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A Three-step Hybrid Block Method for Direct Integration of Third Order Ordinary Differential Equations

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Abstract **Original Research Article**

This article presents a novel continuous numerical method designed for the numerical integration of general third-order initial value problems (IVPs) of ordinary differential equations (ODEs). A combination of power series and exponential function was formulated for the purpose of collocation and interpolation at nodal and off-nodal points to generate system of linear equations necessary for the method. The resulting hybrid linear multistep method was implemented using block mode approach. Consistency, stability, and convergence of the method were verified using established criteria. The developed method was applied directly to solve linear and nonlinear third order ODEs without reducing them to systems of first-order equations. Computational results demonstrated better accuracy when compared with existing numerical methods in the literature.

Keywords: Linear Multistep Method, Nodal Points, Off-Nodal Points, Block Mode, Collocation, Interpolation
MSC Subject Classification, 65L05, 65L06, 65L07, 65L60.

1. INTRODUCTION

High-order linear and nonlinear initial value problems (IVPs) frequently arise in engineering and scientific applications, particularly in fields such as Biological Sciences and Control Theory, where their solutions are critically important. Conventionally, these high-order IVPs are often addressed by the reduction method (see Lambert [1973] and Fatula [1988]), which transforms the high-order equation into a system of first-order ordinary differential equations (ODEs). The reduction approach has several limitations, including unnecessary computational burden, excessive computer subroutines, and high computational costs (see Mehrkanoon [2011], Kayode [2011], Kayode and Adeyeye [2013], Awoyemi et al [2014], Kayode and Obarhua [2015]).

This paper discusses the development of approximate solution of general third-order ordinary differential equations of the form:

$$y''' = f(x, y, y', y''), y(x_0) = \xi_0, y'(x_0) = \xi_1, y''(x_0) = \xi_2. \tag{1}$$

Where $x, y \in \mathbb{R}^n$ and $f \in \mathbb{C}'[a, b]$.

Many authors has highlighted the advantages of direct methods, in solving higher-order IVPs that avoid the reduction process, with improved computational efficiency and accuracy. In this regard, several continuous collocation and interpolation techniques have been extensively studied, For example, Kayode [2011] investigated a three-step one point method based on collocation at selected both one off-grid and grid points to approximate second order ordinary differential equations but with low order of accuracy. Kayode and Adeyeye [2013], however, propagated a two-step two-point hybrid method for general second order differential equations with application of Chebyshev series as an approximate solution. The computational results showed that the method is better in accuracy than some existing methods. Areo and Adeniyi [2013] investigated a self-starting linear multistep method for direct solution of IVPs of second order ODEs. Kayode and Obarhua [2015] constructed a 3-step y-function Hybrid Methods for Direct Numerical Integration of second Order IVPs in ODEs. In all these methods, third order ordinary differential equations cannot be solved unless reduced to second order ODEs.

In this development, several direct methods have been proposed for solving (1) in literature. Allogmany and Ismail [2020] examined an Implicit Three-Point Block Numerical Algorithm for Solving Third Order Initial Value Problem Directly with Applications. Adeyefa and Olanegan [2022] proposed an Accurate Four-Step Hybrid Block Method for solving Higher-Order Initial Value Problems. Duromola [2022] developed a Single-Step Block Method of P-Stable for solving Third-Order Differential Equations (IVPs) having Ninth Order of Accuracy. Other works in literature on this topic include Ramos and Rufai [2018], Abolarin *et al*., [2020]. In Obarhua [2023] a ninth-order three-step, four-point optimized hybrid block method was developed all with the intention to solving same problem. Motivated by the ongoing quest for higher accuracy and efficiency in numerical integration, this study presents a three-step with six-point Hybrid Block Method. This novel method is problem-independent, providing high degree of freedom in the choice of interpolation points based on the order of the differential equation with greater adaptability in application compared to other problem-dependent block methods. Specifically, the research introduces an order-ten block integrator with six off-step points, designed for third-order ODEs, a significant advancement in the direct numerical solution of high-order ODEs.

2. Mathematical Formulation

In this work, the approximation of the exact solution $y(x)$ of the third-order initial value problem of ordinary differential Equation (1) is considered by a combination of power series polynomial and exponential functions of the type

$$p(x) = \sum_{j=0}^{n-1} a_j x^j + a_{c+i-1} \sum_{j=0}^{n} \frac{\alpha_j x^j}{j!} \tag{2}$$

The third derivatives of (2) is obtained as

$$p'''(x) = \sum_{j=3}^{n-4} j(j-1)(j-2) a_j x^{j-3} + a_{j-3} \sum_{j=3}^{n} \frac{\alpha_j x^{j-3}}{(j-3)!} \tag{3}$$

Equations (1) and (3) yields a differential system:

$$f(x, y, y', y'') = \sum_{j=3}^{n-4} j(j-1)(j-2) a_j x^{j-3} + a_{j-3} \sum_{j=3}^{n} \frac{\alpha_j x^{j-3}}{(j-3)!} \tag{4}$$

where x is continuous and differentiable, parameters a_j 's in (2), (3), and (4) are linear terms to be determined. To get the system of algebraic equations in equations (5) and (6), $= x_{n+j}, j = 0, \tau_1, \tau_2, \tau_3, 1,2, \psi_1, \psi_2,$ and ψ_3 was applied to equation (2) and $x = x_{n+j}, j = 0(1)3$ applied to equation (4).

$$y_{n+j} = \sum_{j=3}^{9} a_j x^{j-1} + \sum_{j=3}^{10} \frac{\alpha_j x^{j-3}}{(j-3)!}, j = 0, \tau_1, \tau_2, \tau_3, 1,2, \psi_1, \psi_2, \psi_3 \tag{5}$$

$$f_{n+j} = \sum_{j=3}^{6} j(j-1)(j-2) a_j x^{j-3} + a_{j-3} \sum_{j=3}^{10} \frac{\alpha_j x^{j-3}}{(j-3)!} j = 0(1)3 \tag{6}$$

Using the relation $x_{n+\frac{j}{4}} = x_n + \frac{jh}{4}$, (5) and (6) were written as matrix form and solved using CAS in Wolfram Mathematical to obtain the parameters a_j 's for $\mathrm{j} = 0,1,2, \cdots, 12$ which were then substituted back into (2) to yields the following continuous scheme after some simplifications:

$$y(x) = \sum_{j=0}^{2} \alpha_j y_{n+j} + \sum_{i>0}^{<1} \alpha_{\tau_i} y_{n+\tau_i} + \sum_{v>2}^{<3} \alpha_{\psi_v} y_{n+\psi_v} + h^3 \sum_{j=0}^{3} \beta_j f_{n+j} \tag{7}$$

where $x = x_{n+t} = x_n + th, \alpha_j' s$ and $\beta_j' s$ are the coefficients that defined the scheme. Evaluating (7) at $t = 3$ yields the main formula of the developed Three-Step Hybrid Block method. This gives

$$y_{n+3}+\frac{12298770837}{433189561}y_{n+2}+\frac{12406261536}{433189561}y_{n+\frac{5}{2}}+\frac{3309410304}{433189561}y_{n+\frac{1}{4}} \\ +\frac{20091557888}{433189561}y_{n+\frac{3}{4}}-\frac{12298770837}{433189561}y_{n+1}-\frac{12406261536}{433189561}y_{n+\frac{1}{2}}-\frac{20091557888}{433189561}y_{n+\frac{9}{4}} \tag{8}$$

2.1 Block Formulation of the Derived Formula

In keeping with [7], the normalized form of the general block method is given by

$$GY_i = Ey_n + h^{\mu-\rho}df(y_n) + h^{\mu-\rho}HF(y_n) \tag{9}$$

To derive the block formula described in (9), we combine the formulas in (8) with the first, and second derivative formulas obtained from (7), and write them in block form using the definition of the implicit block method in (9)

$$h^a\sum_{j=0}^{q}\varphi_{m,j}y_{n+j}^{\phi} = h^{\phi}\sum_{r=0}^{q}\nabla_{m,j}y_n^{\phi} + h^{p-\phi}\left(\sum_{j=0}^{q}\Delta_{m,j}f_n + \sum_{r=0}^{q}\eta_{m,j}f_{n+j}\right) \tag{10}$$

where ρ represent the power of the derivative of the continuous method and p represent the order of the problem to be solved. Equation (10) was solved for $j = 0\left(\frac{1}{4}\right)3$ in order to obtain the following block formulas that constitute the derived Three-Step Hybrid Block Method.

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy_n' + \frac{1}{32}h^2y_n'' + \frac{h^3}{18884159078400}\Big(27324004827df_n + 44926673520f_{n+\frac{1}{4}} - 48487649276f_{n+\frac{1}{2}} \\ + 40628682160f_{n+\frac{3}{4}} - 16678823337f_{n+1} + 8023499385f_{n+2} - 12626777328f_{n+\frac{9}{4}} + 8495097468f_{n+\frac{5}{2}} \tag{11}$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy_n' + \frac{1}{8}h^2y_n'' + \frac{h^3}{73766246400}\Big(544380697f_n + 1554906880f_{n+\frac{1}{4}} - 1198319100f_{n+\frac{1}{2}} \\ + 1015669248f_{n+\frac{3}{4}} - 416275827f_{n+1} + 199365507f_{n+2} - 313570048f_{n+\frac{9}{4}} + 210868988f_{n+\frac{5}{2}} \tag{12}$$

$$y_{n+\frac{3}{4}} = y_n + \frac{1}{4}3hy_n' + \frac{9}{32}h^2y_n'' + \frac{h^3}{77712588800}\Big(1392449289f_n + 4924186128f_{n+\frac{1}{4}} - 2503958292f_{n+\frac{1}{2}} \\ + 2639875920f_{n+\frac{3}{4}} - 1083763395f_{n+1} + 522575955f_{n+2} - 822426000f_{n+\frac{9}{4}} + 553319316f_{n+\frac{5}{2}} \tag{13}$$

$$y_{n+1} = y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{h^3}{576298800}\Big(19059894f_n + 74491008f_{n+\frac{1}{4}} - 24717880f_{n+\frac{1}{2}} + 40870016f_{n+\frac{3}{4}} \tag{14}$$

$$y_{n+2} = y_n + 2hy_n' + 2h^2y_n'' + \frac{h^3}{36018675}\Big(5150159f_n + 21707648f_{n+\frac{1}{4}} + 2233968f_{n+\frac{1}{2}} + 10158720f_{n+\frac{3}{4}} \tag{15}$$

$$y_{n+\frac{9}{4}} = y_n + \frac{9}{4}hy_n' + \frac{81}{32}h^2y_n'' + \frac{h^3}{77712588800}\Big(14343221529af_n + 58964284368bf_{n+\frac{1}{4}} + 13713933420f_{n+\frac{1}{2}} \\ + 19966540176f_{n+\frac{3}{4}} + 32642582445f_{n+1} + 25358652099f_{n+2} - 30971088720hf_{n+\frac{9}{4}} \tag{16}$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5}{2}hy_n' + \frac{25}{8}h^2y_n'' + \frac{h^3}{2950649856}\Big(684728625f_n + 2737344000f_{n+\frac{1}{4}} + 984494500f_{n+\frac{1}{2}} \\ + 553696000f_{n+\frac{3}{4}} + 2111905125f_{n+1} + 1686460875f_{n+2} - 1902912000f_{n+\frac{9}{4}} + 1123633500f_{n+\frac{5}{2}} \tag{17}$$

$$y_{n+\frac{11}{4}} = y_n + \frac{11}{4}hy'_{\,n} + \frac{121}{32}h^2yn'' - \frac{h^3}{156067430400}(44552319427f_n + 169736963583f_{n+1} + 140184427185f_{n+2} \\ + 3259394061f_{n+3} + 83289779364f_{n+\frac{1}{2}} + 89233205468f_{n+\frac{5}{2}} + 173160863920f_{n+\frac{1}{4}} + 11557019760f_{n+\frac{3}{4}} \tag{18}$$

$$y_{n+3} = y_n + 3hy_n' + \frac{9}{2}h^2y_n'' + \frac{h^3}{2371600}\Big(817698f_n + 3093120f_{n+\frac{1}{4}} + 1841400f_{n+\frac{1}{2}} - 198528f_{n+\frac{3}{4}} \tag{19}$$

$$y'_{n+\frac{1}{4}} = y'_n + \frac{hy''_n}{4} + \frac{h^2}{1180259942400}\Big(16937905691af_n + 38008281600bf_{n+\frac{1}{4}} - 37530351452cf_{n+\frac{1}{2}} + 31025613520f_{n+\frac{3}{4}} - 12663740133f_{n+1} - 9506522432f_{n+\frac{9}{4}} + 6391847484f_{n+\frac{5}{2}} \quad (20)$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{hy''_n}{2} + \frac{h^2}{9220780800}\Big(303963961f_n - 241231749f_{n+\frac{1}{4}} + 116562303f_{n+2} + 5428045f_{n+3} - 632649600f_{n+\frac{1}{2}} + 123431264f_{n+\frac{5}{2}} + 1119154480f_{n+1} + 582089552f_{n+\frac{3}{4}} - 183455536f_{n+\frac{9}{4}} \quad (21)$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{3}{4}hy''_n + \frac{h^2}{4857036800}\Big(249574059f_n + 1051054992f_{n+\frac{1}{4}} - 262025676f_{n+\frac{1}{2}} + 510456320f_{n+\frac{3}{4}} - 200583405f_{n+1} - 151454160f_{n+\frac{9}{4}} + 101899116f_{n+\frac{5}{2}} - 33596352f_{n+\frac{11}{4}} + 4481281f_{n+3} \quad (22)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{288149400}\Big(20118959f_n + 89638464f_{n+\frac{1}{4}} - 9493880f_{n+\frac{1}{2}} + 56968384f_{n+\frac{3}{4}} \quad (23)$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{36018675}\Big(5564524f_n + 21561472f_{n+\frac{1}{4}} + 13396944f_{n+\frac{1}{2}} - 415360f_{n+\frac{3}{4}} \quad (24)$$

$$y'_{n+\frac{9}{4}} = y'_n + \frac{9}{4}hy''_n + \frac{h^2}{4857036800}\Big(865279233f_n + 3157351488f_{n+\frac{1}{4}} + 2638732140f_{n+\frac{1}{2}} - 917045712f_{n+\frac{3}{4}} + 5006381985f_{n+1} + 3968707347f_{n+2} - 4211412480f_{n+\frac{9}{4}} + 2411056692f_{n+\frac{5}{2}} \quad (25)$$

$$y'_{n+\frac{5}{2}} = y'_n + \frac{5}{2}hy''_n + \frac{h^2}{368831232}\Big(74431525f_n + 258750000f_{n+\frac{1}{4}} + 263516000f_{n+\frac{1}{2}} - 134926000f_{n+\frac{3}{4}} + 492438375f_{n+1} + 421867875f_{n+2} - 404734000f_{n+\frac{9}{4}} + 243619200f_{n+\frac{5}{2}} - 71090000f_{n+\frac{11}{4}} \quad (26)$$

$$y'_{n+\frac{11}{4}} = y'_n + \frac{11}{4}hy''_n + \frac{h^2}{9754214400}\Big(2199000881f_n + 7346374640f_{n+\frac{1}{4}} + 8635086108f_{n+\frac{1}{2}} - 5289500480f_{n+\frac{3}{4}} + 15988806537f_{n+1} - 12899775152f_{n+\frac{9}{4}} + 8545717444f_{n+\frac{5}{2}} - 2272089600f_{n+\frac{11}{4}} \quad (27)$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{1185800}\Big(295491f_n + 953280f_{n+\frac{1}{4}} + 1255320f_{n+\frac{1}{2}} - 856768f_{n+\frac{3}{4}} + 2307096f_{n+1} \quad (28)$$

$$y''_{n+\frac{1}{4}} = y''_n + \frac{h}{2438553600}\Big(185791571f_n + 695236504f_{\frac{1}{4}+n} - 553656388f_{\frac{1}{2}+n} + 447678040f_{\frac{3}{4}+n} \quad (29)$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{h}{419126400}\Big(30683051f_n + 161654560f_{\frac{1}{4}+n} - 11064856f_{\frac{1}{2}+n} + 47535136f_{\frac{3}{4}+n} \quad (30)$$

$$y''_{n+\frac{3}{4}} = y''_n + \frac{h}{331161600}\Big(24534209f_n + 124341768f_{\frac{1}{4}+n} + 32949972f_{\frac{1}{2}+n} + 88230472f_{\frac{3}{4}+n} \quad (31)$$

$$y''_{n+1} = y''_n + \frac{h}{26195400}\Big(1922599f_n + 10009472f_{\frac{1}{4}+n} + 1804000f_{\frac{1}{2}+n} + 11302016f_{\frac{3}{4}+n} + 1049037f_{1+n} \quad (32)$$

$$y''_{n+2} = y''_n + \frac{h}{3274425}\Big(310387f_n + 669824f_{\frac{1}{4}+n} + 2256496f_{\frac{1}{2}+n} - 2341504f_{\frac{3}{4}+n} + 4002900f_{1+n} \quad (33)$$

$$y''_{n+\frac{9}{4}} = y''_n + \frac{h}{110387200}\Big(10432107f_n + 22827096f_{\frac{1}{4}+n} + 75289500f_{\frac{1}{2}+n} - 77703912f_{\frac{3}{4}+n} + 134096391f_{1+n} \quad (34)$$

$$y''_{n+\frac{5}{2}} = y''_n + \frac{h}{16765056}\Big(1588195f_n + 3437600f_{\frac{1}{4}+n} + 11525800f_{\frac{1}{2}+n} - 11941600f_{\frac{3}{4}+n} + 20459175f_{1+n} \quad (35)$$

$$y''_{n+\frac{11}{4}} = y''_n + \frac{h}{2438553600}\Big(230031571f_n + 507428504f_{\frac{1}{4}+n} + 1653703612f_{\frac{1}{2}+n} - 1702561960f_{\frac{3}{4}+n} + 2953570191f_{1+n} + 3220055025f_{2+n} - 2284389976f_{\frac{9}{4}+n} + 2297471044f_{\frac{5}{2}+n} - 217479064f_{\frac{11}{4}+n} \tag{36}$$

$$y''_{n+3} = y''_n + \frac{h}{29400}\Big(2821f_n + 5760f_{\frac{1}{4}+n} + 21024f_{\frac{1}{2}+n} - 22144f_{\frac{3}{4}+n} + 36639f_{1+n} + 36639f_{2+n} - 22144f_{\frac{9}{4}+n} + 21024f_{\frac{5}{2}+n} + 5760f_{\frac{11}{4}+n} + 2821f_{3+n}\Big) \tag{37}$$

3. Analysis of the Properties of the Derived Method

In this section, the analysis of the basic properties of the developed three-step Hybrid Block method is presented.s

3.1 Order of the Method

Assuming the linear operator $\mathcal{L}$ associated with the $k - step$ hybrid scheme is defined as

$$\mathcal{L}\{y(x):h\} = \sum_{j=0}^{k} \left[\alpha_j y_{n+j} - h^3\left(\beta_j f(x_{n+j})\right)\right] \tag{38}$$

where α_0 and β_0 are not both zero and $y(x) \in C^{(n)}[a,b]$. Expanding y_{n+j} and f_{n+j} as Taylors series expansion gives

$$y_{n+j} = y(x_n + jh) = y(x_n) + jhy'(x_n) + \frac{(jh)^2}{2!}y''(x_n) + \cdots + \frac{(jh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \tag{39}$$

$$f_{n+j} = y^{(iii)}(x_n + jh) = y^{(iii)}(x_n) + jhy(iv)(x_n) + \frac{(jh)^2}{2!}y(v)(x_n) + \frac{(jh)^3}{3!}y(vi)(x_n) + \cdots + \frac{(jh)^p}{(p)!}y^{p+3}(x_n) \tag{40}$$

Substituting and collecting like terms gives

$$\mathcal{L}\{y(x),h\} = C_0 y(x) + C_1 hy'(x_n) + C_2 h^2 y''(x_n) + \cdots C_p h^p y^p(x) \tag{41}$$

Therefore, applying the linear operator $L(41)$ to determine the order and error constant of the main method

$$y_{n+3} + \frac{12298770837}{433189561}y_{n+2} + \frac{12406261536}{433189561}y_{n+\frac{5}{2}} + \frac{3309410304}{433189561}y_{n+\frac{1}{4}} + \frac{20091557888}{433189561}y_{n+\frac{3}{4}} - \frac{12298770837}{433189561}y_{n+1} - \frac{12406261536}{433189561}y_{n+\frac{1}{2}} - \frac{20091557888}{433189561}y_{n+\frac{9}{4}}$$

where C_p are constants. Since $C_0 = C_1 = C_2 = \cdots = C_{p+2} = 0, C_{p+3} \neq 0$ is the error constant.
Hence the method is of order 10 with error constant $c_{p+3} = -\frac{2850317}{642252800}$

Consistency of the Method

The first and second characteristics polynomial (ρ) and (σ) respectively of the main method are given as:

$$\rho(r) = r^3 - r^0 - \frac{3309410304}{433189561}r^{\frac{1}{4}} + \frac{12406261536}{433189561}r^{\frac{1}{2}} - \frac{20091557888}{433189561}r^{\frac{3}{4}} + \frac{12298770837}{433189561}r^1 - \frac{12298770837}{433189561}r^2 + \frac{20091557888}{433189561}r^{\frac{9}{4}}$$

$$-\frac{12406261536}{433189561}r^{\frac{5}{2}} + \frac{3309410304}{433189561}r^{\frac{11}{4}} = 0$$

$$\sigma(r) = \frac{229905}{247536892}r^0 + \frac{44132445}{247536892}r^1 + \frac{44132445}{247536892}r^2 + \frac{229905}{247536892}r^3$$

It shows by appling the following conditions that the method developed in this article is consistent
(i) The method is of order $p = 10 > 1$
which is obvious condition (i) is satisfied

$$(ii)\sum_{j=0}^{k} \alpha_j = 0$$

$$\sum \alpha_j = -1 - \frac{3309410304}{433189561} + \frac{12406261536}{433189561} - \frac{20091557888}{433189561} + \frac{12298770837}{433189561} - \frac{12298770837}{433189561} + \frac{20091557888}{433189561} - \frac{12406261536}{433189561} + \frac{3309410304}{433189561} + 1 = 0$$

since

$\alpha_0 = -1, \alpha_1 = \frac{3309410304}{433189561}, \alpha_2 = -\frac{12406261536}{433189561}, \alpha_3 = \frac{20091557888}{433189561}, \alpha_4 = \frac{20091557888}{433189561},$

$\alpha_5 = -\frac{12406261536}{433189561}, \alpha_6 = -\frac{3309410304}{433189561}, \alpha_7 = -\frac{12298770837}{433189561}, \alpha_8 = \frac{12298770837}{433189561}$

$\alpha_9 = 1$ condition (ii) is satisfied

(iii) $\rho(r) = \rho'(r) = 0$

$$\rho(r) = r^3 - 1 - \frac{3309410304}{433189561} r^{\frac{1}{4}} + \frac{12406261536}{433189561} r^{\frac{1}{2}} - \frac{20091557888}{433189561} r^{\frac{3}{4}} + \frac{12298770837}{433189561} r^1 - \frac{12298770837}{433189561} r^2 + \frac{20091557888}{433189561} r^{\frac{9}{4}} - \frac{12406261536}{433189561} r^{\frac{5}{2}} + \frac{3309410304}{433189561} r^{\frac{11}{4}} = 0$$

and

$$\rho'(r) = 3r^2 - \frac{827352576}{433189561} r^{\frac{3}{4}} + \frac{6203130768}{433189561} r^{\frac{1}{2}} - \frac{15068668416}{433189561} r^{\frac{1}{4}} + \frac{12298770837}{433189561} - \frac{24597541674}{433189561} r^1 + \frac{45206005248}{433189561} r^{\frac{5}{4}} - \frac{31015653840}{433189561} r^{\frac{3}{2}} + \frac{9100878336}{433189561} r^{\frac{7}{4}} = 0$$

Also

$$\rho(1) = 1 - 1 - \frac{3309410304}{433189561} + \frac{12406261536}{433189561} - \frac{20091557888}{433189561} + \frac{12298770837}{433189561} - \frac{12298770837}{433189561} + \frac{20091557888}{433189561} - \frac{12406261536}{433189561} + \frac{3309410304}{433189561} = 0$$

and

$$\rho'(1) = 3 - \frac{827352576}{433189561} + \frac{6203130768}{433189561} - \frac{15068668416}{433189561} + \frac{12298770837}{433189561} - \frac{24597541674}{433189561} + \frac{45206005248}{433189561} - \frac{31015653840}{433189561} + \frac{9100878336}{433189561} = 0$$

it follows that $\rho(1) = \rho'(1) = 0$ showing that the condition (iii) is satisfied as well

$\rho'''(r) = n!\,\sigma(r)$ and for the principal root r = 1

$$\rho'''(r) = 6 - \frac{155128608}{61884223} r^{\frac{11}{4}} + \frac{4652348076}{433189561} r^{\frac{5}{2}} - \frac{4708958880}{433189561} r^{\frac{9}{4}} + \frac{14126876640}{433189561} r^{\frac{3}{4}} - \frac{23261740380}{433189561} r^{\frac{1}{2}} + \frac{1706414688}{61884223} r^{\frac{1}{4}} = \frac{133087050}{61884223}$$

$$3!\,\sigma(r) = 3!\left(\frac{229905}{247536892} r^0 + \frac{44132445}{247536892} r^1 + \frac{44132445}{247536892} r^2 + \frac{229905}{247536892} r^3\right)$$

since

$$\rho'''(1) = 6 - \frac{155128608}{61884223} + \frac{4652348076}{433189561} - \frac{4708958880}{433189561} + \frac{14126876640}{433189561} - \frac{23261740380}{433189561} + \frac{1706414688}{61884223} = \frac{133087050}{61884223}$$

and

$$3!\,\sigma(1) = 3!\left(\frac{229905}{247536892} + \frac{44132445}{247536892} + \frac{44132445}{247536892} + \frac{229905}{247536892}\right) = 3!\frac{22181175}{61884223} = \frac{133087050}{61884223}$$

Therefore for the principal root $r = 1$; it is observed that last condition above is satisfied, hence the method is consistent. condition (iv) is satisfied.

Zero Stability of the Method

Using (11)-(37) as $h \to 0$, we have

$\det\left[\gamma A^{(0)} - A^{(0)}\right]$

$$= \det\left[\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\end{bmatrix}\right]$$

$$= \gamma^9 - \gamma^8 = 0$$

By solving for γ we have

$\gamma^8(\gamma - 1) = 0$

Solving the above equation for $\gamma, \gamma = 1, \gamma = 0, \gamma = 0, \gamma = 0, \gamma = 0, \gamma = 0, \gamma = 0, \gamma = 0, \gamma = 0$

Hence, the method is zero-stable.

3.2 Convergence

For a numerical method to converge, it must be both consistent and zero-stable [10]. Therefore, since it has been obviously seen that the three-step hybrid block method is consistent and zero stable. Hence the method is convergent.

Region of Absolute Stability of the Method

The region of absolute stability of the method is examined via the procedure discussed in Lambert (1973). The stability matrix can be expressed as

$$J(z) = zH(I - zG)^{-1}Q + R \tag{42}$$

together with the Stability function

$$p(n, z) = \det(-J(z) + nI) \tag{43}$$

for the Stability properties, the method (3.142) - (3.150) was formulated as a general linear method of the form,

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} G & Q \\ --- & ------ \\ H & R \end{bmatrix} \begin{bmatrix} h^3 f(u) \\ --- \\ Y_{i-1} \end{bmatrix} \tag{44}$$

where n represents the roots of the first characteristics polynomial, and

$$Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_n \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+3} \end{bmatrix}$$

$$G = \begin{bmatrix}
\frac{187194473}{78683996160} & -\frac{1101992029}{429185433600} & \frac{46168957}{21459271680} & -\frac{168472963}{190749081600} & \frac{48627269}{114449448960} & -\frac{23914351}{35765452800} & \frac{64356799}{143061811200} & -\frac{35009879}{236051988480} & \frac{53368643}{2697737011200} \\
\frac{1214771}{57629880} & -\frac{363127}{22353408} & \frac{60113}{4365900} & -\frac{12614419}{2235340800} & \frac{2013793}{745113600} & -\frac{10123}{2381400} & \frac{4792477}{1676505600} & -\frac{9049}{9604980} & \frac{17651}{140507136} \\
\frac{307761633}{4857036800} & -\frac{56908143}{1766195200} & \frac{2999859}{88309760} & -\frac{19704789}{1412956160} & \frac{9501381}{1412956160} & -\frac{186915}{17661952} & \frac{12575439}{1766195200} & -\frac{1036503}{441548800} & \frac{3476001}{11101798400} \\
\frac{1551896}{12006225} & -\frac{5107}{119070} & \frac{232216}{3274425} & -\frac{30259}{1164240} & \frac{109999}{8731800} & -\frac{4328}{218295} & \frac{29119}{2182950} & -\frac{158408}{36018675} & \frac{9659}{16465680} \\
\frac{21707648}{36018675} & \frac{67696}{1091475} & \frac{61568}{218295} & \frac{43742}{218295} & \frac{11839}{72765} & -\frac{28288}{130977} & \frac{39376}{297675} & -\frac{492416}{12006225} & \frac{998}{190575} \\
\frac{335024343}{441548800} & \frac{62336061}{353239040} & \frac{113446251}{441548800} & \frac{593501499}{1412956160} & \frac{2305332009}{7064780800} & -\frac{35194419}{88309760} & \frac{418322799}{1766195200} & -\frac{349786593}{4857036800} & \frac{1830519}{201850880} \\
\frac{1782125}{1920996} & \frac{22374875}{67060224} & \frac{17875}{95256} & \frac{21332375}{29804544} & \frac{51104875}{89413632} & -\frac{112625}{174636} & \frac{8512375}{22353408} & -\frac{1318375}{11525976} & \frac{6019625}{421521408} \\
\frac{2164510799}{1950842880} & \frac{6940814947}{13005619200} & \frac{48154249}{650280960} & \frac{56578987861}{52022476800} & \frac{3115209493}{3468165120} & -\frac{9236758003}{9754214400} & \frac{22308301367}{39016857600} & -\frac{109305713}{650280960} & \frac{155209241}{7431782400} \\
\frac{38664}{29645} & \frac{837}{1078} & -\frac{1128}{13475} & \frac{331047}{215600} & \frac{140859}{107800} & -\frac{17592}{13475} & \frac{22059}{26950} & -\frac{6696}{29645} & \frac{393}{13552}
\end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1301143087}{899245670400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{77768671}{10538035200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{18083757}{100925400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50423}{1524600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{735737}{5145525} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2049031647}{11101798400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32606125}{140507136} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6364617061}{2229537200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58407}{169400} \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} y_{n+1/4} \\ y_{n+1/2} \\ y_{n+3/4} \\ y_{n+1} \\ y_{n+2} \\ y_{n+9/4} \\ y_{n+5/2} \\ y_{n+11/4} \\ y_{n+3} \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f(y) = \begin{bmatrix} f_{n+1/4} \\ f_{n+1/2} \\ f_{n+3/4} \\ f_{n+1} \\ f_{n+2} \\ f_{n+9/4} \\ f_{n+5/2} \\ f_{n+11/4} \\ f_{n+3} \end{bmatrix}$$

Now, putting the values of the variables G, H, Q, R, J and I in equations (43) and (44), to obtain the Stability function. The stability polynomial (45) and its first derivatives (46) are then plotted in MATLAB (R2012a) environment. It should be noted that J is 9 by 9 identity matrix. The region of absolute stability (RAS) of the method is shown in the Figure 1 below;

$$f(z) = \left(\delta + \frac{7493751z^2}{54208000} + \frac{322743z}{84700} - 1\right)\delta^8 \tag{45}$$

$$f'(z) = \frac{(7493751z + 103277760)\delta^8}{27104000} \tag{46}$$

The region of absolute stability of the method is P-stable, since the region consists of the complex plane outside the enclosed figure and its interval of periodicity lies between $(0,0.52)$ which falls within the interval of periodicity for P-stability. $(0, \infty)$.

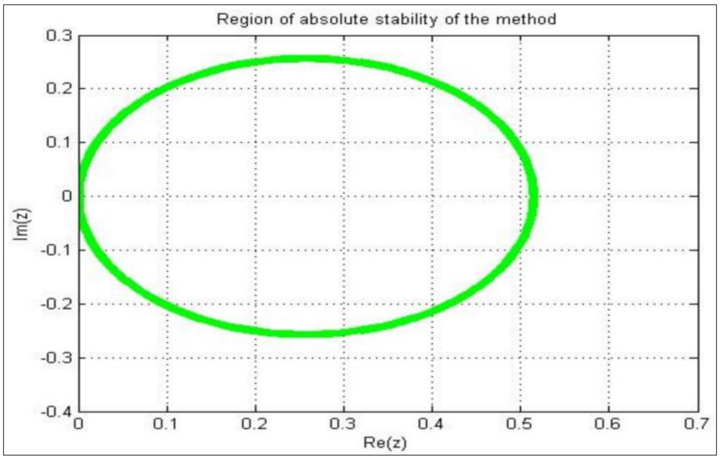

Figure 1: Region of Absolute Stability of the new method. The figure shows the area where the method is stable

4 Numerical Experiments

To test how well the proposed method works, the authors used three sample problems as numerical examples. They measured the accuracy of the method by calculating the absolute error it generated when applied to the sample problems.

4.1 Problem 1

The first sample problem considered in this work is
$y'''(x) = x - 4y'(x); y(0) = 0; y'(0) = 0; y''(0) = 1; h = 0.1$

Exact Solution $y(x) = \frac{3}{16}\left(1 - \cos(x) + \frac{x^2}{8}\right)$
Source: Obarhua (2022)

Table 1: Numerical Results for problem 2, $k = 3, p = 10, h = 0.1$ for problem 1

x	Exact Solution	Computed Solution	Error in 3-step
0.10	0.0049875166547671941642130	0.00498751665476719433453626	1.70323E − 19
0.20	0.0198010636244590469752760	0.01980106362445904816948220	1.19421E − 18
0.30	0.0439995722044353192673220	0.04399957220443532468283120	5.41551E − 18
0.40	0.0768674919974064835773590	0.07686749199740651444272070	3.08653E − 17
0.50	0.1174433176497238029873240	0.11744331764972386434452000	6.13572E − 17
0.60	0.1645579210356237041928050	0.16455792103562381289780000	1.08705E − 16
0.70	0.2168811607062048240093600	0.21688116070620502252007000	1.98511E − 16
0.80	0.2729749104314916361635820	0.27297491043149193735952200	3.01196E − 16
0.90	0.3313503927549538228718760	0.33135039275495425500857300	4.32137E − 16
1.00	0.3905275318525891975620440	0.39052753185258981085163100	6.13290E − 16

Table 2: Comparison of the errors of the 3 -step with other existing method for problem 1.

x	3-step, $p = 10, h = 0.1$	Adeyefa and Olanegan $(2022), p = 10, h = 0.1$
0.10	1.70323E − 19	3.0000E − 10
0.20	1.19421E − 18	2.1560E − 10
0.30	5.41551E − 18	3.9810E − 10
0.40	3.08653E − 17	7.2860E − 09
0.50	6.13572E − 17	4.6470E − 09
0.60	1.08705E − 16	9.0400E − 09
0.70	1.98511E − 16	1.7320E − 08
0.80	3.01196E − 16	2.6640E − 08
0.90	4.32137E − 16	4.2960E − 08
1.00	6.13290E − 16	6.2790E − 08

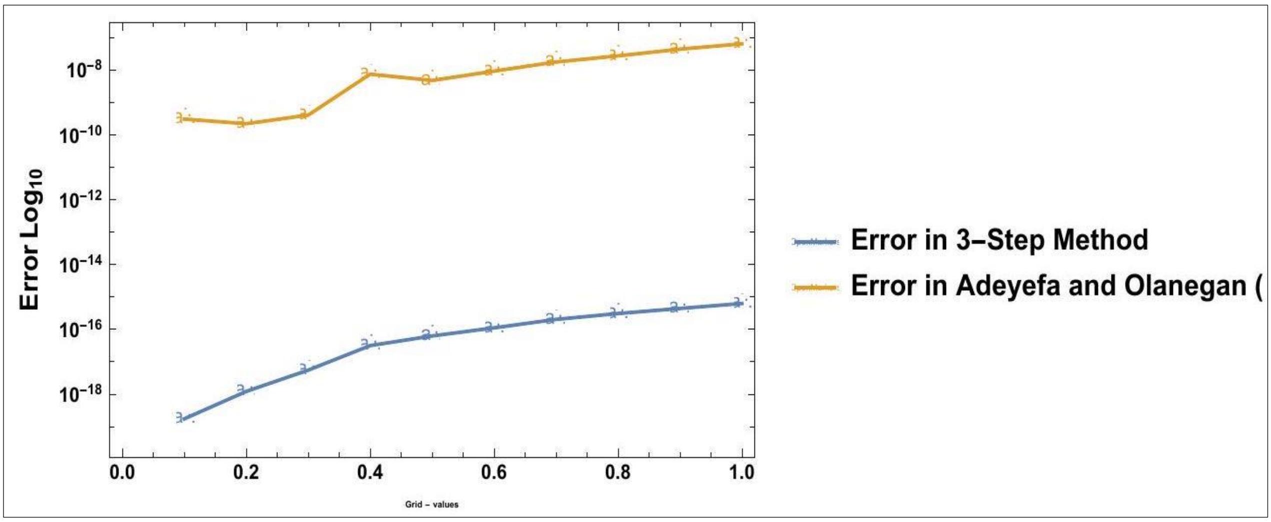

Figure 2: Comparison of absolute errors of the proposed method on problem 1 as compared with Adeyefa and Olanegan (2022)

Problem 2:
$y''' = -e^x, y(0) = 1; y'(0) = -1; y''(0) = 3; h = 0.1$
Exact Solution
$y(x) = 2 + 2x^2 - e^x$
Source: Adoghe *et al*., (2016)

Table 3: Numerical Results for $k = 3, p = 10, h = 0.1$ for problem 2

x	Exact Solution	Computed Solution	Error in 3-step
0.10	0.91482908192435237518829	0.914829081924352375187823	$4.67000E-22$
0.20	0.85859724183983016607893	0.858597241839830166075687	$3.24300E-21$
0.30	0.83014119242399689601626	0.830141192423996896001787	$1.44730E-20$
0.40	0.82817530235872968217515	0.828175302358729682114330	$6.08200E-20$
0.50	0.85127872929987185315135	0.851278729299871853034126	$1.17224E-19$
0.60	0.89788119960949102512463	0.897881199609491024921742	$2.02888E-19$
0.70	0.96624729252952347837545	0.966247292529523478021593	$3.53857E-19$
0.80	1.05445907150753239542046	1.054459071507532394884220	$5.36240E-19$
0.90	1.16039688884305033619987	1.160396888843050335423870	$7.76000E-19$
1.00	1.28171817154095476463971	1.281718171540954763517960	$1.12175E-18$

Table 4: Comparison of errors in the 3-step with other methods for test problem 2.

x	3 -step, $p = 10, h = 0.1$	Omole *et al*., $(2024), p = 10, h = 0.1$
0.10	$4.67000E-22$	$8.1100E-17$
0.20	$3.24300E-21$	$1.4010E-16$
0.30	$1.44730E-20$	$2.0410E-16$
0.40	$6.08200E-20$	$2.7010E-16$
0.50	$1.17224E-19$	$3.4810E-16$
0.60	$2.02888E-19$	$4.4310E-16$
0.70	$3.53857E-19$	$5.3510E-16$
0.80	$5.36240E-19$	$6.4410E-16$
0.90	$7.76000E-19$	$7.6410E-16$
1.00	$1.12175E-18$	$8.8410E-16$

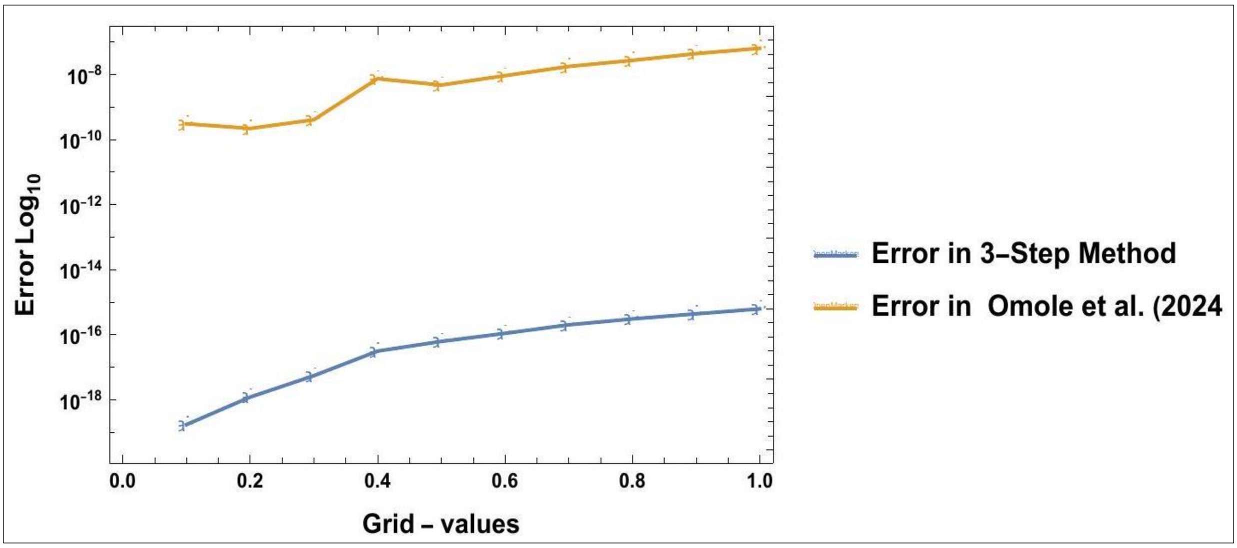

Figure 3: Comparison of absolute errors of the proposed method on problem 2 as compared with Omole (2024)

Problem 3:
$y''' = -y, y(0) = 1; y'(0) = -1; y''(0) = 1; h = 0.1$
Exact Solution
$y(x) = e^{-x}$
Source: Abolarin *et al*., (2020)

Table 5: Numerical Results for problem 3, $k = 3, p = 10, h = 0.1$

x	Exact Solution	Computed Solution	Error in 3-step
0.10	0.904837418035959573164249	0.904837418035959573163890	3.5900E − 22
0.20	0.818730753077981858669936	0.818730753077981858667492	2.4440E − 21
0.30	0.740818220681717866066874	0.740818220681717866056065	1.0809E − 20
0.40	0.670320046035639300744433	0.670320046035639300705584	3.8849E − 20
0.50	0.606530659712633423603800	0.606530659712633423531536	7.2264E − 20
0.60	0.548811636094026432628459	0.548811636094026432507914	1.2055E − 19
0.70	0.496585303791409514704800	0.496585303791409514511279	1.9352E − 19
0.80	0.449328964117221591430102	0.449328964117221591149638	2.8046E − 19
0.90	0.406569659740599111883454	0.406569659740599111495248	3.8821E − 19
1.00	0.367879441171442321595524	0.367879441171442321071750	5.2377E − 19

Table 6: Comparison of the errors in the 3 -step with other methods for problem 3

x	3-step, $p = 10, h = 0.1$	Abolarin *et al.*, (2020), $p = 15, h = 0.1$
0.10	3.5900E − 22	3.450699E − 06
0.20	2.4440E − 21	6.169050E − 05
0.30	1.0809E − 20	1.532998E − 04
0.40	3.8849E − 20	3.687668E − 04
0.50	7.2264E − 20	7.117489E − 04
0.60	1.2055E − 19	1.199891E − 03
0.70	1.9352E − 19	1.845664E − 03
0.80	2.8046E − 19	4.036620E − 03
0.90	3.8821E − 19	3.638358E − 03
1.00	5.2377E − 19	6.859964E − 03

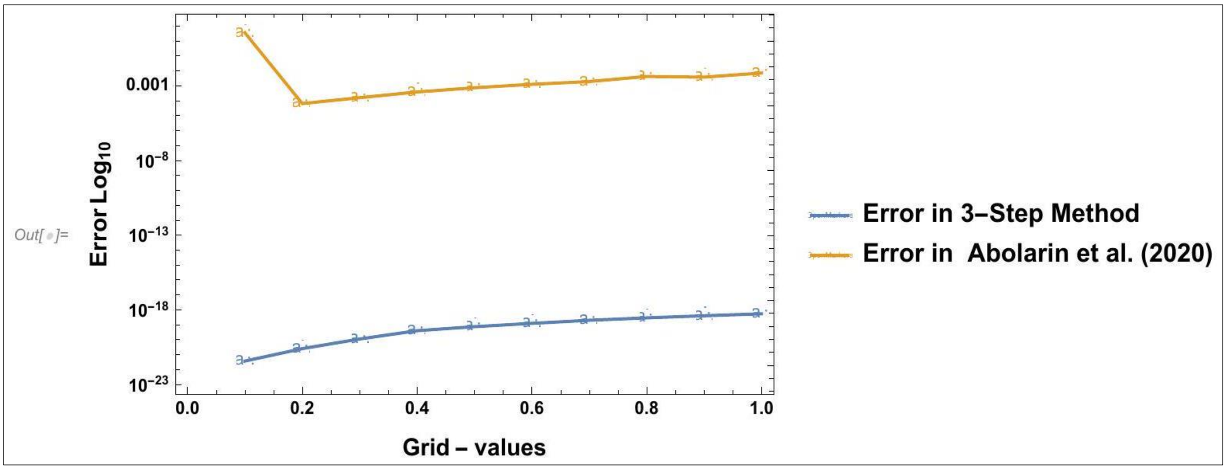

Figure 4: Comparison of absolute errors of the proposed method on problem 3 as compared with Abolarin (2020)

5. CONCLUSION

In this study, a three-step, P-stable, order ten hybrid block method that solve initial value problems of third order ordinary differential equations was developed. The method was zero stable and consistent satisfying basic requirements for convergence of Linear Multistep Methods (LMM). As shown by the region of absolute stability. The accuracy and the usability of developed method was tested by applying it to solve three numerical examples and was found to be efficient as it gives a minimal error, hence has higher accuracy for handling the direct solution of third-order initial value problem of ordinary differential equations.

Competing interests.

Authors have declared that no competing interests exist.

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