \{1, 2,\dots, n\}$, then Sn, is called the *symmetric group on n letters*, or simply the *symmetric group of degree n*. The subgroup of a symmetric group is the permutation group.

The elements of S_n are the permutations of X not the elements of X itself. Example if X = {1, 2, 3}. (i.e. |X|=3), then; S_3 = {(1 2 3), (1 3 2), (2 1 3), (2 3 1), (3 1

2), (3 2 1)} not {1, 2, 3}. For k ≥ 3, the group S_n is not commutative (non-abelian), so in general $f \circ g \neq g \circ f$

Lagrange Theorem 2.2. If *G* is a finite group of order k (k ∈ N, k < ∞) With *H* a subgroup of order m (m ∈ N, m <∞) then *m* divides *n.*

Proof:

Trivially, if *H*=*G*, the result follows. Otherwise *m*<*n* and ∃ a ∈ G\H ∴ a ∉ H, aH ≠ H and so, aH ∩ H = ∅ if G = aH ∪ H, the k ∘(G) = ∘(aH) + ∘(H) = 2 ∘(H) => the theorem holds.

Otherwise ∃ b ∈ G\(H ∪ aH), with bH ∩ H = ∅ and bH ∩ aH = ∅ and of course ∘(bH) = ∘(H). If G = bH ∪ aH ∪ H => ∘(G) = 3 ∘(H) => the theorem holds.

Otherwise we're back to an element c ∈ G with c ∉ bH ∪ aH ∪ H. But since *G* is finite, this process must terminate. So, G = a_1H ∪ a_2H ∪. . .∪ a_kH, say i.e ∘(G) = k∘(H) (k ∈ N, k < ∞) i.e. $n = k.m \Rightarrow m|n$ ■

Symmetric Groups of Degree Three (S_3) 2.3. Let consider the symmetric group of degree three S_3 given by;

$$S_3 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\right\}$$

Show that S_3 group with respect to the composition of function "∘" and identify all it subgroups.

SOLUTION:

Observe that $|S_3| = 3! = 6$. We wish to show that S_3 is indeed a group, and for this purpose we form a group multiplication table. It is necessary to first label the elements of S_3 as follows;

$$e = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, a = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, b = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, c = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, d = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, f = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

Obviously, *e* is the identity element.

$$a \circ a = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = e$$

$$b \circ a = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = c$$

$$d \circ b = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = a$$

$$c \circ f = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = a, etc$$

Table 2.3.1: Caley's table for S_3

∘	*a*	*b*	*c*	*d*	*e*	*f*
*a*a	*e*	*d*	*f*	*b*	*a*	*c*
*b*b	*c*	*e*	*a*	*f*	*b*	*d*
*c*c	*b*	*f*	*d*	*e*	*c*	*a*
*d*d	*f*	*a*	*e*	*c*	*d*	*b*
*f*e	*a*	*b*	*c*	*d*	*e*	*f*
*e*f	*d*	*c*	*b*	*a*	*f*	*e*

From the Cayley's table above we can see that S_3 is a group with respect to. "∘" From table 2.3.1 we identify the inverse of each element by tracing the element on the first row that corresponds with the identity element in that coordinates (row and column wise).

Table 2.3.2: Elementandinverseelementof*S3*

ELEMENT	INVERSE
a	a
b	b
c	d
d	c
e	e
f	*f*

Then to find the subgroups we have in mine by Lagrange theorem that the order of a subgroup must divides the order of the group.

We proceed by finding the factors of 6 (i.e. the order of $|S_3|$)

Factors of 6 are: 1,2,3,6. Where the subgroups with order 1 and 6 are the two trivial subgroups of any group of order six. For the case of S_3 the two trivial subgroups are $(e,\circ)$ and $(S_3,\circ)$

Table 2.3.3: subgroups of S_3

Subgroups	Order of the subgroup (factors of 6)
$(e,\circ)$	1
$(\{a,e\},\circ),(\{f,e\},\circ),(\{b,e\},\circ)$	2
$(\{c,d,e\},\circ)$	3
$(\{a,b,c,d,e,f\},\circ) = (S_3,\circ)$	6

Symmetric Groups of Degree Four (*S*4) 2.3. Let us look at the case of X = {1, 2, 3, 4}. From theorem 1.1 we know that Symmetric group of degree four (*S4*) has 24 elements (i.e. $n! = 24$). We start by labeling the element of (*S4*). This is done in the table below;

Table 2.3.1: Elements of S_4

S/N (n)	Labelling	Two way notation	Product of disjoint of cycles
1	a	$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$	$(1) = e$
2	b	$\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}$	$(3\quad 4)$
3	c	$\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$	$(2\quad 3)$
4	d	$\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}$	$(2\quad 3\quad 4)$
5	f	$\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$	$(2\quad 4\quad 3)$
6	g	$\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$	$(2\quad 4)$
7	h	$\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}$	$(1\quad 2)$
8	i	$\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$	$(1\quad 2)(3\quad 4)$
9	j	$\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}$	$(1\quad 2\quad 3)$
10	k	$\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$	$(1\quad 2\quad 3\quad 4)$
11	l	$\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}$	$(1\quad 2\quad 4\quad 3)$
12	m	$\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}$	$(1\quad 2\quad 4)$
13	n	$\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}$	$(1\quad 3\quad 2)$
14	p	$\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$	$(1\quad 3\quad 4\quad 2)$
15	q	$\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$	$(1\quad 3)$
16	r	$\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}$	$(1\quad 3\quad 4)$
17	s	$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$	$(1\quad 3)(2\quad 4)$
18	t	$\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}$	$(1\quad 3\quad 2\quad 4)$
19	u	$\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$	$(1\quad 4\quad 3\quad 2)$
20	v	$\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$	$(1\quad 4\quad 2)$
21	w	$\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$	$(1\quad 4\quad 3)$
22	x	$\begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}$	$(1\quad 4)$
23	y	$\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}$	$(1\quad 4\quad 2\quad 3)$
24	z	$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$	$(1\quad 4)(2\quad 3)$

We now construct a Caley table to show that the element of *S4* form a group with composition of function ($'o'$) as the binary operation.

Table 2.3.2. Cayley' stable for symmetric group of degree 4 (S_4)

∘	a	b	c	d	*f*	g	*h*	i	j	k	l	m	k	p	q	r	s	t	u	v	w	x	y	z
a	e	b	c	d	*f*	g	*h*	i	j	k	l	m	k	p	q	r	s	t	u	v	w	x	y	z
b	b	e	*f*	g	c	d	i	*h*	l	m	j	k	u	v	w	x	y	z	k	p	q	r	s	t
c	c	d	e	b	g	*f*	k	p	q	r	s	t	*h*	i	j	k	l	m	v	u	y	z	w	x
d	d	c	g	*f*	e	b	p	k	s	t	q	r	v	u	y	z	w	x	*h*	i	j	k	l	m
f	*f*	g	b	e	d	c	u	v	w	x	y	z	i	*h*	l	m	j	k	p	k	s	t	q	r
g	g	*f*	d	c	b	e	v	u	y	z	w	x	p	k	s	t	q	r	i	*h*	l	m	j	k
h	*h*	i	j	k	l	m	e	b	c	d	*f*	g	q	r	k	p	t	s	w	x	u	v	z	y
i	i	*h*	m	l	j	k	b	e	*f*	g	c	d	w	x	u	v	z	y	q	r	k	p	t	s
j	j	k	*h*	i	m	l	q	r	k	p	t	s	e	b	c	d	*f*	g	x	w	z	y	u	v
k	k	j	m	l	*h*	i	r	q	t	s	k	p	x	w	z	y	u	v	e	b	c	d	*f*	g
l	l	m	i	*h*	k	j	w	x	u	v	z	y	b	e	*f*	g	c	d	r	q	t	s	k	p
m	m	l	k	j	i	*h*	x	w	z	y	u	v	r	q	t	s	k	p	b	e	*f*	g	c	d
k	k	p	q	r	s	t	c	d	e	b	g	*f*	j	k	*h*	i	m	l	y	z	v	u	x	w
p	p	k	s	t	q	r	d	c	g	*f*	e	b	y	z	v	u	x	w	j	k	*h*	i	m	l
q	q	r	k	p	t	s	j	k	*h*	i	m	l	c	d	e	b	g	*f*	z	y	x	w	v	u
r	r	q	t	s	k	p	k	j	m	l	*h*	i	z	y	x	w	v	u	c	d	e	b	g	*f*
s	s	t	p	k	r	q	y	z	v	u	x	w	d	c	g	*f*	e	b	k	j	m	l	*h*	i
t	t	s	r	q	p	k	z	y	x	w	v	u	k	j	m	l	*h*	i	d	c	g	*f*	e	b
u	u	v	w	x	y	z	*f*	g	b	e	d	c	l	m	i	*h*	k	j	s	t	p	k	r	q
v	v	u	y	z	w	x	g	*f*	d	c	b	e	s	t	p	k	r	q	l	m	i	*h*	k	j
w	w	x	u	v	z	y	l	m	i	*h*	k	j	*f*	g	b	e	d	c	t	s	r	q	p	k
x	x	w	z	y	u	v	m	l	k	j	i	*h*	t	s	r	q	p	k	*f*	g	b	e	d	c
y	y	z	v	u	x	w	s	t	p	k	r	q	g	*f*	d	c	b	e	m	l	k	j	i	*h*
z	z	y	x	w	v	u	t	s	r	q	p	k	m	l	k	j	i	*h*	g	*f*	d	c	b	e

From table 2.3.2. above we see that

∗S_4 = {a = e,b,c,d,*f*,g,*h*,i,j,k,l,,m,k,p,q,r,s,t,u,v,w,x,y,z,} is close under "∘" i.e. compositions of functions.

*Associativity holds

*There exist identity element e ∈ S_4

*Each element has a unique inverse

Hence (S_4, ∘) ≡ a group

Table 2.3.3: Element and inverse element of *S₄*

Element	(Inverse)
a=e	a=e
b	b
c	c
d	*f*
f	d
g	g
h	*h*
i	i
j	k
k	u
l	p
m	v
k	j
p	l
q	q
r	w
s	s
t	y
u	k
v	m
w	r
x	x
y	t
z	z

We compute the subgroups of (S_4 ∘) using Lagrange theorem.

Lagrange theorem states that the order of a subgroup divides the order of the group. So we start by finding the factors of 24 (4! = 24) and examining **table 2.3.3** for closure of any element to be consider as a member of the subgroup.

Table 2.3.4: Subgroups of S_4

Elements	Divisors of 24(order)
{e}	***1***
{e,x},{e,c},{e,q},{e,g},{e,*h*},{e,b},{e,i},{e,s},{e,z}	***2***
{e,j,k},{e,m,v},{e,r,w},{e,d,*f*}	***3***
{e,i,s, z},{e, i,t,y}, {e,s, k,u},{e,z, l,p},{e,*h*,i,b},{e,x, z,c},{e, q,s, q}	***4***
{e,*h*,q,c,j,k},{e,*h*,x,g,m,v},{e,q,x,b,r,w},{e,c,g,b,d,*f*}	***6***
{e,*h*,i,s,z,b,t,y},{e,i,s,x,z,c,l,p},{e,i,s,z,g,k,u,q}	***8***
{e,i,s,z,j,m,k,r,w,d,*f*,v}	***12***
{e,b,c,d,*f*,g,*h*,i,j,k,l,m,k,p,q,r,s,t,u,v,w,x,y,z,}	***24***

Table 2.3.5.: Caley's table for ({e,i,s,z,j,m,k,r,w,d,*f*,v},∘)

∘	e	d	*f*	i	j	m	k	r	s	v	w	z
e	e	d	*f*	i	j	m	k	r	s	v	w	z
d	d	*f*	e	k	s	r	v	z	w	i	j	m
f	*f*	e	d	v	w	z	i	m	j	k	s	r
i	i	m	j	e	*f*	d	w	v	z	r	k	s
j	j	i	m	r	k	s	e	d	*f*	w	z	v
m	m	j	i	w	z	v	r	s	k	e	*f*	d
k	k	r	s	d	e	*f*	j	i	m	z	v	w
r	r	s	k	j	m	i	z	w	v	d	e	*f*
s	s	k	r	z	v	w	d	*f*	e	j	m	i
v	v	z	w	*f*	d	e	s	k	r	m	i	j
w	w	v	z	m	i	j	*f*	e	d	s	r	k
z	z	w	v	s	r	k	m	j	l	*f*	d	e

From **table 2.3.5.** above we see that ({e,i,s,z,j,m,k,r,w,d,*f*,v}∘) ≡ a group We then apply Lagrange theorem to find the associate subgroups

3. APPLICATION OF SYMMETRIC GROUP IN INVESTICATING THE CONVERSE OF LAGRANGE THEOREM.

Lagrange theorem state that If G is a finite group of order k (k∈N, k<∞) With H a subgroup of order m (m∈N, m<∞) then m divides n. Then, is the converse true? (i.e is every divisor of the order of a group the order of some subgroup?).

Let us look at the subgroups of the group ({e,i,s,z,j,m, k,r,w,d,*f*,v}∘)
Considertable4.1below;

Table 3.1: Subgroups of the group ({e,i,s,z,j,m,k,r,w,d,*f*,v},∘)

Subgroups	Divisors of 12 (order of subgroup)
({e,},∘)	1
({e,i,},),({e,s,},),({e,z,},∘)	2
({e,j,k,},),({e,d,*f*,}),({e,r,w,},),({e,m,v,}∘,)	3
({e,i,s,v},∘)	4
none	6
({e,i,s,z,j,m,k,r,w,d,*f*,v} ∘,)	12

We see that 6 is a divisor of 12 but there is no subgroup of ({e, i, s, z, j, m, k, r, w, d, *f*,v}, ∘) that has order 6, hence from symmetric group of degree four we can show that the converse of the Lagrange theorem does not hold in general.

4. APPLICATION OF SYMMETRIC GROUP TO CARD TRICK

One of the interesting applications of permutation groups is card tricks. This project work only touches on a couple different card tricks that were explained in the article titled "Invariants Under Group Actions to Amaze Your Friends" written by Douglas E. Ensley (1999). The basic idea behind these card tricks is to make the audience feel as if the cards are being mixed, when the moves are actually designed. The cards are basically the permutation group and the tricks involve certain rotations such that a desired outcome may be achieved.

The first card trick is very simple. Have a volunteer to pick out any card from a standard 52-card deck. They must remember the card and place it on top of the deck. This is followed by cutting the cards as many times as they wish. The dealer then takes the deck and fans it out face up and picks out the volunteer's card. The logic behind this is that the top card and the bottom card will be side-by-side beginning once the deck is cut once. As long as the dealer knows what the bottom card is, he/she can identify the volunteer's card as the card next to the former bottom card. This card trick is directly related to permutation groups since a permutation group is a cycle that can be written starting with any permutation as long as the arrangement of permutations stays the same. In his article, Ensley (1999) showed that since this card trick only allows cutting the deck, the arrangement of cards will always remain the same, just with different starting numbers, which allows the dealer to know the volunteer's card will always be next to the former bottom card. S52 represents the deck of cards and each card is assigned a number 1 - 52 such that S52 = (1 2 3 4 … 50 51 52).

The next card trick seems simple on the surface; however, it is somewhat complicated. The dealer hands the volunteer four aces (any card will work as long as they are each of a different suit), and gives the following instructions:

i. Stack the four cards face-up with the diamond at the bottom, then the club, then the heart, and finally the spade.

ii. Turn the spade (the uppermost card) face down.

iii. Perform any of the following three operations as many times and in any order that you wish:

1. Cut any number of cards from the top to the bottom.
2. Turn the top two cards over as one.
3. Either turn the entire stack over or do not – your choice.

iv. Finish the rearrangement of cards by turning the topmost card over, then the top two cards over as one, and then the top three cards over as one.

All of the previous operations are to be performed with the dealer having his back turned.

After the volunteer has performed steps 1-4, the dealer can now say with confidence that the ace of clubs is the only card facing in the opposite direction from the other three aces. So while the volunteer thinks that they are mixing up the cards, they are essentially maintaining the 4 properties that the dealer has set up.

In order to understand this card trick, one must understand the permutation that is assigned to this group. In order for this trick to work, Ensley (1999) stated that we need to let the ace of spades be 1, the ace of hearts be 2, the ace of clubs be 3, and the ace of diamonds be 4. Also, let an underlined number be a face-down card. Now, using these definitions, we can determine the orientation and position of each card. Listed below is one example of how a person could go about the trick. Since the cards are to be placed in a certain order, the original four aces will be represented as 1,2,3,4.

•	The original deck	1,2,3,4
•	Turning the spade (the uppermost card) face down	$\underline{1}$,2,3,4
•	Cutting two cards from the top to the bottom	3,4,$\underline{1}$,2
•	Turning the top two cards over as one	$\underline{4},\underline{3},\underline{1}$,2
•	Cutting three cards from the top to the bottom	2,$\underline{4},\underline{3},\underline{1}$
•	Turning the top two cards over as one	4,$\underline{2},\underline{3},\underline{1}$
•	Turning the entire stack over	1,3,2,$\underline{4}$
•	Turning the topmost card over	$\underline{1}$,3,2,$\underline{4}$
•	Turning the top two cards over as one	$\underline{3}$,1,2,$\underline{4}$
•	Turning the top three cards over as one	$\underline{2},\underline{1}$,3,$\underline{4}$

As you can see, we are left with $\underline{2},\underline{1}$,3,$\underline{4}$, which shows that card #3 is facing the opposite way of the rest of the cards. As defined earlier, card #3 is the ace of clubs which means that the trick worked. It can be shown that no matter what choices you make when mixing the cards, the ace of clubs will always be facing the opposite way of the rest of the cards after the final move as long as the directions were followed. The trick seems to be fair but it is not because every permutation cannot be represented using the set of rules given above.

5. CONCLUSION

This research has delved into a thorough analysis of symmetric groups, spanning various orders and degrees. We navigated through different representations of permutations, constructed a Cayley table for the symmetric group of degree four, and methodically identified all its subgroups employing Lagrange's theorem. An intriguing revelation emerged as we demonstrated that the universal applicability of the converse of Lagrange's theorem is challenged by Sylow's. This paper seamlessly blended theoretical exploration with practical applications by utilizing fundamental concepts of permutations and the product of disjoint cycles to tackle a real-world problem—the Card Trick game. This seamless integration exemplifies the versatility and profound depth of symmetric group theory, highlighting its relevance not only in abstract mathematical realms but also in addressing practical challenges.

REFERENCES

1. Cayley, A. (1854). On the Theory of Groups, as Depending on the Symbolic Equation θn = 1. *Philosophical Magazine*, *4*(13), 40-47.
2. Cayley, A. (1878). A Memoir on the Theory of Matrices. *Philosophical Transactions of the Royal Society of London*, *169*, 333-429.
3. Lagrange, J. L. (1771). Réflexions sur la résolution algébrique des équations. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres*, *17*, 535-586.
4. Sylow, L. (1872). Théorèmes sur les groupes de substitutions. *Mathematische Annalen*, *5*(1), 584-594.
5. Diaconis, P., & Graham, R. L. (1983). The Mathematics of Perfect Shuffles. *Advances in Applied Mathematics*, *4*(2), 175-196.
6. Michael, N. J., Udoaka, O. G., & Alex, M. Key Agreement Protocol Using Conjugacy Classes of Finitely Generated group. *International Journal of Scientific Research in Science and technology (IJSRST)*, *10*(6), 52-56.
7. Michael, N. J., Udoaka, O. G., & Alex, M. (2023). SYMMETRIC BILINEAR CRYPTOGRAPHY ON ELLIPTIC CURVE AND LIE ALGEBRA. *GPH - International Journal of Mathematics*, *06*(10), 01–15.
8. Michael, N. J., Edet, E., & Udoaka, O. G. (2023). On Finding B-Algebras Generated by Modulo Integer Groups $\boldsymbol{Z}_n$. *International Journal of Mathematics and Statistics Invention (IJMSI)*, *11*(6), 01-04. E-ISSN: 2321 – 4767 P-ISSN: 2321 – 4759.
9. Michael, N. J., Otobong, G. U. & Alex, M. (2023). Solvable Groups with Monomial Characters of Prime Power Codegree and Monolithic Characters. *BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH*, *98-102*, *11*(7), 01-04.
10. Udoaka, O. G., Asibong-Ibe, U. I., & David, E. E. (2016). Rank of product of certain algebraic classes. *IOSR Journal of Mathematics*,*12*, *6*(1), 123-125., e-ISSN: 2278-5728,
11. Udoaka, O. G. (2022). Generators and inner automorphism. *THE COLLOQUIUM -A Multidisciplinary Thematc Policy Journal*,*10*(1), 102-111. www.ccsonlinejournals.com. CC-BY-NC-SA 4.0 International Print ISSN: 2971-6624 eISSN: 2971-6632.
12. Udoaka, O. G., & David, E. E. (2014). Rank of Maximal subgroup of a full transformation semigroup. *International Journal of Current Research*, *6*(09), 8351-8354.
13. Akpan, F. E., & Udoaka, O. G. (2022). Finite Semi-group Modulo and Its Application to Symmetric Cryptography. *International Journal of Pure Mathematics*, *9*, 90-98. DOI: 10.46300/91019.2022.9.13.