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Numerical Integration of Fourth-Order Initial Value Problems of Odes Using a Seventh-Order Linear Multistep Scheme

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Abstract **Original Research Article**

This paper introduces a seventh-order linear multistep method for the numerical integration of fourth-order initial value problems. The method is derived by constructing a continuous scheme by carefully applying collocation and interpolation to the Chebyshev polynomial at chosen points, and evaluating it at a specific grid point. The stability of the method is analyzed, and its convergence is proven. Numerical examples are provided to demonstrate the accuracy and efficiency of the proposed method compared to existing methods. The results show that the new method is a viable alternative for solving fourth-order initial value problems.

Keywords: linear multi-step method, collocation, interpolation, convergence, zero stability.

1. INTRODUCTION

Differential equations play a crucial role in numerous scientific and engineering disciplines, serving as fundamental tools for modeling and analyzing dynamic systems [1, 2]. Among these, fourth-order ordinary differential equations have garnered considerable interest due to their relevance in areas such as structural mechanics, fluid dynamics, and quantum mechanics [3, 4]. To address the challenges of solving such equations, researchers have developed various numerical approaches, including the linear multistep method [5]. Linear multistep methods are a class of numerical techniques that approximate solutions to differential equations by employing a linear combination of previously computed function values and their derivatives [6]. These methods are particularly advantageous, as they achieve higher-order accuracy compared to single-step methods like Runge-Kutta [7]. This characteristic makes them highly effective for tackling higher-order differential equations.

This study focuses on numerical solution of ordinary differential equation of the form;

$$y^{(4)} = f(x, y, y', y'', y'''), \qquad y(x_0) = \eta_0, y'(x_0) = \eta_1, y''(x_0) = \eta_2, y'''(x_0) = \eta_3 \tag{1}$$

The function f in equation (1) is assumed to be a continuous real-valued function, as noted in [8, 9]. Initially, the reduction approach was the preferred method for solving equations of the form (1), primarily due to the availability of established techniques for handling their equivalent first-order systems, as documented in [5, 10] and other studies. However, over time, the limitations of the reduction approach became apparent, particularly its complexity [11, 12] and inefficiency when applied to larger systems of differential equations [7, 13]. These challenges led to the exploration of direct approaches [8, 14-16] as more suitable alternatives for solving such problems.

In the literature [3, 17-22], various authors have independently developed numerical methods for solving fourth-order ordinary differential equations (ODEs). A common feature of these methods is their use of implicit linear multistep techniques implemented in block form. While many of these methods boast an order of accuracy equal to or higher than the one proposed in this article, their performance often falls short in comparison. This discrepancy may be attributed to the hybrid nature of the method presented here.

Specifically, [9] introduced a fully hybrid linear multistep formula for solving equations of type (1). Although this method directly addresses equation (1), there remains room for improvement in terms of accuracy. Consequently, this article focuses on the development and implementation of an efficient numerical algorithm for directly solving fourth-order ODEs. Additionally, it aims to enhance the accuracy of existing methods, advancing the application of linear multistep method to numerically solve nonlinear fourth-order ordinary differential equations.

2. MATHEMATICAL FORMULATION

This research provided an approximate solution to the general fourth order ordinary differential equations of the form in (1) using partial sum of Chebyshev polynomial of first kind denoted here as

$$y(x) = \sum_{j=0}^{(p+q)-1} a_j T_j(x) \tag{2}$$

where, $y(x)$ is considered an approximation to equation (1), and x is assumed to be continuously differentiable. The fourth derivative of (2) gives

$$y^{(4)}(x) = \sum_{j=4}^{(p+q)-1} a_j T_j^{(4)}(x). \tag{3}$$

Equating equation (1) and equation (3) results in the differential system:

$$f(x, y, y', y'', y''') = \sum_{j=4}^{(p+q)-1} a_j T_j^{(4)}(x). \tag{4}$$

Equations (2) and (4) are the interpolating and collocating equations respectively which shall be used to derive the propose methods. It is important to note that the parameters a_j's must be uniquely determined. Substituting $x = x_{n+r}$ for $r = 1(\frac{1}{2})\frac{5}{2}$ into equation (2), and $x = x_{n+r}$ for $r = 0(\frac{1}{2})3$ into equation (4), produces the following system of equations.

$$y_{n+r} = \sum_{r=0}^{10} a_r T_r^{(4)}(x), \quad r = 1(\frac{1}{2})\frac{5}{2}, \tag{5}$$

and

$$f_{n+r} = \sum_{r=4}^{10} a_r T_r^{(4)}(x), \quad r = 0(\frac{1}{2})3. \tag{6}$$

By applying the scaling function $x_{n+i} = x_n + ih$, the matrix representations of equations (5) and (6) are solved to determine the coefficients a_j's for $j = 0(1)10$. Substituting the parameters a_j's into equation (2) and setting $x = x_n + th$, yields a continuous scheme of the form;

$$y(t) = \sum_{j=2}^{5} \alpha_{\frac{j}{2}}(t) y_{n+\frac{j}{2}} + h^4 \sum_{j=0}^{6} \beta_{\frac{j}{2}}(t) f_{n+\frac{j}{2}} \tag{7}$$

with the following coefficients;

$$\alpha_1 = -\frac{4t^3}{3} + 8t^2 - \frac{47t}{3} + 10,$$

$$\alpha_{\frac{3}{2}} = 4t^3 - 22t^2 + 38t - 20,$$

$$\alpha_2 = -4t^3 + 20t^2 - 31t + 15,$$

$$a_{\frac{5}{2}} = \frac{4t^3}{3} - 6t^2 + \frac{26t}{3} - 4,$$

$$\beta_0 = \frac{h^4}{7257600}\begin{pmatrix} 128t^{10} - 2240t^9 + 16800t^8 - 70560t^7 + 181888t^6 - 296352t^5 \\ +302400t^4 - 183995t^3 + 59469t^2 - 7658t + 120 \end{pmatrix}$$

$$\beta_{\frac{1}{2}} = -\frac{h^4}{3628800}(384t^{10} - 6400t^9 + 44640t^8 - 167040t^7 + 350784t^6 \\ -362880t^5 + 441520t^3 - 483423t^2 + 217965t - 35550),$$

$$\beta_1 = \frac{h^4}{1451520}(384t^{10} - 6080t^9 + 39456t^8 - 132768t^7 + 235872t^6 \\ -181440t^5 - 55219t^3 + 345108t^2 - 369783t + 124470),$$

$$\beta_{\frac{3}{2}} = -\frac{h^4}{181440}(64t^{10} - 960t^9 + 5808t^8 - 17856t^7 + 28448t^6 \\ -20160t^5 + 19615t^3 - 54645t^2 + 71126t - 31440),$$

$$\beta_2 = \frac{h^4}{1451520}(384t^{10} - 5440t^9 + 30816t^8 - 88416t^7 + 133056t^6 \\ -90720t^5 + 10459t^3 + 84549t^2 - 139308t + 64620),$$

$$\beta_{\frac{5}{2}} = -\frac{h^4}{3628800}(34560t^8 - 368640t^7 + 1532160t^6 \\ -3144960t^5 + 3265920t^4 - 1451520t^3 + 144120t - 8706),$$

$$\beta_3 = \frac{h^4}{7257600}(11520t^8 - 115200t^7 + 456960t^6 \\ -907200t^5 + 920640t^4 - 403200t^3 + 40050t - 3012).$$

Evaluating (7) at $t = 3$ yields the main discrete method;

$$y_{n+3} - 4y_{n+\frac{5}{2}} + 6y_{n+2} - 4y_{n+\frac{3}{2}} + y_{n+1} = \frac{h^4}{241920}(5f_n - 30f_{n+\frac{1}{2}} + 54f_{n+1} \\ +2504f_{n+\frac{3}{2}} + 10029f_{n+2} + 2574f_{n+\frac{5}{2}} - 16f_{n+3}) \tag{8}$$

Equation (8) is implemented in block mode by first evaluating equation (7) at $t = 0, \frac{1}{2}$ Type equation here. and it first, second and third derivatives at $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ and 3 to obtain a total of twenty-three formulas. These formulas are combined as a block inline with block matrix given in (9). This yields the block method written explicitly as (9) to (32)

$$y_{n+\frac{1}{2}} + \frac{h^3 y'''_n}{48} + \frac{h^2 y''_n}{8} + \frac{hy'_n}{2} - y_n = \frac{h^4}{58060800}\Big(95929f_n + 112028f_{n+\frac{1}{2}} \\ -115165f_{n+1} + 97320f_{n+\frac{3}{2}} - 53465f_{n+2} + 16876f_{n+\frac{5}{2}} - 2323f_{n+3}\Big) \tag{9}$$

$$y_{n+1}+\frac{h^3y'''_n}{6}+\frac{h^2y''_n}{2}+hy'_n+y_n=\frac{h^4}{226800}\Big(4127f_n+8782f_{n+\frac{1}{2}}-6965f_{n+1}+5820f_{n+\frac{3}{2}}-3175f_{n+2}+998f_{n+\frac{5}{2}}-137f_{n+3}\Big) \quad (10)$$

$$y_{n+\frac{3}{2}}+\frac{9h^3y'''_n}{16}-\frac{9h^2y''_n}{8}-\frac{3hy'_n}{2}-y_n=\frac{9h^4}{716800}\Big(5471f_n+15228f_{n+\frac{1}{2}}-8775f_{n+1}+8120f_{n+\frac{3}{2}}-4455f_{n+2}+1404f_{n+\frac{5}{2}}-193f_{n+3}\Big) \quad (11)$$

$$y_{n+2}-\frac{4h^3}{3}y'''_n-2h^2y''_n-2hy'_n-y_n=\frac{2h^4}{14175}\Big(1220f_n+3904f_{n+\frac{1}{2}}-1580f_{n+1}+1920f_{n+\frac{3}{2}}-1015f_{n+2}+320f_{n+\frac{5}{2}}-44f_{n+3}\Big) \quad (12)$$

$$y_{n+\frac{5}{2}}-\frac{125h^3}{48}y'''_n-\frac{25h^2y''_n}{8}-\frac{5hy'_n}{2}-y_n=\frac{125h^4}{2322432}\Big(6457f_n+22460f_{n+\frac{1}{2}}-6325f_{n+1}+11400f_{n+\frac{3}{2}}-5225f_{n+2}+1708f_{n+\frac{5}{2}}-235f_{n+3}\Big) \quad (13)$$

$$y_{n+3}-\frac{9h^3}{2}y'''_n+\frac{9h^2y''_n}{2}-3hy'_n-y_n=\frac{9h^4}{2800}\Big(191f_n+702f_{n+\frac{1}{2}}-135f_{n+1}+380f_{n+\frac{3}{2}}-135f_{n+2}+54f_{n+\frac{5}{2}}-7f_{n+3}\Big) \quad (14)$$

$$y'_{n+\frac{1}{2}}-\frac{1}{8}h^2y''''_n-\frac{hy''_n}{2}-y'_n=\frac{h^3}{29030400}\Big(343801f_n-494715f_{n+1}-226605f_{n+2}-9809f_{n+3}+506604f_{n+\frac{1}{2}}+414160f_{n+\frac{3}{2}}+71364f_{n+\frac{5}{2}}\Big) \quad (15)$$

$$y'_{n+1}-\frac{1}{2}h^2y''''_n-hy''_n-y'_n=\frac{h^3}{226800}\Big(13774f_n-24465f_{n+1}-11370f_{n+2}-491f_{n+3}+35976f_{n+\frac{1}{2}}+20800f_{n+\frac{3}{2}}+3576f_{n+\frac{5}{2}}\Big) \quad (16)$$

$$y'_{n+\frac{3}{2}}-\frac{1}{8}9h^2y''''_n-\frac{3hy''_n}{2}-y'_n=\frac{9h^3}{358400}\Big(5877f_n-8055f_{n+1}-4905f_{n+2}-213f_{n+3}+19188f_{n+\frac{1}{2}}+8960f_{n+\frac{3}{2}}+1548f_{n+\frac{5}{2}}\Big) \quad (17)$$

$$y'_{n+2}-2h^2y''''_n-2hy''_n-y'_n=\frac{h^3}{14175}\Big(3863f_n-3390f_{n+1}-3255f_{n+2}-142f_{n+3}+13992f_{n+\frac{1}{2}}+6800f_{n+\frac{3}{2}}+1032f_{n+\frac{5}{2}}\Big) \quad (18)$$

$$y'_{n+\frac{5}{2}}-\frac{1}{8}25h^2y''''_n-\frac{5hy''_n}{2}-y'_n=\frac{125h^3}{1161216}\Big(4045f_n-2055f_{n+1}-2865f_{n+2}-149f_{n+3}+15564f_{n+\frac{1}{2}}+8560f_{n+\frac{3}{2}}+1092f_{n+\frac{5}{2}}\Big) \quad (19)$$

$$y'_{n+3}-\frac{1}{2}9h^2y''''_n-3hy''_n-y'_n=\frac{9h^3}{2800}\Big(198f_n-45f_{n+1}-90f_{n+2}-7f_{n+3}+792f_{n+\frac{1}{2}}+480f_{n+\frac{3}{2}}+72f_{n+\frac{5}{2}}\Big) \quad (20)$$

$$y''_{n+\frac{1}{2}}-\frac{hy''''_n}{2}-y''_n=\frac{h^2}{483840}\Big(28549f_n-51453f_{n+1}-23109f_{n+2}-995f_{n+3}+57750f_{n+\frac{1}{2}}+42484f_{n+\frac{3}{2}}+7254f_{n+\frac{5}{2}}\Big) \quad (21)$$

$$y''_{n+1} - hy''''_n - y''_n = \frac{h^2}{7560}(1027f_n - 1680f_{n+1} - 873f_{n+2} - 38f_{n+3} + 3492f_{n+\frac{1}{2}} + 1576f_{n+\frac{3}{2}} + 276f_{n+\frac{5}{2}}) \quad (22)$$

$$y''_{n+\frac{3}{2}} - \frac{3hy''''_n}{2} - y''_n = \frac{3h^2}{17920}(1265f_n - 801f_{n+1} - 1089f_{n+2} - 47f_{n+3} + 4950f_{n+\frac{1}{2}} + 2100f_{n+\frac{3}{2}} + 342f_{n+\frac{5}{2}}) \quad (23)$$

$$y''_{n+2} - 2hy''''_n - y''_n = \frac{2}{945}h^2(136f_n - 9f_{n+1} - 105f_{n+2} - 5f_{n+3} + 564f_{n+\frac{1}{2}} + 328f_{n+\frac{3}{2}} + 36f_{n+\frac{5}{2}}) \quad (24)$$

$$y''_{n+\frac{5}{2}} - \frac{5hy''''_n}{2} - y''_n = \frac{25h^2}{96768}(1409f_n + 375f_{n+1} - 225f_{n+2} - 55f_{n+3} + 6030f_{n+\frac{1}{2}} + 4100f_{n+\frac{3}{2}} + 462f_{n+\frac{5}{2}}) \quad (25)$$

$$y''_{n+3} - 3hy''''_n - y''_n = \frac{3h^2}{280}(41f_n + 18f_{n+1} + 9f_{n+2} + 180f_{n+\frac{1}{2}} + 136f_{n+\frac{3}{2}} + 36f_{n+\frac{5}{2}}) \quad (26)$$

$$y'''_{n+\frac{1}{2}} - y''''_n = \frac{h}{120960}(19087f_n - 46461f_{n+1} - 20211hf_{n+2} - 863f_{n+3} + 65112f_{n+\frac{1}{2}} + 37504f_{n+\frac{3}{2}} + 6312f_{n+\frac{5}{2}}) \quad (27)$$

$$y'''_{n+1} - y''''_n = \frac{h}{7560}(1139hf_n + 33hf_{n+1} - 807hf_{n+2} - 37hf_{n+3} + 5640hf_{n+\frac{1}{2}} + 1328hf_{n+\frac{3}{2}} + 264hf_{n+\frac{5}{2}}) \quad (28)$$

$$y'''_{n+\frac{3}{2}} - y''''_n = \frac{h}{4480}(685f_n + 1161f_{n+1} - 729f_{n+2} - 29f_{n+3} + 3240f_{n+\frac{1}{2}} + 2176f_{n+\frac{3}{2}} + 216f_{n+\frac{5}{2}}) \quad (29)$$

$$y'''_{n+2} - y''''_n = \frac{h}{945}(143f_n + 192f_{n+1} + 87f_{n+2} - 4f_{n+3} + 696f_{n+\frac{1}{2}} + 752f_{n+\frac{3}{2}} + 24f_{n+\frac{5}{2}}) \quad (30)$$

$$y'''_{n+\frac{5}{2}} - y''''_n = \frac{h}{24192}(3715f_n + 6375f_{n+1} + 11625hf_{n+2} - 275f_{n+3} + 17400f_{n+\frac{1}{2}} + 16000f_{n+\frac{3}{2}} + 5640f_{n+\frac{5}{2}}) \quad (31)$$

$$y'''_{n+3} - y''''_n = \frac{1}{280}(41hf_n + 27hf_{n+1} + 27hf_{n+2} + 41hf_{n+3} + 216hf_{n+\frac{1}{2}} + 272hf_{n+\frac{3}{2}} + 216hf_{n+\frac{5}{2}}) \quad (32)$$

2.1 Analysis of the Properties of the Derived Method

This section explored the fundamental properties of the derived method.

Error, Local truncation, and order of the derived method

Proposition1

If $y(x)$ is continuously differentiable and assumed to be $s(x)$ then, the Local Truncation Error (LTE) of each formula of the proposed (8) take the form; $\kappa_{\frac{\rho}{42}}\{s(x): h\} = c_{n+13}s^{(13)}(x_n)h^{13} + 0(h^{14})$.

Proof

Let begin by defining the LTE of the formulas in (8) as

$$\kappa_{\frac{\rho}{2}}\{s(x):h\} \quad = s\left(x_n + \frac{\rho}{2}h\right) - \left\{\sum_{b=1}^{3} \alpha_b\, s^{(b)}(x)h^b - h^4 \sum_{\rho=0}^{6} \beta_r\, s^{(4)}\left(x + \frac{\rho}{2}h\right)\right\} \tag{33}$$

whose Taylor series expansion about the point x yields

$$T_{\frac{r}{42}}\{s(x):h\} \quad = c_0 s(x) + c_1 h s'(x) + c_2 h^2 s''(x) + \cdots + c_{p+3} h^{p+3} s^{(p+3)}(x) + c_{p+4} h^{p+4} s^{(p+4)}(x),$$
$$= c_{p+4} h^{p+4} s^{(p+4)}(x_n) + O h^{p+5}.$$

Here the term c_{p+4} is the error constant. The order, error constant and LTE of (8) were obtained in similar manner. Since $c_0 = c_1 = \cdots = c_{p+3} = 0$, $c_{p+4} \neq 0$, hence (8) has order $p = 8$, (see [5, 9]). This procedure is repeated for the block formulas (9)-(32). The procedure revealed that the block formulas have uniform order $p = 7$.

Consistency of the Method

Definition 1 (see [5, 9]) "The linear multistep method is said to be consistent if it has order $p \geq 1$. It is obvious that the present method is consistent."

Zero Stability of the Block Method

As claimed by [10], [16], and [23] zero stability of a numerical method imitates the dynamics of the methods as $h \to 0$. This required setting h to zero in (9)-(32) which reduces to

$$\bar{U}_0 \bar{Y} = \bar{U}_1 Y_n \tag{34}$$

where, $\bar{U_0}$ and $\bar{U_1}$ are as defined before.

Definition 2 (see [5]) A linear multistep method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple.

The characteristic polynomial of (34) is,

$$Det(\lambda \bar{U}_0 - \bar{U}_1) = 0 \tag{35}$$

This gives $\lambda^{21}(\lambda_1)^3 = 0$, that is λ_1 to λ_{21} are error 1. Furthermore, other results indicated that the roots of the characteristics polynomial of the method are all equal to one (i.e not exceeding the order of the differential equation) and simple, hence by Def 2, the method is zero stable.

Convergence

We further the analysis by stating the fundamental Dahlquist theorem. (See [5, 9]) "The necessary and sufficient conditions for a linear multi- step method to be convergent are that it be consistent and zero-stable". Having shown that the proposed method is consistent and zero stable, hence, it is also convergent.

Region of Absolute Stability of the method

We consider the stability polynomials written in general form:

$$\pi(r, \hbar) = \rho(r) - \hbar \sigma(r) = 0 \tag{36}$$

where $\hbar = h^2\lambda$ and $\lambda = \frac{\partial f}{\partial Y}$ is assumed to be a constant. The stability polynomial of the main method (8) becomes

$$(r^3 - 4r^{\frac{5}{2}} + 6r^2 - 4r^{\frac{3}{2}} + r) - \hbar\left(\frac{5}{241920} - \frac{30}{241920}r^{\frac{1}{2}} + \frac{54}{241920}r + \frac{2504}{241920}r^{\frac{3}{2}} + \frac{10029}{241920}r^2 + \frac{2574}{241920}r^{\frac{5}{2}} - \frac{16}{241920}r^3\right) = 0 \tag{37}$$

Obviously, the first characteristics polynomial is,

$$\rho(r) = r^3 - 4r^{\frac{5}{2}} + 6r^2 - 4r^{\frac{3}{2}} + r \tag{38}$$

and the second characteristics polynomial is

$$\sigma(r) = \frac{5}{241920} - \frac{30}{241920}r^{\frac{1}{2}} + \frac{54}{241920}r + \frac{2504}{241920}r^{\frac{3}{2}} + \frac{10029}{241920}r^2 + \frac{2574}{241920}r^{\frac{5}{2}} - \frac{16}{241920}r^3 \tag{39}$$

Adopting the boundary locus method, whose equation is given by

$$\hbar = \frac{\rho(r)}{\sigma(r)} \tag{40}$$

By inserting $\rho(r)$ and $\sigma(r)$ into (40), the boundary locus equation is obtained for the method as:

$$\hbar(r) = \frac{r^3 - 4r^{\frac{5}{2}} + 6r^2 - 4r^{\frac{3}{2}} + r}{\frac{1}{241920}(5 - 30r^{\frac{1}{2}} + 54r + 2504r^{\frac{3}{2}} + 10029r^2 + 2574r^{\frac{5}{2}} - 16r^3)} \tag{41}$$

using $e^{i\theta} = cos\theta + isin\theta$, we obtain after some simplification:

$$\hbar(\theta) = \frac{241920(39900 - 44868cos\frac{1}{2}\theta + 99cos\theta + 4726cos\frac{3}{2}\theta + 188cos2\theta - 50cos\frac{5}{2}\theta + 5cos3\theta)}{113480430 + 102039048cos\frac{1}{2}\theta + 13803526cos\theta - 378836cos\frac{3}{2}\theta - 55828cos2\theta + 26700cos\frac{5}{2}\theta - 160cos3\theta}$$

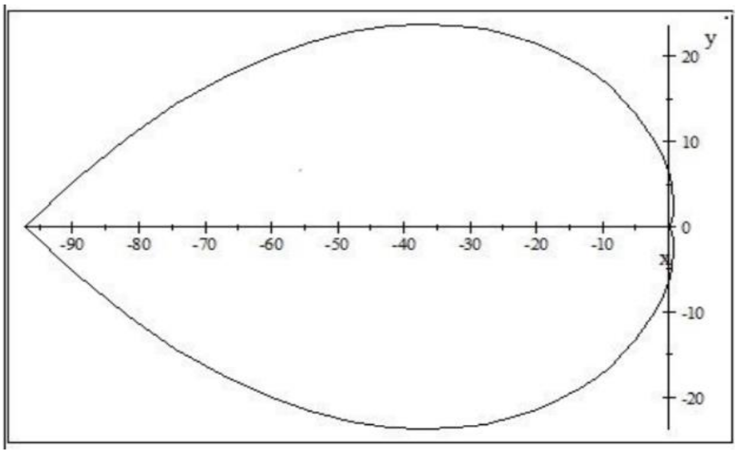

Figure 1: Region of absolute stability of the proposed method

3. Numerical Experiment

This section deals with the application of the methods to solve initial value problems of fourth-order ordinary differential equation to establish its usefulness. We adopted the following in

Problem 1

We first consider the fourth-order ordinary differential equation

$$y^{(4)}(x) = y''' + y'' + y' + 2y, y(0) = 0, y'(0) = 1, y''(0) = 0, \quad y'''(0) = 30,$$

with exact solution in reference [17] as $y(x) = 2e^{2x} - 5e^{-x} + 3cosx - 9sinx.$ The problem is solved numerically over the interval [0,2] in ten iterations. Table 1 compares the numerical results from our proposed method with the exact solution. Columns 4-6 present the absolute errors, while Figure 2 visually demonstrates the error behavior. The results show that our method achieves comparable accuracy to [3], confirming its effectiveness for solving high-order ODEs.

Table 1: Comparison of results of problem 1

X	Numerical y(x)	Exact y(x)	Error in y(x)	Error in [18]	Error in [17]
0.2	0.0421714	0.0421714	8.70415 E-14	3.5129 E-13	2.3190 E-13
0.4	0.3579	0.3579	8.04246 E-13	4.1833 E-12	2.2603 E-12
1.8	62.9237	62.9237	2.93012 E-10	5.4334 E-10	9.1180 E-09
2	99.0875	99.0875	5.11550 E-10	8.0796 E-10	1.7409 E-08

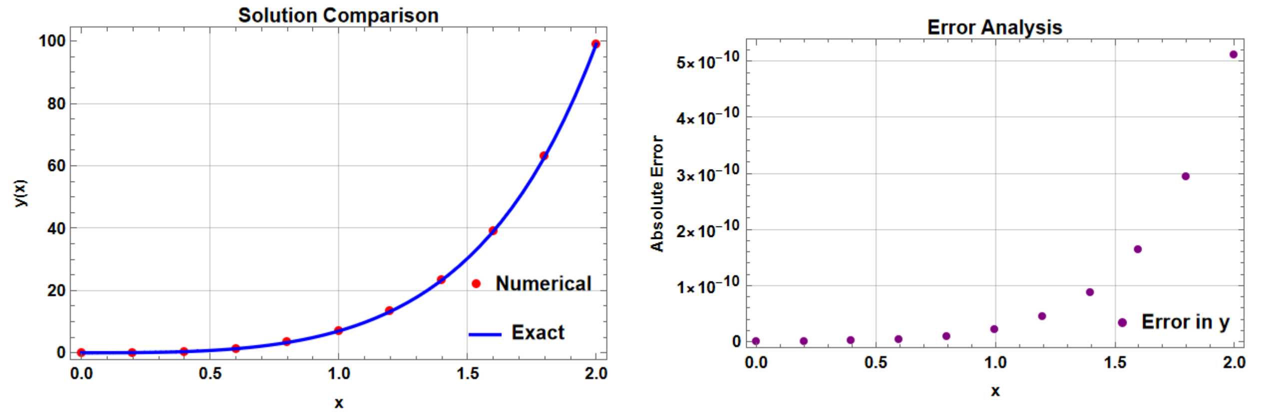

Figure 2: Solution curve of problem 1

Problem 2

Another fourth-order ordinary differential equation consider in this work is

$$y^{(4)}(x) = -y'', y(0) = 0, y'(0) = -\frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, \quad y'''(0) = \frac{1.2}{144 - 100\pi}$$

which has featured in with the exact solution $y(x) = \frac{1 - x - \cos(x) - 1.2\sin(x)}{144 - 100\pi}$.

Table 2: Comparison of results of problem 2

x	Numerical y(x)	Exact y(x)	Error in y(x)	Error in [9]	Error in [22]
0	0	0	0	0	0
0.2	0.00245928	0.00245928	4.33681 E -19	2.60208 E-18	3.40060 E-15
0.4	0.00463309	0.00463309	1.73472 E-18	4.33680 E-18	7.40519 E-14
1.6	0.0104037	0.0104037	2.77556 E-17	5.37764 E-17	5.09100 E-11
1.8	0.010234	0.010234	3.46945 E-17	6.76542 E-17	9.85949 E-11
2	0.00984378	0.00984378	4.16334 E-17	7.97972 E-17	1.83206 E-10

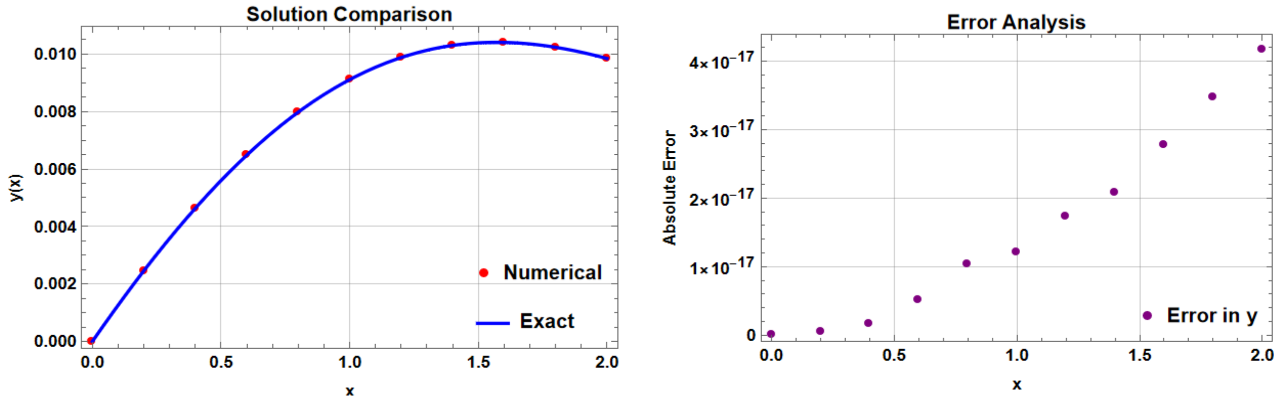

Figure 3: Solution curve of the problem 2

Problem 3

Thirdly, we considered the nonlinear fourth-order ordinary differential of the type

$$y^{(4)} - (y')^2 - yy''' = -4x^2 + e^t(1 + x^2 - 4x), \; y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1,$$

whose exact solution is given by: $y(x) = x^2 + e^x$. The solution was obtained in [0,1] over for 10 iterations. **Table 3** and **Figure 4** detailed the results of the problem.

Table 3: Comparison of results of problem 3

X	Numerical y(x)	Exact y(x)	NDSolve y(x)	Error in y(x)	Error in NDSolve
0	1	1	1	0	0
0.1	1.11517	1.11517	1.11517	0	1.18048 E-8

0.2	1.2614	1.2614	1.2614	0	8.46974 E-9
0.3	1.43986	1.43986	1.43986	2.22045 E-16	2.60246 E-8
0.4	1.65182	1.65182	1.65182	2.22045 E-16	1.45802 E-8
0.5	1.89872	1.89872	1.89872	2.22045 E-16	1.83166 E-8
0.6	2.18212	2.18212	2.18212	2.22045 E-16	2.43914 E-8
0.7	2.50375	2.50375	2.50375	4.44089 E-16	4.37086 E-8
0.8	2.86554	2.86554	2.86554	4.44089 E-16	3.77695 E-8
0.9	3.2696	3.2696	3.2696	4.44089 E-16	4.19098 E-8
1	3.71828	3.71828	3.71828	8.88178 E-16	4.51259 E-8

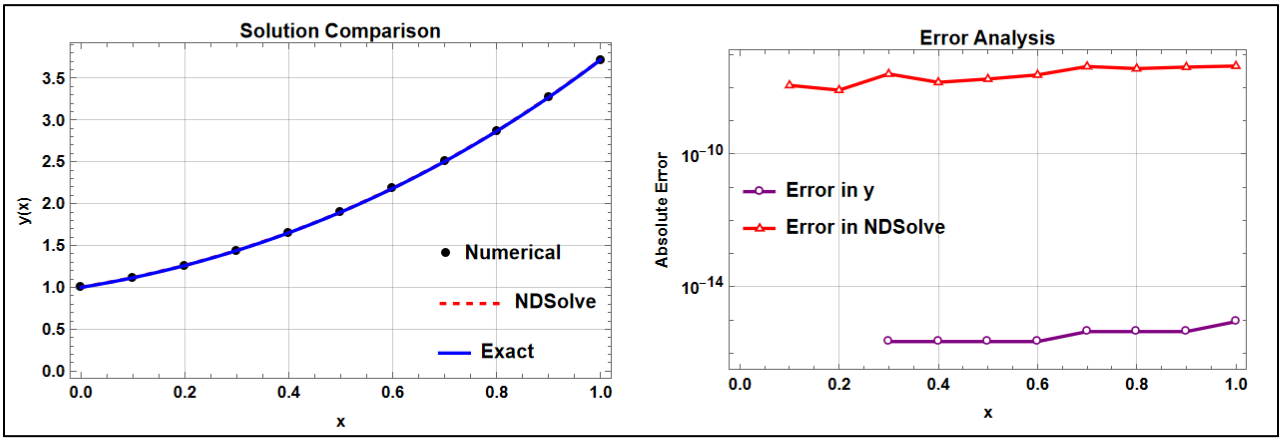

Figure 4: Solution curve of the problem 3

Problem 4

Finally, the inhomogeneous fourth-order initial value problem:

$$y^{(4)} = y'(x) - \cos(x), y(0) = -\frac{1}{2}, \quad y'^{(0)} = \frac{1}{2}, \quad y''^{(0)} = \frac{1}{2}, \quad y'''(0) = \frac{1}{2}$$

whose exact solution $y(x) = \frac{1}{2}(\sin(x) - \cos(x))$ was given by Saleh et al. [26]. The numerical solution, computed over [0,2] in 10 iterations, achieves a maximum absolute error of 1.15×10^{-14}matching the error at $x = 2$. This confirms the method's stability and accuracy for such problems. Results are detailed in **Table 4 and Figure 5**.

Table 4: Comparison of results of problem 4

X	Numerical y(x)	Exact y(x)	NDSolve y(x)	Error in y(x)	Error in NDSolve
0	-0.5	-0.5	-0.5	0	0
0.2	-0.390699	-0.390699	-0.390699	1.11022 E-16	2.3393 E-8
0.4	-0.265821	-0.265821	-0.265821	3.05311 E-16	2.46752 E-8
0.6	-0.130347	-0.130347	-0.130347	5.55112 E-16	2.12746 E-8
0.8	0.0103247	0.0103247	0.0103247	8.88178 E-16	1.71087 E-8
1	0.150584	0.150584	0.150584	1.55431 E-15	4.63302 E-10
1.2	0.284841	0.284841	0.284841	2.60902 E-15	1.18341 E-8
1.4	0.407741	0.407741	0.407741	4.05231 E-15	2.3636 E-8
1.6	0.514387	0.514387	0.514387	5.82867 E-15	3.86825 E-8
1.8	0.600525	0.600525	0.600525	8.43769 E-15	5.63421 E-8
2	0.662722	0.662722	0.662722	1.16573 E-14	6.45155 E-8

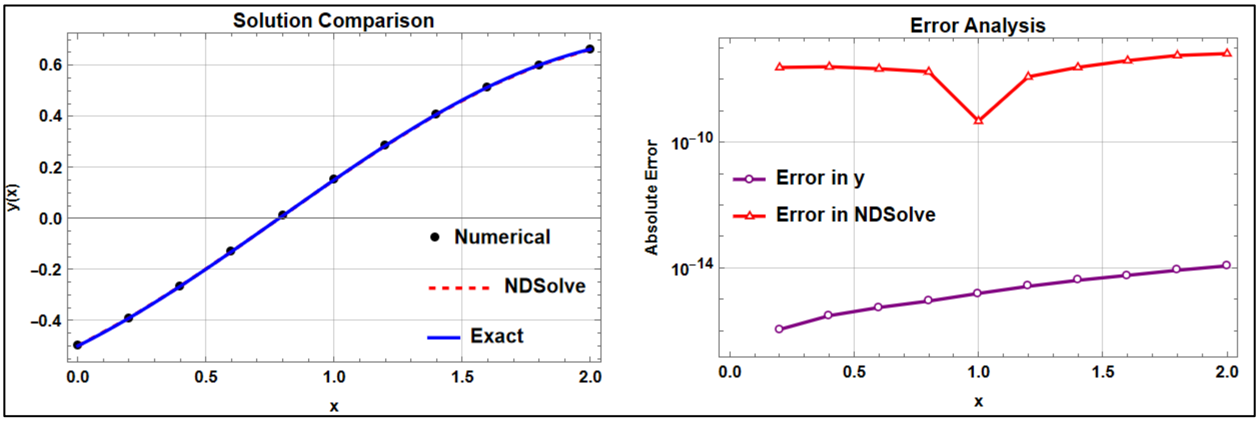

Figure 5: Solution curve of the problem 4

3.1 DISCUSSION OF RESULTS

Four test problems were selected to evaluate the relevance and efficacy of the derived block method. The results, presented in Tables 2–5, demonstrate the superiority of the proposed method over those in [9], [17], [18], and [22]. Tables 1–2 further confirm that the new block method provides a more accurate approximation than the cited methods, as evidenced by smaller errors. Additionally, Tables 3–4 and Figures 4–5 reveal that the proposed method achieves higher accuracy than Mathematica's NDSolve for test problems 3 and 4. The recorded errors indicate that the block method yields solutions very close to the exact solutions in all cases.

4. CONCLUSION

In this paper, we have presented a novel seventh-order linear multistep scheme for the direct numerical integration of fourth-order initial value problems. The method's derivation, based on a continuous scheme and evaluation at a specific grid point, ensures its high order of accuracy. We have rigorously analyzed the stability properties of the proposed method, demonstrating its zero-stability and consistency, which are crucial for guaranteeing convergence. Numerical examples were provided to illustrate the method's better accuracy and efficiency when compared to existing numerical techniques for solving fourth-order ODEs. The results consistently show that this new seventh-order linear multistep scheme offers a robust, accurate, and computationally efficient alternative for tackling complex fourth-order initial value problems encountered in various scientific and engineering disciplines. Future work will focus on extending this method to handle stiff and oscillatory problems, as well as developing adaptive step-size control mechanisms to further enhance its practical utility.

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