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Pentagoning the Circle with Straightedge & Compass

Tran Dinh Son[1*]

[1]Independent Researcher in Mathematics in the United Kingdom

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***Corresponding author:** Tran Dinh Son
Lewisham College, Lewisham Way, London SE4 1UT, United Kingdom

Abstract **Original Research Article**

This study idea came from the exact solution "Circling the Regular Pentagon with Straightedge & compass in Euclidean Geometry", published by the Scholars Journal of Physics, Mathematics and Statistics on 22/08/2024 [1]. In this research, the ANALYSIS method is adopted to prove the process of solving this new challenge problem, which has not existed in the Mathematics field till today. The process is an inverse/converse solution solving the inverse problem "Circling the Regular Pentagon with Straightedge & compass in Euclidean Geometry" problem, using a straightedge & a compass. I hereby commit that this is my own personal research project.

Keywords: Pentagoning the circle; make circle pentagon; circle becomes pentagon; make circle pentagoned; find regular pentagon area same as circle; make a circle regularly pentagoned.

1. INTRODUCTION

"Pentagoning The Circle" is a phrase that includes the word "Pentagoning", a new word proposed to refer to the construction of a regular pentagon with an area equal to the area of a given circle. Therefore, this research study article can be titled "PENTAGONING THE CIRCLE WITH STRAIGHTEDGE AND COMPASS".

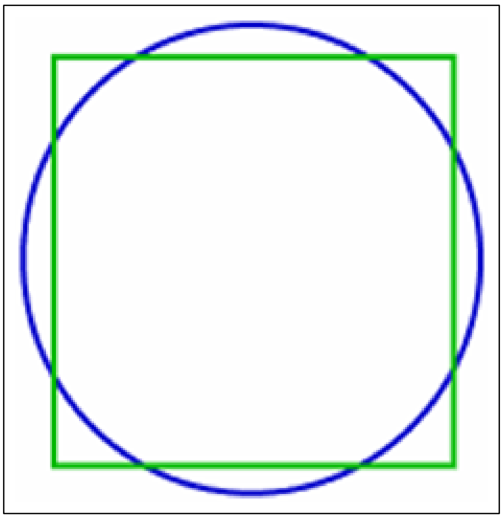
In the past, knowledge was often called scientific if it could be confirmed by specific evidence or experiments. But Karl Popper, in his book "Logik der Forschung" (The Law of Scientific Discovery), published in 1934, showed that an essential characteristic of scientific hypotheses is that they can be proven wrong (falsifiability). Anything that cannot be refuted by evidence is temporarily considered true until new evidence is found. For example, in astronomy, now everyone believes in the Big Bang theory, but in the future, whoever finds a loophole in that theory will be rewarded by the whole physics community! Also, no great theory lasts forever in science, but only specific research and discoveries continuously complement each other [11].

There are three classical problems remaining from ancient Greek mathematics which are extremely influential in the development of Geometry. They are Trisecting an Angle, Squaring the Circle, and Doubling the Cube problems. These problems were solved, and of which papers were published in the International Journal of Mathematics Trends and Technology in 2023 [4-6]. Upstream from these exact solutions, one can deduce, conversely/inversely, to get some new Mathematical challenges using a straightedge & a compass in Euclidean Geometry. Two of these challenge problems, that are "Circling the Square" and "Circling the Regular Pentagon" were also solved in 2024 [1, 3].

While in prison for radical ideas about the sun, mathematician Anaxagoras of Clazomenae worked on the now-iconic problem involving a compass and straightedge [12]. In geometry, "straightedge and compass" construction is also known as Euclidean construction or classical construction [13].

From the oldest mathematical documents known up to today, the problems and related problems concerning π have interested both professional and non-professional mathematicians.

Despite the proof of the impossibility of "squaring the circle," the problem has continued to capture the imaginations of mathematicians and the general public alike, and it remains an important topic in the history and philosophy of mathematics [14, 14]. In 2023, I solved the "Trisecting an Angle" problem with a straightedge and compass" problem, and published it in the IJMTT journal as a counter-proof to the Wantzel, L.'s proof in 1837 [4, 14].



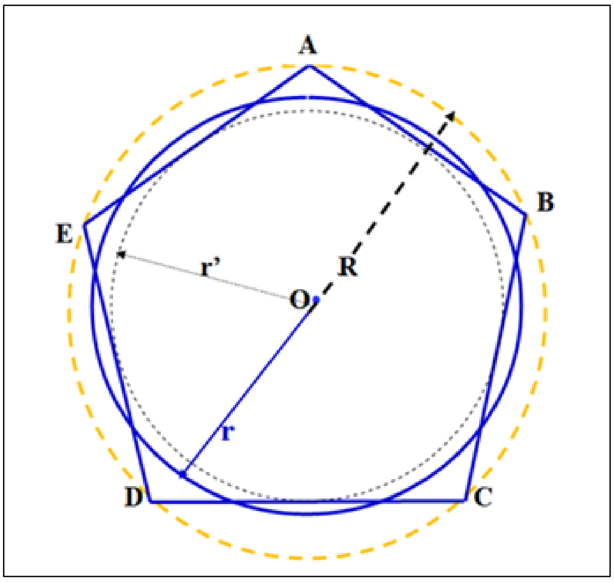
Although the ancient Greek Maths Challenges have close links, I chose to solve the "Squaring The Circle" problem as my second research study after solving the "Trisecting An Angle" exactly and successfully. They were then published in the IJMTT [4, 5].

*In July 2024, I solved exactly and accurately the new challenged problem "*Circling a Regular Pentagon with Straightedge and Compass in Euclidean Geometry*", then, the* Scholars Journal of Physics, Mathematics and Statistics (SJPMS *published this paper on 22/08/2024 [1]. After the paper was published, an idea derived from the solved result to create another new mathematical challenge, which had not existed before. The idea is as follows:*

"If we can circle a given regular pentagon then how about to pentagon a given circle, inversely/conversely". The following diagram illustrates this concept.

"***Circling The Regualar Pentagon***"
"***Pentagoning the Circle***"
(inversely/conversely from one another)

In seeking the solutions to many mathematical problems, geometers developed a special technique, which they called "analysis". They assumed that the problem had been solved, and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved based on the givens. To obtain a formally correct solution to the original problem, the geometers reversed the procedure. First, the data were used to solve the equivalent problem derived in the analysis, and from the solution obtained, the original problem was solved. In contrast to this analysis, this reversed procedure is called "synthesis". I adopted the technique "ANALYSIS" to solve accurately the "Circling the Regular Pentagon" problem with only a straightedge & compass [1]. In this paper, I also used the technique "ANALYSIS" to solve this current challenge problem "PENTAGONING THE CIRCLE", derived from my "Squaring the Circle" solution mentioned above. *I hereby commit that this is my own personal research project*.



2. PROPOSITION

2.1 Definition 1: "Conical-Arc" shape

Given a circle (O,r) and an angle $\widehat{BAC}$ with its vertex outside the circle such that the bisector of the angle passes through the centre O of the circle, then the special shape formed by the 2 sides of the angle and arc $\widehat{DE}$ can be called a Conical-Arc (in Figure 1 below, the red shape ADE is a Conical-Arc). If $\widehat{BAC}$ is a right angle then the shape ADE is called a Right-Conical-Arc.

Note that one can see a Conical-Arc looks like an isosceles triangle with a Conical-Arc base and not a straight line base, in the figures below.

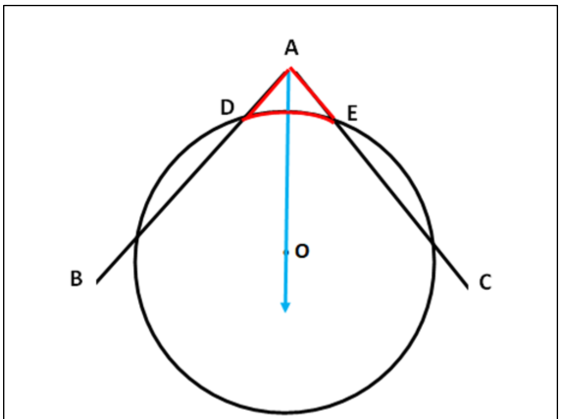
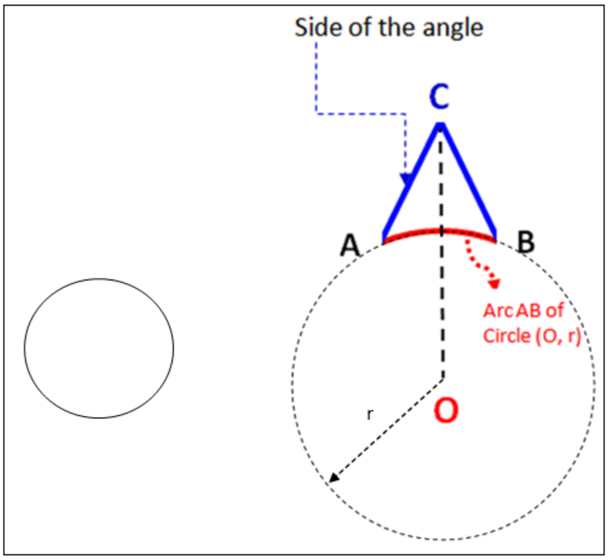


Figure 1: The Conical-Arc ADE



In the Conical-Arc ***ACB*** *above, AB (coloured red) is the circle arc of the circle (O, r) and* ***CO*** *is the bisector of the angle* $\widehat{ACB}$*, illustrated in the above figures.*

2.2 Theorem 1:

If there exists a concentric regular pentagon, area πr^2, to a given circle (O, r), then

a. The assumed regular pentagon intersects the circumference of the circle at 10 points a, b, c, d, e, f, g, h, i & j.
b. The above pentagon and the circle form 5 equal circle segments and 5 equal Conical-Arcs at vertices of the pentagon
c. Areas of the above 5 equal circle segments and 5 equal Conical-Arcs are also equal.

PROOF:

a. ➢ It is obvious the circumscribed regular pentagon A'B'C'D'E' has an area larger than its inscribed circle (O, r), area πr^2, and the area of the inscribed regular pentagon A''B''C''D''E'' of the given circle is smaller than πr^2. In terms of the concentric, this shows that the assumed regular pentagon has to be located in between A'B'C'D'E' and A''B''C''D''E''. Therefore, ABCDE cuts the circle (O, r) at a, b, c, d, e, f, g, h, i & j, as required (Figure 2, below) required (Figure 2, below).

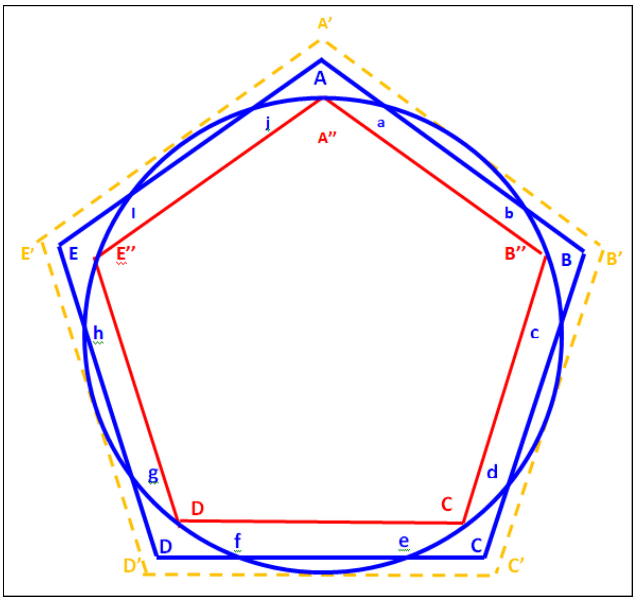


Figure 2: ***The circumscribed regular pentagon A'B'C'D'E' and the inscribed regular pentagon A''B''C''D''E'' to the given circle. Pentagon ABCDE is the assumed regular pentagon, that is concentric to the given circle***

b. ➢ The circle (O, r) and the pentagon ABCDE are concentric, and this concentric property shows that 5 circle segments **ab, cd, ef, gh** & **ij** are equal, and 5 Conical-Arcs **Aja**, **Bbc**, **Cde**, **Dfg** & **Ehi** are equal, as required (Figure 3, below).

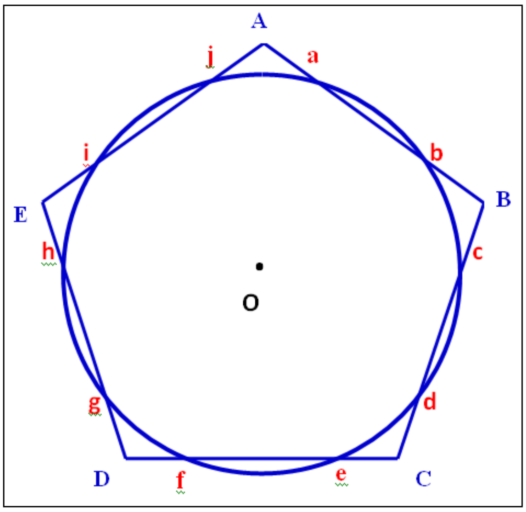


Figure 3: ***The intersection of the given circle (O, r) and the assumed regular pentagon, which has the same area πr² as the area of the circle***

c. ➢ Consider the intersection area between the given circle (O, r), area πr^2 and the assumed regular pentagon, area πr^2 to get the following properties (Figure 3, above):

- ***Property a:*** To equalise this intersection area to the area πr^2 of the given circle, it is necessary to add the areas of the 5 circle segments **ab, cd, ef, gh** & **ij** to the intersection area.
- ***Property b:*** To equalise this intersection area to the area πr^2 of the assumed regular pentagon, it is necessary to add the areas of the 5 Conical-Arcs **Aja**, **Bbc**, **Cde**, **Dfg** & **Ehi** to the intersection area.

The above properties a & b show each area of each circle segment is equal to each area of each Conical-Arc (Figure 3, above).

By the above Section b. of this proof an expression concludes this Theorem 1, as follows:

Area of segment ab = Area of segment cd = Area of segment ef = Area of segment gh = Area of segments ij = Area of Conical-Arc Aja = Area of Conical-Arc Bbc = Area of Conical-Arc Cde = Area of Conical-Arc Dfg = Area of Conical-Arc Ehi.

2.3 Theorem 2: *Core Theorem*

If there exists a regular pentagon ABCDE which is concentric and has the same area πr^2 as a given circle (O, r) then the circle and the pentagon form a regular decagon abcdefghij *which inscribes in the given circle (O, r), and has 5 non-consecutive sides overlapping the 5 sides of the pentagon.*

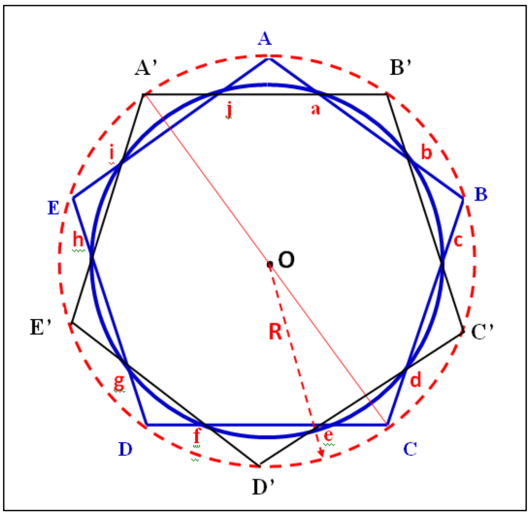


Figure 4: ***Symmetric axe A'O of the angles $\widehat{OA'B'}$ & $\widehat{OA'E'}$.***

PROOF

Consider the intersected points j & a of the given circle and the assumed pentagon ABCDE, and prolong the straight line ja to meet the circle circumference at A' & B'. Then connect OA' and draw a straight line A'E' that meets the circle circumference at E' and makes angle E'A'O equal to angle OA'B' (Figure 4 above). This shows A'E' is symmetric to the line

segment A'jaB' via the axis of symmetry OA', and distances from A'E' and A'jaB' to centre O of the circle (O, r) are the same. Therefore A'E' overlaps chord hi or chord ja = chord hi. Similar proof results:
Chord **bc** = Chord **hi**
Then, **bc** = **hi** = **ja** and then **ab = bc = cd = de = ef = fg = gh = hi = ij**

Therefore, abcdefghij is a decagon, as required.

2.4 Theorem 3: *Resulting Theorem*

Given a circle (O, r) and its inscribed regular decagon, then 5 non-consecutive sides of the decagon, which are prolonged/extended, form a regular pentagon, of which area is equal to the area of the given circle.

PROOF:

Consider an inscribed pentagon of the given circle (O, r), there exists an inscribed decagon abcdefghij in the circle (this decagon is constructive with a straightedge & compass as Figure 5 below). By Theorems 1 & 2, the resulting regular pentagon ABCDE or A'B'C'D'E', the area of which is equal to the area πr^2 of the given circle, is formed by the prolongation/extension of the 5 non-consecutive sides of the inscribed regular decagon of the circle (O, r) as in the Figure 5 below. Note that the circumscribed circle (O, R) of these pentagons is marked by dashed red in Figure 5 below.

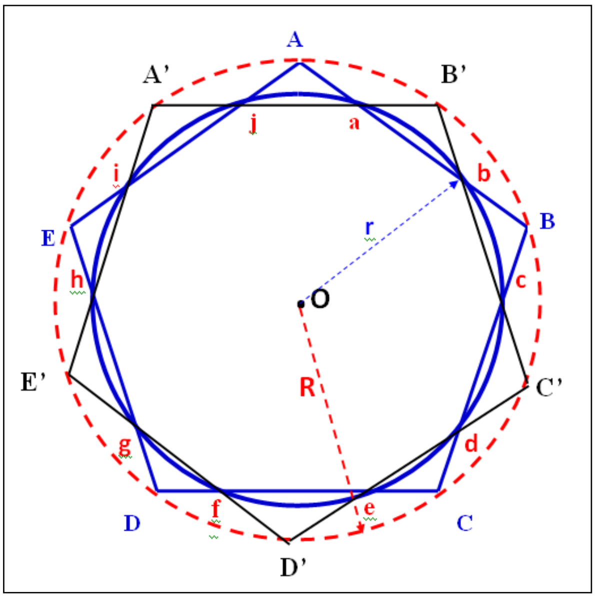


Figure 5: ***The resulting pentagon ABCDE or A;B'C'D'E' of which area πr^2 is equal to the area of the given circle (O, r)***

2.5 Construction Method of the Pentagoning:

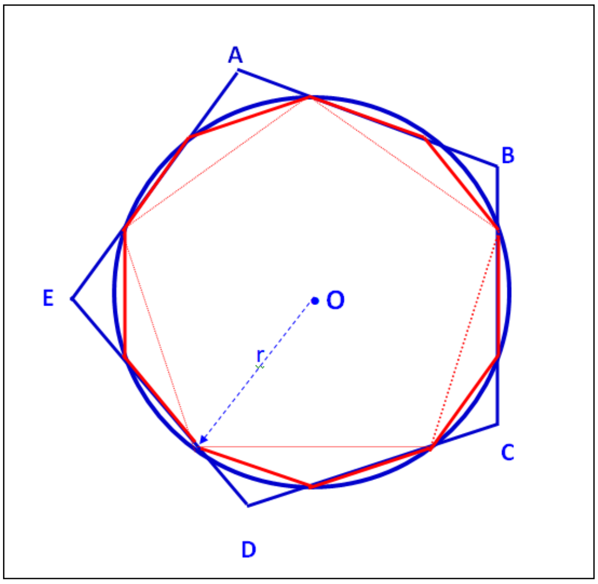


Figure 6: ***A given circle (O, r) and its resulting regular pentagon with the same area***

Aim to construct the regular pentagon with the same area πr^2 as the area of a given circle (O, r), exactly and strictly.

Given a circle (O, r) then use a straightedge and a compass to construct an inscribed regular pentagon (red colour in Figure 6 above) in the circle [9]. Then use also the straightedge and the compass to construct the required pentagon following:

1. From 5 vertices of the inscribed regular pentagon, draw 5 lines that meet the circle circumference at 5 points located at the mid-points of the 5 circles segments (Figure 6). Then connect these 10 points in the circumference with straight line to obtain a regular decagon inscribed in the circle in Figure 6 above.
2. Prolong/extend 5 non-consecutive sides of the decagon to obtain the resulting regular pentagon in Figure 6 above.

3. DISCUSSION & CONCLUSION

The Analysis Method or the inverse problem or the last-up backward research method is a good solution for complex problems involving a lot of logic. Complexity and logic here mean that the data of the problem can constantly change and be closely related to each other.

It is obvious that the problem "Pentagoning the Circle" did not ever exist in the Mathematics field till 2024 when I solved the newly challenged problem "Circling The Square" [3]. This "Circling the Square" problem came from my successful study research to construct exactly a square with the same area as a given circle without any difficulty from the irrational Pi π, only using a straightedge & compass. It also shows the possibility of solving exactly the 3 ancient Greek challenges (Trisecting an Angle, Doubling The Cube, and Squaring The Circle) with straightedge & compass [4-6]. "Pentagoning The Circle" is a topic related to geometry, specifically focusing on the geometric properties and constructions involving a regular pentagon, an inscribed circle, and a circumscribed circle or vice versa. While there may not be an extensive body of research specifically titled "Pentagoning the Circle", there are likely NO mathematical studies, papers, and articles that explore various aspects of this topic within the broader field of geometry and mathematical constructions. Research in this area might cover topics such as:

1. Geometric properties of regular pentagons and circles.
2. Constructions involving regular pentagons inscribed in circles or circumscribed around circles.
3. Relationships between the side length of a regular pentagon and the radius of the circumscribing or inscribing circle.
4. Applications of regular pentagons and circles in architecture, art, or other fields.

Can mathematicians use a compass and a straightedge to construct a regular pentagon having the same area as a given circle exactly/accurately? This question is the inverse problem for finding an accurate solution for the new challenge Mathematical problem "**CIRCLING THE REGULAR PENTAGON**". Surprisingly, only I, myself, have been still working on this question because the challenge problem had just arisen contemporarily when I ended my original research article ***"Circling the Square With Straightedge and Compass in Euclidean Geometry"***, published by IJMTT in January 2024 [3].

The "**Analysis**" Method is applied correctly to Geometry to complete this research study to gain an exact/accurate solution to this new challenge "**PENTAGONING THE CIRCLE**" problem in Mathematics. My construction method is quite different from approximation and is based on using a straightedge and compass within the Euclidean Geometry. Moreover, this method shows that the value π can be expressed accurately, in terms of Geometry. This Geometrical expression of the irrational number π could be an interesting field for mathematicians in this 21st century. In other words, algebraic geometry can express exactly any irrational number πr^2, $r \subset \mathbb{R}$**.**

The results of this independent research show that the correct answer **"the resulting regular pentagon"**, constructed by a compass and straightedge, has the area A = area πr^2 of the given circle. Therefore if the given circle is a unit circle (O, r = 1), area π, then in terms of geometry, π can be constructive/expressed by a regular pentagon with area π. Subsequently, this regular area π can be further researched to find out a nearer value for π, compared to the traditional value of π. **In practice**, if the ***International Bureau of Weights and Measures (BIPM),*** the *International System of Units*, or any accurate laser measurement is used to measure the arithmetic area of the resulting regular pentagon ABCDE of the problem "**Pentagoning the Circle**" then we can have another further research in Applied Mathematics, as 2 following formulations:

$A = 5R^2 \times \sqrt{[(5 + \sqrt{5})/2]} / 4 = \pi r^2$, where R is the radius of the circumscribed circle of the resulting regular pentagon.

$$A = 5R^2 \frac{\sqrt{5 + \sqrt{5}/2}}{4} = \pi r^2 \quad \Longrightarrow \quad \pi = 5R^2 \frac{\sqrt{5 + \sqrt{5}/2}}{4r^2}$$, r is the radius of the given circle.

Or

$A = a^2 \times \sqrt{(25 + 10\sqrt{5})} / 4 = \pi r^2$, where a is a side of the resulting regular pentagon.

$$A = a^2 \frac{\sqrt{25 + 10\sqrt{5}}}{4} = \pi r^2 \Longrightarrow \pi = a^2 \frac{\sqrt{25 + 10\sqrt{5}}}{4r^2}$$, r is the radius of the given circle.

This arithmetic value of π could be the nearest arithmetical value of the irrational number π ever seen.

In addition, this research result can be used for further research in the topics "EXAGONING THE CIRCLE", "HEPTAGONING THE CIRCLE", "OCTAGONING THE CIRCE", etc …, using only a straightedge and compass in Euclidean Geometry.

Therefore, the question "Does there exist any other research on the topic title "Pentagoning the Circle?" is answered, "No, there is not, except in this article paper".

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