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Numerical Analysis of a BDF2 finite Element Scheme for Navier-Stokes-Omega Model

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Abstract **Review Article**

In this paper, we study a second-order backward difference formula (BDF2) scheme for the Navier-Stokes-omega (NS-omega) model. By employing the stabilization scheme for space discretization and the BDF2 method for time discretization of NS-omega model, we obtain the fully discrete approximation of them. The paper provides an analysis of the unconditional stability and convergence of the approximate solutions. Furthermore, the numerical experiments are conducted to validate the theoretical findings and demonstrate the efficiency of the proposed scheme.
Keywords: Navier-Stokes-omega model; BDF2 scheme; Error analysis.

1. INTRODUCTION

The challenge of simulating fluid flow is primarily attributed to the extensive range of spatial eddy scales. For turbulent flows, Kolmogorov's universal framework provides significant insights into the behavior of eddies within these turbulent flow regimes. Accurate simulation of these eddies necessitates a substantial number of spatial mesh points per time step in direct numerical simulation (DNS), typically on the order of $Re^{9/4}$. Constrained by finite computational resources, this approach to the Navier-Stokes equations (NSE) poses a significant challenge in achieving precise simulations, particularly at high Reynolds numbers. Therefore, we should use the turbulence model for numerical solution of flows with a high Reynolds number. The common turbulence model is the large eddy simulation (LES). The LES approach is apply a spatial filter to the NSE. Furthermore, α-models are also turbulence models.

The research on finite element methods (FEM) for α-models of fluid flows has demonstrated their superior effectiveness in obtaining accurate solution to flow problems with coarser spatial and temporal discretization compared to direct numerical simulation of the NSE [1-3]. In these models, the NS-omega model is unique in that in addition to its superior theoretical properties such as energy conservation, model helicity conservation and well-posedness [4, 5], it can be computed efficiently by using unconditionally stable algorithm. This model modifies the vorticity term $\omega = \nabla\times u$ in the nonlinearity of the Navier-Stokes equations. It controls turbulent cascades at scales smaller than a certain length without introducing any additional dissipation [1].

The convective term of the α-models is represented in rotational form. For the NS-omega model, we write the rotational form as follow

$$\begin{aligned} & u_t - u\times(\nabla\times\bar{u}) + \nabla p - \nu\Delta u = f, \\ & \nabla\cdot u = 0, \qquad \qquad \ldots\ldots\ldots\ldots\ (1) \\ & -\alpha^2\Delta\bar{u} + \bar{u} = u. \end{aligned}$$

where $u = u(x,t)$ is the velocity, $p = p(x,t)$ is the pressure, ν is the kinematic viscosity and $f = f(x,t)$ is an external force applied to the fluid and $(-\alpha^2 + I)^{-1}$ is the Helmholtz filter, where $\alpha > 0$ is filtering radius, with suitable initial conditions in a bounded, polyhedral domain $\Omega \in \mathbb{R}^d (d = 2,3)$ and no-slip boundary condition $u = 0$ on $\partial\Omega$. Some mathematical theories of the continuous NS-omega model can be found in [5].

Numerical methods for solving the numerical approximation of the NSE have been investigated extensively. Such as the backward difference method, projection method, discontinuous Galerkin method. Li [22] proposed a second-order mixed stabilized finite element method for solving the variable density Navier–Stokes equations. The method employs second-order backward difference for temporal discretization. In [23], Dokken and Johansson introduced a second-order projection method for the Navier-Stokes equations, leveraging the BDF2 scheme for temporal discretization and the FEM for spatial discretization. They presents several numerical experiments to varify the convergence. Li and Shen [21] introduced a SAV approach that guarantees unconditionally stable schemes for the Navier-Stokes equations, and prove error estimates.

In this work, we investigate the convergence of the BDF2 scheme in temporal discretization and FEM in spatial discretization for the NS-omega model. Multi-step backward difference formula (BDF) time integration methods are widely used for partial differential equations, see [6-11]. As a result of its stability and high accuracy, BDF2 is one of the most popular BDF schemes [12]. The first-order BDF and second-order BDF temporal semi-discrete scheme for Navier-Stokes equations was presented and analyzed by Girault and Raviart in [24]. Rebholz and Tone study the H1-stability for positive time of the BDF2 scheme for 2D Navier-Stokes equations [13]. Rong and Fiordilino introduce the BDF2 modular gradient divergence stabilization scheme for Navier-Stokes equation in [14]. To the best of our knowledge, there is no convergence analysis for the NS-omega by using the BDF2 scheme in temporal discretization.

The structure of this paper is as follows. In Section 2, we establish the foundational notation and essential preliminaries that will be employed throughout the paper. Proceeding to Section 3, we delineate the BDF2 discretization scheme for the NS-omega model, accompanied by an in-depth examination of its stability properties. In Section 4, we provide a comprehensive theoretical analysis of the error estimates. Finally, the numerical experiments corroborate the validity of our theoretical findings.

2. Notation and Preliminaries

This section summarizes the notation, definitions and preliminary lemmas needed. In this article, we use the symbol C for positive numbers. It can be different in different places and it is independent of the mesh size h and the time step Δt. We first denote the inner product in $L^2(\Omega)$ and its corresponding norm by $(\cdot,\cdot)$ and $\|\cdot\|$. Similarly, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms [17] norm are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$ and $\|\cdot\|_k$ denotes the norm in H^k. Additionally, we define the function spaces as following

$$
\begin{aligned}
L^p(0,T;L^s(\Omega)) &:= \left\{ v:[0,T]\to L^s(\Omega) \mid \left(\int_0^t \|v(\cdot,t)\|_{L^s}^p dt\right)^{\frac{1}{p}} < \infty \right\},\\
L^\infty(0,T;L^s(\Omega)) &:= \left\{ v:[0,T]\to L^s(\Omega) \mid ess\sup_{0\le t\le T} \|v(\cdot,t)\|_{L^s} < \infty \right\},\\
L^p(0,T;H^k(\Omega)) &:= \left\{ v:[0,T]\to H^k(\Omega) \mid \left(\int_0^t \|v(\cdot,t)\|_{k}^p dt\right)^{\frac{1}{p}} < \infty \right\},\\
L^\infty(0,T;H^k(\Omega)) &:= \left\{ v:[0,T]\to H^k(\Omega) \mid ess\sup_{0\le t\le T} \|v(\cdot,t)\|_{k} < \infty \right\}.
\end{aligned}
$$

Here, $1\le p<\infty, 1\le s\le\infty, 1\le r<\infty$. For the function spaces, we have these norms

$$
\begin{aligned}
\|v\|_{p,k} &:= \left(\int_0^T \|v(\cdot,t)\|_k^p dt\right)^{\frac{1}{p}} \qquad \text{for } v\in L^p(0,T;H^k(\Omega)),\\
\|v\|_{\infty,k} &:= ess\sup_{0\le t\le T}\|v(\cdot,t)\|_k \qquad \text{for } v\in L^\infty(0,T;H^k(\Omega)).
\end{aligned}
$$

We divide the time interval $[0,T]$ into m elements (t_n,t_{n+1}) for $n=0,1,\cdots,M-1$, where $t_n := n\Delta t$ and $\Delta t = \dfrac{T}{m}$. Next, we introduce the analogous norms in the discrete condition

$$
|||v|||_{p,k} = \left(\Delta t\sum_{n=0}^{m}\|v^n\|_k^p\right)^{\frac{1}{p}} \qquad |||v|||_{\infty,k} := \max_{0\le n\le m}\|v^n\|_k,
$$

where $v^n = v(t_n)$ and the same goes for the other variables.

Next, we define the velocity and pressure spaces by

$$X = H_0^1(\Omega) := \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\},$$

$$Q = L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\},$$

We define X^* is the dual space of X, and the norm is denoted by $\|\cdot\|_*$. Then, the space of divergence-free V is denoted by

$$V := \{ v \in X : (\nabla \cdot v, q) = 0 \ \forall q \in Q \}.$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot)$ and $b_\omega(\cdot,\cdot,\cdot)$ by

$$b(u,v,w) := ((\nabla \times u) \times v, w),$$

$$b_\omega(u,v,w) := ((\nabla \times \bar{u}^h) \times v, w).$$

For $u, v, w \in X$, the trilinear form $b(u,v,w)$ satisfy the following inequalities

$$|b(u,v,w)| \le C\| \nabla \times u\| \ \|v\|_\infty \|w\|,$$

$$|b(u,v,w)| \le C\| \nabla \times u\|_\infty \|v\| \ \|w\|,$$

$$|b(u,v,w)| \le C\| \nabla \times u\| \ \|\nabla v\| \ \|\nabla w\|,$$

$$|b(u,v,w)| \le C\|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \|\nabla v\| \ \|\nabla w\|.$$

The weak formulation of (1) is: Find $u : [0,T] \to X$ and $p : [0,T] \to Q$ satisfy

$$\begin{aligned} (u_t, v) + b(u,u,v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v) \quad \forall v \in X, \\ (\nabla \cdot u, q) = 0 \quad \forall q \in Q. \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Suppose Π^h is the uniformly regular triangulations of Ω with $\Omega = \left\{ \bigcup K : K \in \Pi^h \right\}$ as $h \to 0$. Here $h = \max_{K \in \Pi^h} h_K$, where h_K is a diameter of K .We can denote the finite element spaces $X^h \subset X, Q^h \subset Q$. Then, the discrete divergence-free space V^h is denoted by

$$V^h := \{ v^h \in X^h : (\nabla \cdot v^h, q^h) = 0 \ \forall q^h \in Q^h \}.$$

Assume the velocity-pressure finite element spaces X^h and Q^h satisfy the discrete inf-sup condition,

$$\inf_{q^h \in Q^h} \sup_{v^h \in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\| \ \|q^h\|} \ge \beta \ge 0,$$

Where β is a positive constant independent of the mesh size h.

In addition, X^h and Q^h satisfy the approximation properties [16]:

$$\inf_{v \in X^h} \|u - v\| \le Ch^{k+1} \|u\|_{k+1} \qquad v \in H^{k+1}(\Omega)^d,$$

$$\inf_{v \in X^h} \|u - v\|_1 \le Ch^{k} \|u\|_{k+1} \qquad v \in H^{k+1}(\Omega)^d,$$

$$\inf_{q \in Q^h} \|p - q\| \le Ch^{k} \|p\|_{k} \qquad p \in H^{k}(\Omega).$$

The semidiscrete finite element approximation of (2) is: Find $u^h : [0,T] \to X^h$ and $p^h : [0,T] \to Q^h$ satisfy

$$\begin{aligned} (u_t^h, v^h) + b(u^h, u^h, v^h) + \nu(\nabla u^h, \nabla v^h) - (p^h, \nabla \cdot v^h) = (f, v^h) \quad \forall v^h \in X^h, \\ (\nabla \cdot u^h, q^h) = 0 \quad \forall q^h \in Q^h. \end{aligned}$$

To study the discretization of the NS-omega model, we have to deal with the discrete differential filter.
Germano [18] introduced a continuous differential filter into the turbulent model.

Definition 2.1 *Continuous differential filter: For* $\phi \in L^2(\Omega)$ *and* $\alpha > 0$ *is a fixed constant, denote the filter operation on* ϕ *by* $\bar{\phi}$ *, where* $\bar{\phi}$ *is the unique solution of*

$$-\alpha^2\Delta\overline{\phi}+\overline{\phi}=\phi.$$

Then, we define the discete differential filter from Manica [19].

Definition 2.2 *Discrete differential filter: For* $w\in L^2(\Omega)$, *for the filter radius* $\alpha>0$, $\overline{w}^h=(-\alpha^2\Delta+I)^{-1}w$ *is the unique solution of*

$$\alpha^2(\nabla\overline{w}^h,\nabla\psi)+(\overline{w}^h,\psi)=(w,\psi)\ \forall\psi\in X^h. \qquad (3)$$

Now, we introduction some basic properties of discrete differential filters from [19].

Lemma 2.1 *For* $w\in X$, *we have the bound of the discretely filter and approximately* w

$$\|\overline{w}^h\|\le\|w\|,\quad \|\nabla\overline{w}^h\|\le\|\nabla w\|,\quad \|\nabla\times\overline{w}^h\|\le\|\nabla w\|. \qquad (4)$$

Proof The proof of the first inequality in (4), we can set $\psi=\overline{w}^h$ in (3), and use the Cauchy–Schwarz inequality to complete proof. For the second inequality in (4), the filter can be rewritten as

$$-\alpha^2(\Delta\overline{w}^h,\psi)+(\overline{w}^h,\psi)=(w,\psi)\ \forall\psi\in X^h.$$

Set $\psi=\Delta\overline{w}^h$, we can get

$$\alpha^2\|\Delta\overline{w}^h\|^2+\|\nabla\overline{w}^h\|^2=(\nabla w,\nabla\overline{w}^h).$$

By using the Cauchy–Schwarz inequality to complete the proof. The last inequality in (4) use the the second inequality and the inequality $\|\nabla\times\overline{w}^h\|\le\|\overline{w}^h\|$ to complete proof.

The error analysis will use the discrete Gronwall's lemma:

Lemma 2.2 *Suppose that n and N are non-negative integers,* $n\le N$. *The real numbers* $a_n,b_n,c_n,\kappa_n,\Delta t,C$ *are non-negative and satisfy that*

$$a_N+\Delta t\sum_{n=0}^{N}b_n\le\Delta t\sum_{n=0}^{N-1}\kappa_n a_n+\Delta t\sum_{n=0}^{N}c_n+C.$$

Then,

$$a_N+\Delta t\sum_{n=0}^{N}b_n\le\exp\left(\Delta t\sum_{n=0}^{N-1}\kappa_n\right)\left(\Delta t\sum_{n=0}^{N}c_n+C\right).$$

The proof of Lemma 2.2 can be found in [15].

Lemma 2.3 *If* $u_t,u_{tt},u_{ttt}\in L^2(0,T;H^k(\Omega))$, *then we have the following inequalities*

$$\|u_{n+1}-2u_n+u_{n-1}\|_k^2\le C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_k^2dt,$$

$$\left\|\frac{3u_{n+1}-4u_n+u_{n-1}}{2\Delta t}\right\|_k^2\le C\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_k^2dt,$$

$$\left\|\frac{3u_{n+1}-4u_n+u_{n-1}}{2\Delta t}-u_t(t_{n+1})\right\|_k^2\le C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|_k^2dt.$$

We have used the following identity in the later proof

$$2(3a-4b+c,a)=|a|^2-|b|^2+|2a-b|^2-|2b-c|^2+|a-2b+c|^2. \qquad (5)$$

3. Numerical scheme and its stability

In this section, we introduce the BDF2 scheme for the NS-omega model and provide the analysis for its stability. The method is given by: Given $u_n^h,u_{n-1}^h\in V^h$, find $(u_{n+1}^h,p_{n+1}^h)\in X^h\times Q^h$, for all $(v^h,q^h)\in X^h\times Q^h$, satisfy

$$\frac{1}{2\Delta t}(3u_{n+1}^h - 4u_n^h + u_{n-1}^h, v^h) + b_\omega(2u_n^h - u_{n-1}^h, u_{n+1}^h, v^h) + \nu(\nabla u_{n+1}^n, \nabla v^h)$$
$$-(p_{n+1}^h, \nabla \cdot v^h) = (f_{n+1}, v^h), \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$
$$(\nabla \cdot u_{n+1}^h, q^h) = 0.$$

Theorem 3.1 *Consider the NS-omega model with the BDF2 scheme. A solution* $u_l^h, l = 1, \cdots, M-1$, *exists at each time-step. The scheme is unconditionally stable: the solution satisfy the following á priori bound:*

$$\frac{1}{2}\| u_M^h \|^2 + \frac{1}{2}\|2u_M^h - u_{M-1}^h\|^2 + \nu\Delta t \sum_{n=1}^{M-1} \| \nabla u_{n+1}^h \|^2$$
$$\leq \frac{\Delta t}{\nu}\sum_{n=1}^{M-1} \| f_{n+1}\|_*^2 + \frac{1}{2}\| u_1^h \|^2 + \frac{1}{2}\|2u_1^h - u_0^h\|^2$$

Proof Set $v^h = u_{n+1}^h, q^h = p_{n+1}^h$ in (6) and use the identity (5). The nonlinear term in the scheme vanishes.Thus, for every n we can obtain

$$\frac{1}{4\Delta t}(\| u_{n+1}^h\|^2 - \| u_n^h\|^2 + \|2u_{n+1}^h - u_n^h\|^2 - \|2u_n^h - u_{n-1}^h\|^2 + \| u_{n+1}^h - 2u_{n+1}^h + u_{n-1}^h\|^2)$$
$$+\nu \| \nabla u_{n+1}^h\|^2 \leq \frac{1}{2\nu}\| f_{n+1}\|_*^2 + \frac{\nu}{2}\| \nabla u_{n+1}^h\|^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

i.e.,

$$\frac{1}{2\Delta t}(\| u_{n+1}^h\|^2 - \| u_n^h\|^2 + \|2u_{n+1}^h - u_n^h\|^2 - \|2u_n^h - u_{n-1}^h\|^2) + \nu \| \nabla u_{n+1}^h\|^2 \leq \frac{1}{\nu}\| f_{n+1}\|_*^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Multiplying (8) by Δt and then summing it from $n=1$ to $M-1$, we have

Remark 3.1 Since the energy $E(u_n^h) := \frac{1}{2}\| u_n^h\|^2 + \frac{1}{2}\|2u_n^h - u_{n-1}^h\|^2$ and the energy dissipation $\varepsilon(u_n^h) := \nu\| \nabla u_n^h\|^2$ of NS-omega model, we can get

$$E(u_M^h) + \nu\Delta \sum_{n=1}^{M-1}\varepsilon(u_{n+1}^h) = E(u_1^h) + \Delta t\sum_{n=1}^{M-1}(f_{n+1}, u_{n+1}^h). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Assume $\nu = 0$ and $f = 0$, we can get $E(u_M^h) = E(u_1^h)$. Hence, this scheme is energy conserving.

4. Error Analysis

In this section, we present a complete error analysis for method (6). To give the error estimate of the method (6), we firstly denote the error by $e_{n+1} = u_{n+1} - u_{n+1}^h$. Next, we give the error estimate by the following theorem.

Theorem 4.1 *Let* (X^h, Q^h) *be chosen as* (P_2, P_1) *Taylor–Hood elements, and further suppose that* (u, p) *is a solution of the NS-omega model for* $\alpha > 0, \nu > 0$ *, with given* $f \in L^\infty(0,T; H^{-1}(\Omega))$ *, satisfies the following regularity*

$$u \in L^\infty(0,T;(H^{k+1}(\Omega)^d)), \quad u_t \in L^2(0,T;(H^{k+1}(\Omega)^d)), \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$u_{tt} \in L^2(0,T;(H^1(\Omega)^d)), \quad u_{ttt} \in L^2(0,T;(L^2(\Omega)^d)), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$p \in L^2(0,T;H^{s+1}(\Omega)). \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Then, (u_{n+1}^h, p_{n+1}^h) *is given by method* (6) *with* $n \in 1, 2, \cdots, m-1$ *, the error in the discrete solution satisfies*

$$\frac{1}{2}\| e_m\|^2 + \nu\Delta t\sum_{n=1}^{m-1}\| \nabla e_{n+1}\|^2 \leq O(h^{2k} + \alpha^4 + (\Delta t)^4). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Proof Note it for $u, v, w \in X$, we can define the filtering error accordingly:

$$FE = FE_\omega(u, v, w) := ((\nabla \times u - \nabla \times \bar{u}^h) \times v, w).$$

At time t_{n+1} the solution of the $NSE(u,p)$ satisfies,

$$\begin{aligned}&\frac{1}{2\Delta t}(3u_{n+1}-4u_n+u_{n-1},v^h)+b_\omega(u_{n+1},u_{n+1},v^h)+\nu(\nabla u_{n+1},\nabla v^h)\\&\quad-(p_{n+1},\nabla\cdot v^h)=(f_{n+1},v^h)+E^1_{n+1}(v^h)-FE(u_{n+1},u_{n+1},v^h),\end{aligned} \qquad \text{.............................. (13)}$$

where $E^1_{n+1}(v^h)$ is denoted by

$$E^1_{n+1}(v^h)=(\frac{3u_{n+1}-4u_n+u_{n-1}}{2\Delta t}-u_t(t_{n+1}),v^h).$$

We decompose the error as

$$e_{n+1}=\eta_{n+1}+\phi^h_{n+1},\quad \eta_{n+1}=u_{n+1}-U_{n+1},\quad \phi^h_{n+1}=U_{n+1}-u^h_{n+1}.$$

Here, U_{n+1} is the interpolation of u_{n+1} in V^h.

Subtracting (14) from (6), for all $(v^h,q^h)\in(X^h,Q^h)$, we have

$$\begin{aligned}&\frac{1}{2\Delta t}(3e_{n+1}-4e_n+e_{n-1},v^h)+\nu(\nabla e_{n+1},\nabla v^h)-(p_{n+1}-p^h_{n+1},\nabla\cdot v^h)\\&\qquad=E^1_{n+1}(v^h)-E^2_{n+1}(v^h)-FE(u_{n+1},u_{n+1},v^h),\end{aligned} \qquad \text{.......................... (14)}$$

Where E^2_{n+1} is denoted by

$$E^2_{n+1}=b_\omega(u_{n+1},u_{n+1},v^h)-b_\omega(2u^h_n-u^h_{n-1},u^h_{n+1},v^h).$$

Setting $v^h=\phi^h_{n+1}$ in (15), we can get

$$\begin{aligned}\frac{1}{2\Delta t}(3\phi^h_{n+1}-4\phi^h_n+\phi^h_{n-1},\phi^h_{n+1})+\|\nabla\phi^h_{n+1}\|^2=&-\frac{1}{2\Delta t}(3\eta_{n+1}-4\eta_n+\eta_{n-1},\phi^h_{n+1})\\&-(\nabla\eta_{n+1},\nabla\phi^h_{n+1})+(p_{n+1}-\lambda^h_{n+1},\nabla\cdot\phi^h_{n+1})\\&+E^1_{n+1}(\phi^h_{n+1})-E^2_{n+1}(\phi^h_{n+1})\\&-FE(u_{n+1},u_{n+1},\phi^h_{n+1}),\end{aligned} \qquad \text{.................. (15)}$$

for every $\lambda^h_{n+1}\in Q^h$.

i.e.,

$$\begin{aligned}&\frac{1}{4\Delta t}(\|\phi^h_{n+1}\|^2-\|\phi^h_n\|^2+\|2\phi^h_{n+1}-\phi^h_n\|^2-\|2\phi^h_n-\phi^h_{n-1}\|^2+\|\phi^h_{n+1}-2\phi^h_n+\phi^h_{n-1}\|^2)\\&\quad+\nu\|\nabla\phi^h_{n+1}\|^2=-\frac{1}{2\Delta t}(3\eta_{n+1}-4\eta_n+\eta_{n-1},\phi^h_{n+1})-\nu(\nabla\eta_{n+1},\nabla\phi^h_{n+1})\\&\qquad\qquad+(p_{n+1}-\lambda^h_{n+1},\nabla\cdot\phi^h_{n+1})+E^1_{n+1}(\phi^h_{n+1})-E^2_{n+1}(\phi^h_{n+1})\\&\qquad\qquad\qquad-FE(u_{n+1},u_{n+1},\phi^h_{n+1}),\end{aligned} \qquad \text{................. (16)}$$

for every $\lambda^h_{n+1}\in Q^h$.

The terms on the right-hand side of (17) can be bounded as follows. $\forall\varepsilon_1>0$,

$$\begin{aligned}\frac{1}{2\Delta t}(3\eta_{n+1}-4\eta_n+\eta_{n-1},\phi^h_{n+1})&\le C\frac{1}{\Delta t}\|3\eta_{n+1}-4\eta_n+\eta_{n-1}\|^2+\varepsilon_1\|\nabla\phi^h_{n+1}\|^2\\&\le C\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2dt+\varepsilon_1\|\nabla\phi^h_{n+1}\|^2,\end{aligned} \qquad \text{.................. (17)}$$

Where we use the second inequality in lemma 2.3.

$$\begin{aligned}-\nu(\nabla\eta_{n+1},\nabla\phi^h_{n+1})&\le\nu\|\nabla\eta_{n+1}\||\nabla\phi^h_{n+1}\|^2\\&\le C\nu\|\nabla\eta_{n+1}\|^2+\varepsilon_1|\nabla\phi^h_{n+1}\|^2,\end{aligned} \qquad \text{.. (18)}$$

$$(p_{n+1}-\lambda^h_{n+1},\nabla\cdot\phi^h_{n+1})\le C\|p_{n+1}-\lambda^h_{n+1}\|^2+\varepsilon_1|\nabla\phi^h_{n+1}\|^2, \qquad \text{................................... (19)}$$

For the term $E_{n+1}^{1}(\phi_{n+1}^{h})$,

$$\begin{aligned}E_{n+1}^{1}(\phi_{n+1}^{h}) &= (\frac{3u_{n+1}-4u_{n}+u_{n-1}}{2\Delta t}-u_{t}(t_{n+1}),\phi_{n+1}^{h}) \\ &\le C\|\frac{3u_{n+1}-4u_{n}+u_{n-1}}{2\Delta t}-u_{t}(t_{n+1}\|^{2}+\|\ \phi_{n+1}^{h}\|^{2} \quad \text{.......................................} \ (21) \\ &\le C(\Delta t)^{3}\int_{t_{n-1}}^{t_{n+1}}\|\ u_{ttt}\|^{2}dt+\varepsilon_{1}\,|\,\nabla\phi_{n+1}^{h}\|^{2},\end{aligned}$$

Where we use the third inequality in lemma 2.3.

Next, we bound the term $-E_{n+1}^{2}(\phi_{n+1}^{h})$,

$$\begin{aligned}E_{n+1}^{2}(\phi_{n+1}^{h}) &= b_{\omega}(u_{n+1},u_{n+1},v^{h})-b_{\omega}(2u_{n}^{h}-u_{n-1}^{h},u_{n+1}^{h},v^{h}) \\ &= b_{\omega}(u_{n+1}-2u_{n}+u_{n-1},u_{n+1},\phi_{n+1}^{h})+b_{\omega}(2e_{n}-e_{n-1},u_{n+1}^{h},\phi_{n+1}^{h}) \\ &\quad +b_{\omega}(2u_{n}^{h}-u_{n-1}^{h},\eta_{n+1},\phi_{n+1}^{h}) \qquad \text{....................} \ (22) \\ &= b_{\omega}(u_{n+1}-2u_{n}+u_{n-1},u_{n+1},\phi_{n+1}^{h})+b_{\omega}(2\eta_{n}-\eta_{n-1},u_{n+1}^{h},\phi_{n+1}^{h}) \\ &\quad +b_{\omega}(2\phi_{n}^{h}-\phi_{n-1}^{h},u_{n+1}^{h},\phi_{n+1}^{h})+b_{\omega}(2u_{n}^{h}-u_{n_{1}}^{h},\eta_{n+1},\phi_{n+1}^{h})\end{aligned}$$

$\forall\varepsilon_{2},\varepsilon_{3}>0$, we have the following estimates.

$$\begin{aligned}b_{\omega}(u_{n+1}-2u_{n}+u_{n-1},u_{n+1},\phi_{n+1}^{h}) &\le C\|\ \nabla(u_{n+1}-2u_{n}+u_{n-1})\|\ \ \|\ \nabla u_{n+1}\|\ \ \|\ \nabla\phi_{n+1}^{h}\| \\ &\le C\|\ \nabla(u_{n+1}-2u_{n}+u_{n-1})\|^{2}\|\ \nabla u_{n+1}\|^{2}+\grave{o}_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2} \quad \text{....................} \ (23) \\ &\le C(\Delta t)^{3}\int_{t_{n-1}}^{t_{n+1}}\|\ \nabla u_{tt}\|^{2}dt\|\ u\|_{\infty,1}^{2}+\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2},\end{aligned}$$

Where we use the first inequality in lemma 2.3.

$$\begin{aligned}b_{\omega}(2\eta_{n}-\eta_{n-1},u_{n+1}^{h},\phi_{n+1}^{h}) &\le 2\,|\,b_{\omega}(\eta_{n},u_{n+1}^{h},\phi_{n+1}^{h})\,|+|\,b_{\omega}(\eta_{n-1},u_{n+1}^{h},\phi_{n+1}^{h})\,| \\ &\le C\|\ \nabla\eta_{n}\|\ \ \|\ \nabla u_{n+1}^{h}\|\ \ \|\ \nabla\phi_{n+1}^{h}\|+C\|\ \nabla\eta_{n-1}\|\ \ \|\ \nabla u_{n+1}^{h}\|\ \ \|\ \nabla\phi_{n+1}^{h}\| \quad \text{..........} \ (24) \\ &\le C(\|\ \nabla\eta_{n}\|^{2}+\|\ \nabla\eta_{n-1}\|^{2})\|\ \nabla u_{n+1}^{h}\|^{2}+\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2},\end{aligned}$$

$$\begin{aligned}b_{\omega}(2\phi_{n}^{h}-\phi_{n-1}^{h},u_{n+1}^{h},\phi_{n+1}^{h}) &\le 2\,|\,b_{\omega}(\phi_{n}^{h},u_{n+1}^{h},\phi_{n+1}^{h})\,|+|\,b_{\omega}(\phi_{n-1}^{h},u_{n+1}^{h},\phi_{n+1}^{h})\,| \\ &\le C\|\ \phi_{n}^{h}\|^{\frac{1}{2}}\|\ \nabla\phi_{n}^{h}\|^{\frac{1}{2}}\|\ \nabla u_{n+1}^{h}\|\ \ \|\ \nabla\phi_{n+1}^{h}\| \\ &\quad +C\|\ \phi_{n-1}^{h}\|^{\frac{1}{2}}\|\ \nabla\phi_{n-1}^{h}\|^{\frac{1}{2}}\|\ \nabla u_{n+1}^{h}\|\ \ \|\ \nabla\phi_{n+1}^{h}\| \\ &\le C\|\ \phi_{n}^{h}\|\ \ \|\ \nabla\phi_{n}^{h}\|\ \ \|\ \nabla u_{n+1}^{h}\|^{2} \qquad \text{...............................} \ (25) \\ &\quad +C\|\ \phi_{n-1}^{h}\|\ \ \|\ \nabla\phi_{n-1}^{h}\|\ \ \|\ \nabla u_{n+1}^{h}\|^{2}+\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2} \\ &\le C\|\ \phi_{n}^{h}\|^{2}\|\ \nabla u_{n+1}^{h}\|^{4}+C\|\ \phi_{n-1}^{h}\|^{2}\|\ \nabla u_{n+1}^{h}\|^{4} \\ &\quad +\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2}+\varepsilon_{2}\|\ \nabla\phi_{n}^{h}\|^{2}+\varepsilon_{3}\|\ \nabla\phi_{n-1}^{h}\|^{2}\end{aligned}$$

$$\begin{aligned}b_{\omega}(2u_{n}^{h}-u_{n_{1}}^{h},\eta_{n+1},\phi_{n+1}^{h}) &\le 2\,|\,b_{\omega}(u_{n}^{h},\eta_{n+1},\phi_{n+1}^{h})\,|+|\,b_{\omega}(u_{n-1}^{h},\eta_{n+1},\phi_{n+1}^{h})\,| \\ &\le C\|\ \nabla u_{n}^{h}\|\ \ \|\ \nabla\eta_{n+1}\|\ \ \|\ \nabla\phi_{n+1}^{h}\|+C\|\ \nabla u_{n-1}^{h}\|\ \ \|\ \nabla\eta_{n+1}\|\ \ \|\ \nabla\phi_{n+1}^{h}\| \quad \text{..........} \ (26) \\ &\le C(\|\ \nabla u_{n}^{h}\|^{2}+\|\ \nabla u_{n-1}^{h}\|^{2})\|\ \nabla\eta_{n+1}\|^{2}+\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2},\end{aligned}$$

Combining (23-26) with (22), we can get

$$\begin{aligned}-E_{n+1}^{2}(\phi_{n+1}^{h}) &\le C(\Delta t)^{3}\int_{t_{n-1}}^{t_{n+1}}\|\ \nabla u_{ttt}\|^{2}dt+C(\|\ \phi_{n}^{h}\|^{2}+\|\ \phi_{n-1}^{h}\|^{2}) \\ &\quad +C(\|\ \nabla\eta_{n}^{h}\|+\|\ \nabla\eta_{n-1}\|^{2})+C(\|\ \nabla u_{n}^{h}\|^{2}+\|\ \nabla_{n-1}^{h}\|^{2})\|\ \nabla\eta_{n+1}\|^{2} \quad \text{...............................} \ (20) \\ &\quad +4\varepsilon_{1}\|\ \nabla\phi_{n+1}^{h}\|^{2}+\varepsilon_{2}\|\ \nabla\phi_{n}^{h}\|^{2}+\varepsilon_{3}\|\ \nabla\phi_{n-1}^{h}\|^{2},\end{aligned}$$

Next, we use the definition of the discrete filter to bound the filtering error

$$\begin{aligned} -FE &\le |(\nabla\times u_{n+1} - \nabla\times \overline{u_{n+1}}^h)\times u_{n+1}, \phi_{n+1}^h| \\ &\le C\|\nabla\times(u_{n+1} - \overline{u_{n+1}}^h)\| \; \|\nabla u_{n+1}\| \; \|\nabla\phi_{n+1}^h\| \\ &\le \varepsilon_1\|\nabla\phi_{n+1}^h\|^2 + C\alpha^4\|\nabla u_{n+1}\|^2\|\nabla\Delta^h\overline{u_{n+1}}^h\|^2 \\ &\le \varepsilon_1\|\nabla\phi_{n+1}^h\|^2 + C\alpha^4\|\nabla u_{n+1}\|^2\|u_{n+1}\|^2. \end{aligned} \qquad \text{(28)}$$

Set $\varepsilon_1 = \frac{1}{16}\nu, \varepsilon_2 = \frac{1}{8}\nu, \varepsilon_3 = \frac{1}{8}\nu$. From the mentioned above estimate (17-21), (27) and (28), we can get

$$\begin{aligned} &\frac{1}{4\Delta t}(\|\phi_{n+1}^h\|^2 - \|\phi_n^h\|^2 + \|2\phi_{n+1}^h - \phi_n^h\|^2 - \|2\phi_n^h - \phi_{n-1}^h\|^2 + \|\phi_{n+1}^h - 2\phi_n^h + \phi_{n-1}^h\|^2) \\ &\quad + \frac{1}{2}\nu\|\nabla\phi_{n+1}^h\|^2 - \frac{1}{8}\nu\|\nabla\phi_n^h\|^2 - \frac{1}{8}\nu\|\nabla\phi_{n-1}^h\|^2 \\ &\le C(\|\phi_n^h\|^2 + \|\phi_{n-1}^h\|^2) + C(\|\nabla\eta_{n+1}\|^2 + \|\nabla\eta_n\|^2 + \|\nabla\eta_{n-1}\|^2) \\ &\quad + C(\|\nabla u_n^h\|^2 + \|\nabla_{n-1}^h\|^2)\|\nabla\eta_{n+1}\|^2 + \|p_{n+1} - \lambda_{n+1}^h\|^2 \\ &\quad + C\alpha^4\|\nabla u_{n+1}\|^2\|u_{n+1}\|^2 + C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2 dt \\ &\quad + C\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2 dt + C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2 dt, \end{aligned} \qquad \text{(21)}$$

for every $\lambda_{n+1}^h \in Q^h$.

Multiplying (29) by $2\Delta t$ and summing it from $n=1$ to $m-1$, we have

$$\begin{aligned} &\frac{1}{2}\|\phi_m^h\|^2 + \frac{1}{2}\|2\phi_m^h - \phi_{m-1}^h\|^2 + \nu\Delta t\sum_{n=1}^{m-1}\|\nabla\phi_{n+1}^h\|^2 \\ &\le \frac{1}{2}\|\phi_1^h\|^2 + \frac{1}{2}\|2\phi_1^h - \phi_0^h\|^2 + C\Delta t\sum_{n=1}^{m-1}(\|\phi_n^h\|^2 + \|\phi_{n-1}^h\|^2) + \nu\frac{\Delta t}{4}\|\nabla\phi_1^h\|^2 \\ &\quad + \nu\frac{\Delta t}{4}\|\nabla\phi_0^h\|^2 + C\Delta t\sum_{n=1}^{m-1}(\|\nabla\eta_{n+1}\|^2 + \|\nabla\eta_n^h\|^2 + \|\nabla\eta_{n-1}\|^2) \\ &\quad + C\Delta t\sum_{n=1}^{m-1}(\|\nabla u_n^h\|^2 + \|\nabla_{n-1}\|^2)\|\nabla\eta_{n+1}\|^2 + C\Delta t\sum_{n=1}^{m-1}\|p_{n+1} - \lambda_{n+1}^h\|^2 \\ &\quad + C\Delta t\sum_{n=1}^{m-1}\alpha^4\|\nabla u_{n+1}\|^2\|u_{n+1}\|^2 + C\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2 dt \\ &\quad + C(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2 dt + C(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2 dt, \end{aligned} \qquad \text{(22)}$$

Applying lemma 2.2 to (30), we have:

$$\begin{aligned} \frac{1}{2}\|\phi_m^h\|^2 + \nu\Delta t\sum_{n=1}^{m-1}\|\nabla\phi_{n+1}^h\|^2 &\le C\|\phi_1^h\|^2 + C\|2\phi_1^h - \phi_0^h\|^2 \\ &\quad + C\nu\Delta t\|\nabla\phi_1^h\|^2 + C\nu\Delta t\|\nabla\phi_0^h\|^2 \\ &\quad + C\Delta t\sum_{n=1}^{m-1}(\|\nabla\eta_{n+1}\|^2 + \|\nabla\eta_n^h\|^2 + \|\nabla\eta_{n-1}\|^2) \\ &\quad + C\Delta t\sum_{n=1}^{m-1}(\|\nabla u_n^h\|^2 + \|\nabla_{n-1}^h\|^2)\|\nabla\eta_{n+1}\|^2 \\ &\quad + C\Delta t\sum_{n=1}^{m-1}\|p_{n+1} - \lambda_{n+1}^h\|^2 + C\Delta t\sum_{n=1}^{m-1}\alpha^4\|\nabla u_{n+1}\|^2\|u_{n+1}\|^2 \\ &\quad + C\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2 dt + C(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2 dt \\ &\quad + C(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2 dt, \end{aligned} \qquad \text{(23)}$$

Then, we bound the terms on the right-hand side of (31).

For the term $\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2 dt,$ we have the following estimate

$$\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2 dt \le Ch^{2k+2}\|u_t\|_{2,k+1}^2. \qquad (32)$$

For the term $(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2 dt$, we have the following estimate

$$(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2 dt \le C(\Delta t)^4\|u_{ttt}\|_{2,0}^2. \qquad (33)$$

For the term $(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2 dt$, we have the following estimate:

$$(\Delta t)^4\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2 dt \le C(\Delta t)^4\|u_{tt}\|_{2,1}^2. \qquad (34)$$

For the term $\Delta t\sum_{n=1}^{m-1}(\|\nabla\eta_{n+1}\|^2+\|\nabla\eta_n\|^2+\|\nabla\eta_{n-1}\|^2)$, we have the following estimate

$$\begin{aligned}\Delta t\sum_{n=1}^{m-1}(\|\nabla\eta_{n+1}\|^2+\|\nabla\eta_n\|^2+\|\nabla\eta_{n-1}\|^2) &\le C\Delta t\sum_{n=0}^{m}\|\nabla\eta_n\|^2\\ &\le C\Delta t\sum_{n=0}^{m}h^{2k}\|u_n\|_{k+1}^2 \qquad (35)\\ &= Ch^{2k}\,\||u\||_{2,k+1}^2 .\end{aligned}$$

For the term $\Delta t\sum_{n=1}^{m-1}(\|\nabla u_n^h\|^2+\|\nabla_{n-1}^h\|^2)\|\nabla\eta_{n+1}\|^2$, we have the following estimate

$$\begin{aligned}\Delta t\sum_{n=1}^{m-1}(\|\nabla u_n^h\|^2+\|\nabla_{n-1}^h\|^2)\|\nabla\eta_{n+1}\|^2 &\le Ch^{2k}\Delta t\sum_{n=1}^{m-1}(\|\nabla u_n^h\|^2+\|\nabla_{n-1}^h\|^2)\|)\|u_{n+1}\|_{k+1}^2\\ &\le Ch^{2k}\,\||u\||_{\infty,k+1}^2\,(\Delta t\sum_{n=0}^{m-1}\|\nabla u_n^h\|^2) \qquad (36)\\ &\le Ch^{2k}\,\||u_{n+1}\||_{\infty,k+1}^2 .\end{aligned}$$

For the term $\Delta t\sum_{n=1}^{m-1}\|p_{n+1}-\lambda_{n+1}^h\|^2$, λ_{n+1}^h is the interpolation of p_{n+1} in Q^h, we have the following estimate

$$\Delta t\sum_{n=1}^{m-1}\|p_{n+1}-\lambda_{n+1}^h\|^2 \le Ch^{2k}\,\||p\||_{2,k}^2 \qquad (37)$$

Combining (32)-(37) with (31)

$$\frac{1}{2}\|\phi_m^h\|^2+\nu\Delta t\sum_{n=1}^{m-1}\|\nabla\phi_{n+1}^h\|^2 \le O(h^{2k}+\alpha^4+(\Delta t)^4). \qquad (38)$$

Using the triangle inequality, we can get

$$\frac{1}{2}\|e_m\|^2+\nu\Delta t\sum_{n=1}^{m-1}\|\nabla e_{n+1}\|^2 \le O(h^{2k}+\alpha^4+(\Delta t)^4). \qquad (39)$$

We've done the proof.

With the results, we can get that for the Taylor-Hood mixed finite element $k=2$, we can get the corollary.

Corollary 4.1 *Suppose that the finite element spaces* X^h, Q^h *are Taylor-Hood elements. Assume that the true solution* (u, p) *satisfies the reqularity in Theorem* (4.1) *with* $k = 2$. *Then, we can get the following estimate.*

$$||| e|||_{\infty,0}^2 + ||| \nabla e|||_{2,0}^2 \le O(h^4 + \alpha^4 + (\Delta t)^4). \quad (24)$$

5. Numerical experiments

In this section, we use two numerical experiments to validate the theoretical analysis. The first experiment computed the errors and convergence rates of our scheme. The second experiment is the benchmark test of channel flow over a step. We use the (P_2, P_1) Taylor-Hood finite elements for velocity spaces and pressure spaces. The software Freefem++ [20] was used for numerical experiments.

5.1 Convergence rate

In order to verify the convergence rate of our scheme, we consider the NS-omega model in unit square domain $\Omega = [0,1]\times[0,1]$ with the solution such that:

$$u_1(x, y, t) = (1 + 0.01t)sin(2\pi y),$$
$$u_2(x, y, t) = (1 + 0.01t)cos(2\pi),$$
$$p(x, y, t) = x + y.$$

In the test, we choose $\nu = 1, T = 0.01$ and $\alpha = h = 1/m$, where m is the number of subdivisions of interval $[0,1]$. By using (P_2, P_1) on uniform meshes, we compute errors and convergence rates by changing mesh size. The result for NS-omega model is presented in Table 1. The values of convergence rates are consistent with the theoretical analysis.

Table 1: Error and convergence rates for NS-omega

m	Δt	$\lVert\lVert u - u^h \rVert\rVert_{\infty,0}$	Rate	$\lVert\lVert \nabla u - \nabla u^h \rVert\rVert_{2,0}$	Rate	$\lVert\lVert p - p^h \rVert\rVert_{2,2}$	Rate
8	T/2	1.3353e-2		3.7090e-1		1.8351e-2	
16	T/4	3.3540e-3	1.9932	1.0663e-1	1.798	4.7546e-3	1.948
32	T/8	8.3941e-4	1.9984	2.8756e-2	1.891	1.2094e-3	1.975
64	T/16	2.1068e-4	1.9943	7.6998e-3	1.901	3.0612e-4	1.982

5.2 Channel flow over a step

In this numerical experiment, we consider the BDF2 scheme for the NS-omega model to simulate the two-dimensional channel flow over a step. The domain Ω of this experiment is a $[0,40]\times[0,10]$ rectangle with a 1×1 step on the bottom for $x \in [5,6]$. The no-slip boundary conditions are applied on the top and bottom boundaries and the steps. The parabolic inflow (left) and outflow (right) conditions are given by:

$$u_1(0, y) = u_1(40, y) = \frac{y(10-y)}{25},$$
$$u_2(0,y) = u_2(40,y) = 0.$$

The flow with a viscosity of $\nu = 1/600$ through the channel from left to right. The force $f = 0$, time step $\Delta t = 0.01$ and run the test until T = 40. In the simulation, the coarse and fine meshes used for the computations of the simulations as shown in Figure 1 and Figure 2. The degrees of freedom is 24544 in the coarse mesh, while 41322 in the fine mesh. With these settings, we anticipate the emergence of eddies down-stream of the obstruction. Over time, these eddies are eventually growing and detaching.

We first present the result of computing the NSE directly. Computations were performed on a mesh using (P_2, P_1) Taylor-Hood elements and the BDF2 finite element formulation. The solution at T = 40 is shown in Figure 3. Although it predicted a smooth flow field, it did not capture the reformation of eddies after detachment.

As the primary goal of fluid flow models is to accurately predict solution on a coarser mesh than that required by DNS, we conducted test of the method on the coarse mesh and obtained the following result using identical parameters as those used for fine mesh NSE computation. The NS-omega model found a smooth flow field and captured the correct behavior.

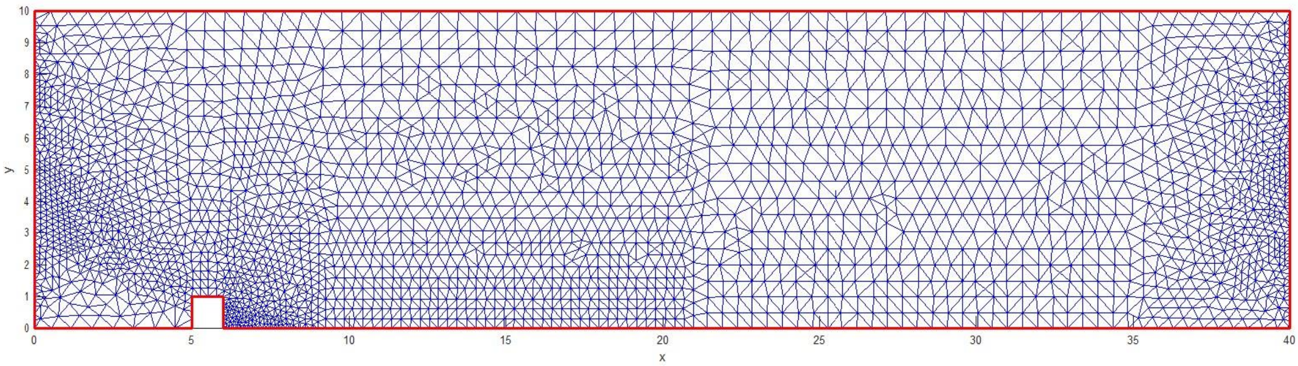

Figure 1: Two-dimensional channel flow over a step, coarse mesh

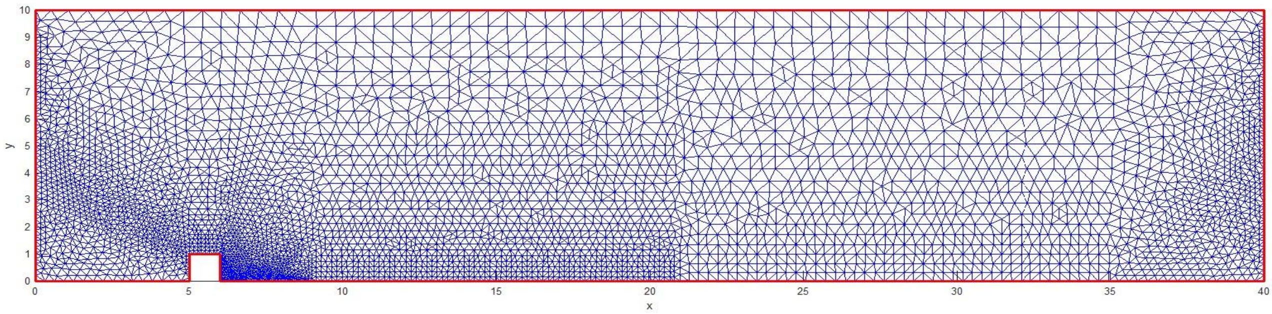

Figure 2: Two-dimensional channel flow over a step, fine mesh

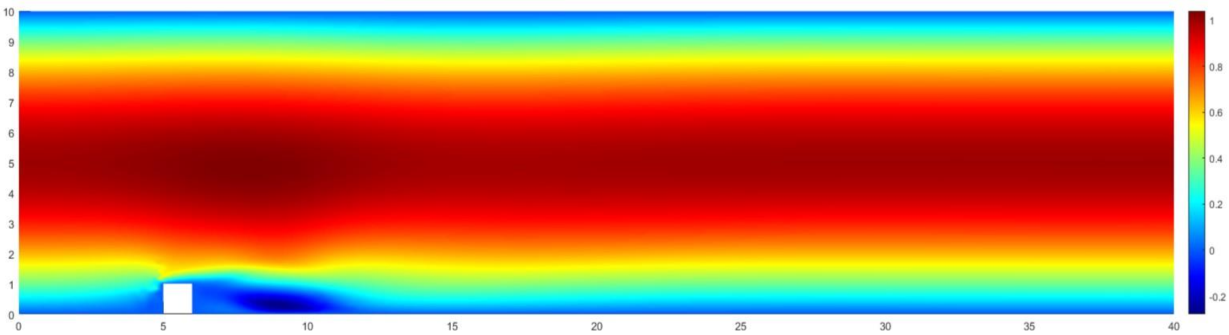

Figure 3: Navier-Stokes equations on the fine mesh, T=40, ν = 1/600

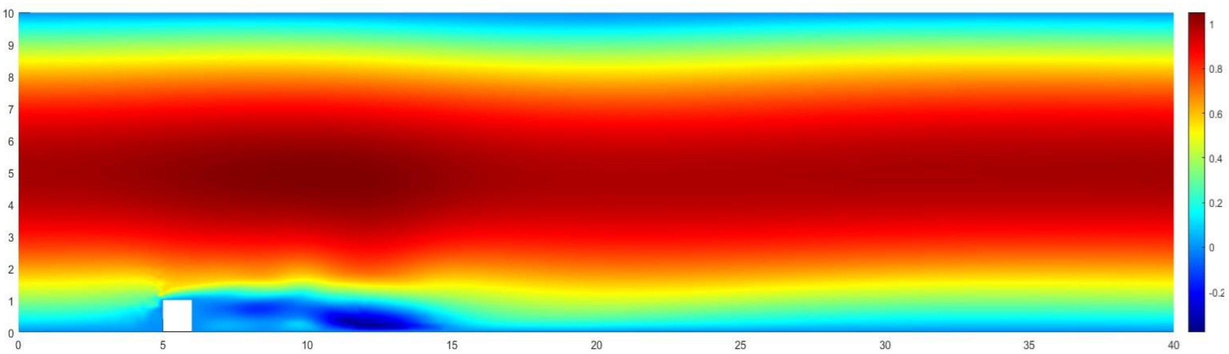

Figure 4: NS-omega model on the coarse mesh, T=40, ν = 1/600, α = 0.125

6. CONCLUSION

In this article, we applied the BDF2 time-stepping scheme with the finite element method to solve the NS-omega model with no-slip boundary condition in $\mathbb{R}^d, d = 2,3$. We provided a complete numerical analysis about the stability and convergence of the solution. In the analysis, we proved the solution is unconditional stability, and converge to the true solution of NSE. The correctness of the theoretical analysis is proved by several numerical experiments.

In the future, we will commit to extending this model across the broader range of application fields. For instance, by incorporating the method with pressure correction and the generalized scalar auxiliary variable

(GSAV) approach, we will delve into the stability and convergence of these methods.

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