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Application of Homotopy Analysis Method on Selected Highly Nonlinear BVPs

Sumra Mugheer Shah[1]*, Johar Abbas[2], Tahir Naveed[3], Arsalan Jafar[4], Qurban Ali Khoso[5]

[1]Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan
[2]Department of Mathematics, University of Education, Lahore, Punjab, Pakistan
[3]Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan
[4]Department of Mathematics, Bahauddin Zakariya University, Multan, Punjab 36350, Pakistan
[5]Department of Information Technology, Shaheed Benazir Bhutto University, Shaheed Benazirabad, Sindh 67450, Pakistan

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***Corresponding author:** Sumra Mugheer Shah
Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan

Abstract **Review Article**

This research explores the application of the Homotopy Analysis Method (HAM) to address selected highly nonlinear boundary value problems (BVPs) commonly found in physical and engineering sciences. Traditional approaches such as the perturbation method, homotopy perturbation method (HPM), and other semi-analytical techniques often face limitations due to the requirement for small parameters or lack of control over convergence. In contrast, HAM provides a flexible framework that introduces an auxiliary parameter, enabling convergence control of the solution series without relying on the existence of small parameters. The study is structured into three core chapters. The first chapter lays a comprehensive foundation, introducing key fluid dynamics concepts, heat transfer principles, types of differential equations, and mathematical laws pertinent to the subsequent analyses. Chapter two investigates the nonlinear convection-radiation heat transfer equation, applying HAM and comparing its effectiveness with the perturbation method and HPM. The analysis reveals that HAM maintains high accuracy even for large parameter values, where perturbative techniques fail due to asymptotic divergence. Using Mathematica, the convergence behavior is examined, and error profiles are plotted to validate the results. Section three presents a novel application of HAM to solve second- and fourth-order Sturm–Liouville eigenvalue problems, which are critical in modeling vibrations, thermal analysis, and elastic stability. The study introduces new algorithmic formulations and solution profiles, capturing multiple eigenvalue solutions and validating them through the appearance of λ-plateaus. These results showcase HAM's capacity to yield multiple accurate eigenfunctions from a single initial approximation, highlighting its robustness and broader applicability compared to traditional methods. The outcomes confirm that the Homotopy Analysis Method is a powerful and adaptable tool for solving complex nonlinear differential systems, offering both analytical precision and computational efficiency.
Keywords: Homotopy Analysis Method (HAM), Nonlinear Differential Equations, Boundary Value Problems (BVPs), Heat Radiation Equation, Convection-Radiation, Sturm–Liouville Problem, Eigenvalue Problems, Homotopy Perturbation Method (HPM), Perturbation Methods, Semi-analytical Methods.

1. INTRODUCTION

The majority of problems encountered in science and engineering are expressed in terms of linear, nonlinear, ordinary, or partial differential equations. Because such an equation cannot be solved analytically due to its various completions. There are several numerical methods that have been developed to solve equations. Shijun Liao [1], is credited with inventing the Homotopy Analysis Method which has proven to be a highly effective approach for solving a wide range of nonlinear differential equations. The HAM is a semi-analytical technique for solving nonlinear ordinary or partial differential equations. It is based on the concept of homotopy in topology, which is a continuous deformation of one function into another. The HAM approach has been shown to be an extremely efficient way of tackling a large range of nonlinear differential equations. It has been used for the equations of physics, chemistry, engineering and other fields. As a result, Homotopy Analysis Methods are applicable to solve nonlinear problems. The auxiliary parameter ħ in Homotopy Analysis Methods controls how quickly the HAM series solution converges. Abbasbandy [2], and other researchers have used it extensively in the past ten years to approximate solutions of highly nonlinear differential equations that have appeared in various fields of research. One of the widely used techniques for solving nonlinear problems is the perturbation approach, which is dependent on the presence of large or small parameters. However, many nonlinear problems do not have a readily available perturbation parameter, which restricts the applicability of perturbation-based approaches. In such cases, alternative non-perturbation techniques, such as the δ-expansion technique discussed by A.V. Karmishin [3], and the homotopy perturbation method discussed by Abbasbandy [4], can be employed. Nevertheless, both perturbation and non-perturbation approaches lack a straightforward strategy for adjusting and controlling the convergence region and rate of an approximate series. Abbasbandy demonstrated the significance of predicting multiple solutions while employing the auxiliary parameter h, which governs the convergence behavior of HAM solutions in general [5, 6].

Section 1 Includes certain prerequisites and essential definitions of fluid, fluid flow, heat, fundamental laws, differential equations, types of differential equation, fundamental concepts of basic terminologies and parameters that are used in the thesis. Also discussed are the fundamental concepts of the homotopy analysis method and the homotopy perturbation method.

Section 2 is a discussion of the non-linear boundary value problem by utilizing the Homotopy analysis method to identify numerous solutions. After introducing the concept of the Homotopy Analysis Method, research is done on how it applies to the heat radiation equation. Additionally, Abbasbandy [7], used the homotopy analysis method (HAM) to solve a heat radiation equation and compare the results to those obtained by Ganji [8], using the numerical solution, perturbation method, and homotopy perturbation method. The solutions of the Homotopy Perturbation method and the perturbation method are frequently the same. The HAM is used to solve the heat radiation equation, which has two small parametersϵ_1, ϵ_2 and $\mathcal{L}$, which is a linear operator. In this method, the auxiliary parameter h allows for adjustment and control of the convergence of the approximation series solution obtained by the Homotopy Analysis Method. Actually, in many cases, especially for the situation studied in [9], the solutions derived using the perturbation approach and the HPM are identical.

In Section 3, we calculate several solutions to the Sturm-Liouville Problem using the auxiliary parameter ħ. For several papers that deal numerically with the Sturm-Liouville Problem discuss by Attili BS [10]. The HAM was described by Liao as handling nonlinear second- order and fourth-order eigenvalue problems. However, in this chapter, we discuss the new application of the homotopy analysis method presented by Abbasbandy [11], and compute the eigenvalue and eigenfunction by initiating HAM with the same initial guess, and $\mathcal{L}$ is the linear operator. This chapter consists of some fundamental laws related to fluid flows and heat transfer, as well as some basic properties regarding fluids and flows. Additionally, discussed the basic concepts regarding the solution methodology (HAM) that is used to find out the solution to the proposed problem in the next chapter.

1.1 Differential Equation and its Types

A differential equation is a mathematical equation that consists of functions and their derivatives. Generally, it can be defined as follows:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{1.1}$$

1.1.1 Ordinary Differential Equation

A differential equation is said to be ordinary differential equation, if it contains derivative of dependent variable with respect to one independent variable.

For example

$$\frac{dy}{dx} = 7\mathrm{x} + 3. \tag{1.2}$$

1.1.2 Partial Differential Equation

An equation that contains partial derivatives of a dependent variable with respect to two or more independent variables is called a partial differential equation, for example

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 7. \tag{1.3}$$

1.1.3 Linear and Non Linear Differential Equation

i. Coefficient of the derivative of the dependent variable should be either constant or a function of independent variable.
ii. The power of dependent variable and it's all derivatives must be one.

For example

$$\frac{dy}{dx} = 7\mathrm{xy} + 3. \tag{1.4}$$

The differential equation is not a linear is called nonlinear differential equation, such as

$$y\frac{dy}{dx} = 7\mathrm{xy} + 3. \tag{1.5}$$

1.2 Basic Fundamentals

1.2.1 Fluid Mechanics

Fluid mechanics is the branch of science concerned with the mechanics of fluids (gases, liquids, and plasmas), internal and external forces acts on them. Normally fluid mechanics can be divided into static fluid and dynamic fluid.

1.2.2 Fluid

A substance that may have the potential to flow is called fluid when the external force is applied to it.

1.2.3 Newton's Law of Viscosity

Newton's law of viscosity tells us that the shear stress(τ) is directly proportional to the deformation/shear rate at given pressure and temperature. Mathematically it can be written as

$\tau \propto \left(\frac{du}{dy}\right)$,	(1.6)
$\tau = \mu\left(\frac{du}{dy}\right)$,	(1.7)

where $\frac{du}{dy}$ is the shear rate, μ is the viscosity and τ is the shear stress.

1.3 Types of Fluids

Fluids can be separated into two types.

1.3.1 Ideal Fluid

A fluid is said to be ideal fluid or perfect fluid if its viscosity $\mu = 0$.

1.3.2 Real Fluid

A fluid is said to be real fluid if its viscosity $\mu \neq 0$. Real fluids are further classified into two major categories.

1.3.3 Newtonian Fluids

A fluid that follows Newton's law of viscosity is called Newtonian fluid. Well-known Newtonian fluids include air, alcohol, water, glycerol, and thin motor oil.

1.3.4 Non-Newtonian Fluids

Fluids that do not adhere to Newton's law of viscosity are called Non-Newtonian fluids. Such as soap solutions, toothpaste, jam, butter, yogurt, gum, paint, shampoo, silly putty, molten plastic, and honey are all examples of Non-Newtonian fluids. Biological fluids, such as blood, saliva, mucus, and sperm, are also Non-Newtonian fluids.

1.4 Flow and its Types

Flow is a branch of fluid mechanics that studies the motion of fluids subject to various forces. There are various flow types, expressed as steady flow and unsteady flow.

1.4.1 Steady Flow

In steady flow, the fluid characteristics such as pressure, density, velocity, and temperature remain constant at all points in the fluid over time. It can be written as:

$\frac{\partial V}{\partial t} = 0,$	(1.8)

where, V represents the velocity of fluid flows.

1.4.2 Unsteady Flow

The flow is unsteady if the fluid characteristics change over time. The flow of water in the dam is an example of unsteady flow. Mathematically, it is expressed as:

$\frac{\partial V}{\partial t} \neq 0.$	(1.9)

1.5 Heat

Heat is the form of thermal energy and transfer due to the temperature difference. It involves the transport of kinetic energy due to the temperature difference between the two mediums. Heat can be transferred in three different ways:

1.5.1 Convection

In the convection mode, the fluid velocities/movements are involved in heat transfer from one place to another. When a fluid is heated, the warmer fluid moves upward and the cooler fluid moves downward, due to density change.

1.5.2 Conduction

In the conduction mode, the direct contact between objects or substances is caused by heat transfer. When objects of different temperatures collide, higher-energy particles transmit energy to lower-energy particles through molecular collisions. This type of heat transfer occurs within solid materials. For instance, when you touch a hot metal surface, heat is transferred from the metal to your hand.

1.5.3 Radiation

Radiation is the mode of transfer of heat from one place to another in the form of waves. The radiation is emitted by all bodies. The rate at which radiation is emitted depends upon various factors such as the color and texture of the surface, surface temperature, and surface area.

1.6 Specific Heat

The amount of heat energy necessary to increase the temperature of one kilogram of substance up to one Celsius or Kelvin is called specific heat. It is measured in joules per kilogram per Kelvin (J/kg/K).

1.7 Fluid Properties

1.7.1 Pressure

Pressure is defined as the force applied per unit area perpendicular to the surface on which the force is acting. In mathematical terms, it can be represented by an equation:

$\boldsymbol{P} = \frac{\boldsymbol{F}}{\boldsymbol{A}},$	(1.10)

where $\boldsymbol{P}$ represents pressure, $\boldsymbol{F}$ indicates force, and $\boldsymbol{A}$ denotes area.

1.7.2 Shear Stress

The amount of shear stress is proportional to the force applied and the area of the surface that is being deformed. Shear stress is denoted by the Greek letter tau (τ). The formula for shear stress can be written as:

$\boldsymbol{\tau} = \frac{\boldsymbol{F}}{\boldsymbol{A}},$	(1.11)

where $\boldsymbol{F}$ indicates force, and $\boldsymbol{A}$ denotes area.

1.7.3 Density

Density is defined as mass per unit volume. It is represented by $\boldsymbol{\rho}$, its SI unit is kg/m^3, and mathematically written as:

$\boldsymbol{\rho} = \frac{\mathbf{M}}{\mathbf{V}},$	(1.12)

where V is volume and M is mass.

1.7.4 Viscosity

Viscosity is a measure of a fluid's resistance to flow. The ratio of shear stress and rate of strain in the fluids motion is referred to as viscosity.

$\mu = \frac{\tau}{\frac{du}{dy}},$	(1.13)

where $\frac{du}{dy}$ is the rate of deformation is, μ is the dynamic viscosity, and τ is the shear stress.

1.7.5 Kinematic Viscosity

The ratio of dynamic viscosity to the fluid density is called kinematic viscosity.It is mathematically defined as:

$\nu = \frac{\mu}{\rho},$	(1.14)

where μ is the viscosity,ρ is density and ν is kinematic viscosity.

1.8 Boundary Layer and Its Thickness

Boundary layer theory is a branch of fluid mechanics that studies the behaviour of fluid flows near solid surfaces. Boundary layer theory aims to understand and predict the behaviour of fluid flows in the boundary layer region, including the thickness of the boundary layer, the velocity and temperature profiles within the boundary layer, and the forces acting on the surface. This information is useful for a wide range of applications, including the design of aircraft wings, ships, and heat exchangers. Boundary layer flow is the term for a thin layer of fluid that is present close to a solid body's/walls and interacts with a stream that is moving in that direction. The boundary layer thickness is the distance from the solid surface to the point where the fluid velocity is attained 99% of the free stream velocity. There are two main types of boundary layer flows: laminar and turbulent. Laminar boundary layer flows are characterized by smooth, orderly flow patterns. Turbulent boundary layer flows are characterized by chaotic, disordered flow patterns.

1.9 Some Basic Laws

1.9.1 Law of Conservation of Mass

The law of conservation of mass is a fundamental principle of nature is states that mass cannot be created or destroyed. It means that the fluid's mass will remain constant during flows over a control volume. This is an essential principle for understanding the behaviour of fluids, for their applied in many different applications, from engineering to environmental science.

1.9.2 Law of Conservation of Momentum

The law of conservation of momentum states that the total momentum of an isolated system remains constant. This means that the sum of the momentum of all the particles in the system does not change over time, mathematically it can be defined as

$$\rho \frac{DV}{Dt} + \nabla.\tau + f = 0, \qquad (1.15)$$

τ is the Cauchy stress tensor, which varies and depending on the fluid.

1.9.3 Law of Conservation of Energy

This law indicates that the energy of the entire system cannot be created or destroyed; this means the total amount of energy in the system is also constant. It is mathematically defined as,

$\rho c_p \frac{D\theta}{Dt} = \mathrm{T.L} - \nabla.\mathrm{q}.$	(1.16)

Here, T is the temperature at the boundary, L is the vector normal to the boundary and $\nabla.\mathrm{q}$ represents the divergence of the heat flux within the material.

1.10 Sturm-Liouville Problem

In mathematics, Sturm-Liouville problem/eigenvalue problem is a certain class of partial differential equations (PDEs) subject to additional constraints, known as boundary values. The current study examined the determination of eigenvalues and eigenfunctions for 2nd order linearly homogeneous differential equation, is known as the Sturm-Liouville equation. The general form of the Sturm-Liouville equation is given as:

$\frac{d}{dx}\left\{g(x)\frac{dy}{dx}\right\} + h(x)y = \lambda \mathrm{v}(x)y.$	(1.17)

Here,$h(x)$, $g(x)$, $g'(x)$ and $v(x)$ are continue on close interval [a, b], $h(x)$, $g(x)$and $v(x)$ all are positive, and λ is parameter not depends on x.

The goal of the Sturm-Liouville problem is to determine the eigenvalues and the relevant eigenfunction $y(x)$ that satisfies the Sturm-Liouville equation and the specified boundary conditions.

1.11 The Fundamental Concept of the Homotopy Perturbation Method

The homotopy perturbation method is a powerful mathematical technique for solving nonlinear differential equations that combine the homotopy method with traditional perturbation techniques. It provides an iterative method for finding approximate solutions to a wide range of nonlinear problems. We take a certain nonlinear differential problem into consideration to explain the basic idea of the homotopy perturbation method for solving nonlinear differential equations.

$$C(v) - f(s) = 0, s \in \Omega, \quad (1.18)$$

where, f is a known function, C is a differential operator and Ω represents the domain of the problem. To solve equation(1.18) using the HPM, we introduce the concept of a homotopy. We split the operatorinto two components: a linear component $L(v)$ and a nonlinear component $N(v)$, so equation(1.18) can be written as:

$$L(v) + N(v) - f(s) = 0, \quad (1.19)$$

We construct the homotopy equation by adding and subtracting the linear component evaluated at an initial guess v_0. The homotopy equation is given by:

$$H(v, p) = L(v) - L(v_0) + pL(v_0) + pN(v) - f(s) = 0, \quad (1.20)$$

Here, p is a homotopy parameter. The approximate solution of equation (1.20) can be written as a series of powers of p by using the homotopy perturbation method, i.e.

$$v = v_0 + pv_1 + p^2v_2 + \cdots, \quad (1.21)$$

This series represents an iterative process where each term represents an approximation of the solution at a specific order. As p approaches 1, the series simplifies to approximation solution given in equation (1.20).

$$v = v_0 + v_1 + v_2 + \cdots, \quad (1.22)$$

The above convergence is discussed in [12, 13].

1.12 Homotopy Analysis Method

The Homotopy Analysis is a powerful mathematical tool for solving nonlinear differential equations and system of equation. The HAM [14], is applied to a nonlinear system represented by:

$$\mathbb{N}[v(\delta)] = 0, \quad (1.23)$$

$\mathbb{N}$stands for the nonlinear operator, and δis the independent variable. In the context of the broadened scope of the conventional HAM, the zero-order deformation equation as presented by Liao follows.

$$(1-p)\,\mathcal{L}[\psi(\delta;p) - v_0(\delta)] = p\hbar H(\delta)\mathbb{N}[\psi(\delta;p)], \quad (1.24)$$

where $\hbar \neq 0$ denotes an auxiliary parameter, p represents the embedding parameter for the range$0 \leq p \leq 1$,$H(x) \geq 0$ is auxiliary function, $v_0(\delta)$ is an initial estimate of $v(\delta)$, $\mathcal{L}$ represents an auxiliary linear operator, and $\psi(\delta;p)$ is solution function. At $p = 0$ and $p = 1$, theequation (1.24) becomes as

$\psi(\delta;0) = v_0(\delta), \quad \psi(\delta;1) = v(\delta),$

when $p \epsilon (0,1)$ the solution $\psi(\delta;p)$ changes from the initial approximation $v_0(\delta)$,and solution function $\psi(\delta;p)$ can be generalized as Taylor's series, and one has

$$\psi(\delta;p) = v_0(\delta) + \sum_{m=1}^{+\infty} v_m(\delta)p^m, \quad (1.25)$$

where

$$v_m(\delta) = \frac{1}{m!}\frac{\partial^m \psi(\delta;p))}{\partial p^m}\Big|_{p=0}, \quad (1.26)$$

the series (1.25) converges at $p = 1$.

$$u(\delta) = v_0(\delta) + \sum_{m=1}^{+\infty} v_m(\delta), \quad (1.27)$$

Based on Liao's evidence, this solution seems to be one of the possible solutions to the original nonlinear equation.

As$h = -1$ and $H(\delta) = 1$, equation (1.24) becomes

$$(1-p)\,\mathcal{L}[\psi(\delta;p) - v_0(\delta)] = (p)(-1)(1)\,\mathbb{N}[\psi(\delta;p)],$$
$$(1-p)\,\mathcal{L}[\psi(\delta;p) - v_0(\delta)] + p\,\mathbb{N}[\psi(\delta;p) = 0,$$

According to equation (1.26)the zero-order deformation equation(1.24)provides the governing equation is defining a vector

$\vec{v_n} = \{v_0(\delta), v_1(\delta), v_2(\delta), \dots, v_1(\delta)\}.$

At $p = 0$ and after dividing by m!, the zero-order deformation equation differentiates m-times with regard to the embedding parameter p, then the equation becomes the mth-order deformation equation is given as:

$$\mathcal{L}[v_m(\delta) - v_{m-1}(\delta)] = \hbar H(\delta)R_m(v_{m-1}), \quad (1.28)$$

$R_m(\vec{v}_{m-1})$ is the Residual function.

$$R_m(\vec{v}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbb{N}[\psi(\delta;p)]}{\partial p^{m-1}}\Big|_{p=0}, \quad (1.29)$$

$$\chi_m = \begin{cases} 0, form \leq 1, \\ 1, form > 1, \end{cases} \quad (1.30)$$

It is worth emphasizing that the linear boundary conditions from the original problem are applied to

$v_m(\delta)$ for $m \geq 1$. Equation(1.29) can be effectively solved using various symbolic computation programs, such as Mathematica or Maple. These programs offer powerful tools for finding solutions to mathematical equations and can handle the given linear equation against the specified boundary conditions. In HAM, a convergence parameter, commonly referred to as "ħ" is offered to regulate and modify the convergence behavior of the series solution obtained by using HAM.

2. Homotopy Analysis Technique for Radiative Heat Transfer

The goal of this chapter is to revise the study of heat radiation equations as discussed by Saied Abbasbandy [7], and compare the results with those obtained by Ganji [8]. The study of heat radiation equations plays a significant role in understanding and analyzing various thermal phenomena. Saied Abbasbandy which makes an additional contribution to the advancement of Homotopy Analysis Method. Liao created the Homotopy Analysis Method to find out the solution to a nonlinear problem. Recently, HAM became a very effective method to find out the solution to the non-linear problem. The other approaches such as the Method of Adomian Decomposition, the Method of Lyapunov Small Artificial Parameters, Homotopy Perturbation Method [15], and δ-Expansion Method, etc. are also well-known approaches to finding the solution to the problem. Expansion Method is very simple. This method was also used for the solution of the linear problem by Liao SJ. In order to tackle heat transfer problems characterized by a high degree of nonlinearity, the HAM is utilized. The Perturbation and Homotopy Perturbation Methods are also used to compare the results. In fact, the solutions of the Homotopy Perturbation technique and the perturbation technique are frequently the same. The HAM is used to solve the under consider problem, which has two small parameters ϵ_1, ϵ_2 and $\mathcal{L}$, which is a linear operator. In this method, the auxiliary parameter $\hbar$ allows for adjustment and control of the convergence of the approximation series solution obtained by the Homotopy Analysis Method. This flexibility allows for fine-tuning the solution process and enhancing the accuracy of the results.

2.1 The Fundamental Concept of the Homotopy Analysis Method:

The Homotopy Analysis Method is an effective mathematical method for simplifying nonlinear differential equations. In the context of this method, the HAM is used to solve a particular nonlinear system represented by an equation given as:

$\mathbb{N}[v(\delta)] = 0,$	(2.1)

$\mathbb{N}$ stands for the nonlinear operator, δ stands for the independent variable, and $v(\delta)$ is the unknown function.

2.1.1 Zero-Order Deformation Equation

In the expanded framework of the Homotopy Analysis Method, Liao introduces the zero-order deformation equation can be formulated as follows:

$(1-p)\mathcal{L}[\psi(\delta;p) - v_0(\delta)] = p\hbar H(\delta)\mathbb{N}[\psi(\delta;p)],$	(2.2)

The embedding parameter p controls the deformation from the known linear problem ($p = 0$) to the desired nonlinear problem ($p = 1$). Where $\hbar \neq 0$ denotes an auxiliary parameter, p represents the embedding parameter with range $0 \leq p \leq 1$, and $v_0(\delta)$ is an initial estimation of $v(\delta)$.

2.1.2 Mth-Order Deformation Equation

The zero-order deformation equation is differentiated m times with respect to the establishing parameter p when p is set to zero, creating the mth-order deformation equation after dividing by m! can be written as:

$\mathcal{L}[v_m(\delta) - v_{m-1}(\delta)] = \hbar H(\delta)R_m(v_{m-1}),$	(2.3)

The auxiliary linear operator denoted by the letter $\mathcal{L}$ is defined based on a well-known linear problem connected to the original nonlinear problem.

2.1.3 Residual Function

The term $R_m(v_{m-1})$ in the HAM indicates the mth order residual function, which is utilized to create the mth order deformation equation is represented as

$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbb{N}[\psi(\delta;p)]}{\partial p^{m-1}}\Big\vert_{p=0}$	(2.4)

2.2 Nonlinear Unsteady Convection-Radiation Equation

In a lumped system with simultaneous convective and radiation heat exchanges, the specific heat coefficient is linearly related to temperature is discussed in [8] can be written as:

$c = c_a[1 + \beta(T - T_a)],$	(2.5)

c_a is the specific heat at T_a, and β is the constant.

$\rho Vc\frac{dT}{dt} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0,$ here, E is the emissivity ρ is the density, and V is the volume.	(2.6)
$T(0) = T_i,$	(2.7)

by utilizing

$u = \frac{T}{T_i}, u_a = \frac{T_a}{T_i}, \delta = \frac{t(hA)}{\rho Vc_a}, \epsilon_1 = \beta T_i, \epsilon_2 = \frac{E\sigma T_i^3}{h}, u_s = \frac{T_s}{T_i},$	(2.8)

We have

$[1 + \epsilon_1(u - u_a)]\frac{du}{d\delta} + (u - u_a) + \epsilon_2(u^4 - u_s^4) = 0, u(0) = 1,$	(2.9)

Currently, a fundamental issue is determining how to estimate $\epsilon_1 and\ \epsilon_2$. By using the homotopy analysis approach, we may select auxiliary parameters in the HAM and arrive at a suitable solution for each of the ϵ_1 and ϵ_2. However; only for small values of ϵ_1 and ϵ_2 are acceptable for the solution using HPM or the perturbation approach, so assumed that $u_a = u_s = 0$. The resultant equation (2.9) becomes;

$[1 + \epsilon_1 u]\frac{du}{d\delta} + u + \epsilon_2 u^4 = 0,$	(2.10)

2.3 Perturbation Method

In equation (2.10) there are two different parameters $\epsilon_1 and\ \epsilon_2$, expressed as

$u(\delta) = u_{00}(\delta) + \epsilon_1 u_{01}(\delta) + \epsilon_2 u_{10}(\delta) + \epsilon_1^2 u_{02}(\delta) + \epsilon_1\epsilon_2 u_{11}(\delta) + \epsilon_2^2 u_{20}(\delta).$	(2.11)

Substituting equation (2.11) into equation (2.10) and arranging it according to coefficients of $1, \epsilon_1, \epsilon_2, \epsilon_1^2, \epsilon_2^2$ and $\epsilon_1\epsilon_2$.

For coefficient 1:

$$\frac{du_{00}(\delta)}{d\delta} + u_{00}(\delta) = 0, u_{00}(0) = 1,$$

$$u_{00}(\delta) = ce^{-\delta}, c = 1, u_{00}(\delta) = e^{-\delta}, \quad (2.12)$$

Equating the coefficients of ϵ_1:

$$\frac{du_{01}(\delta)}{d\delta} + u_{00}(\delta)\frac{du_{00}(\delta)}{d\delta} + u_{01}(\delta) = 0, u_{01}(0) = 0,$$

$$\frac{du_{01}(\delta)}{d\delta} + e^{-\delta}\frac{d(e^{-\delta})}{d\delta} + u_{01}(\delta) = 0,$$

$$\frac{du_{01}(\delta)}{d\delta} + u_{01}(\delta) = e^{-2\delta},$$

$$u_{01}(\delta) = e^{-\delta} - e^{-2\delta}, \quad (2.13)$$

Equating the coefficients of ϵ_2:

$$\frac{du_{10}(\delta)}{d\delta} + u_{10}(\delta) = -(u_{00}(\delta))^4, u_{10}(0) = 0,$$

$$\frac{du_{10}(\delta)}{d\delta} + u_{10}(\delta) = -e^{-4\delta},$$

$$u_{10}(\delta)e^{\delta} = \frac{e^{-3\delta}}{3} + c, c = \frac{-1}{3},$$

$$u_{10}(\delta) = \frac{e^{-4\delta}}{3} - \frac{e^{-\delta}}{3}, \quad (2.14)$$

Equating the coefficients of ϵ_1^2:

$$\frac{du_{02}(\delta)}{d\delta} + u_{01}(\delta)\frac{du_{00}(\delta)}{d\delta} + u_{00}(\delta)\frac{du_{01}(\delta)}{d\delta} + u_{02}(\delta) = 0, u_{02}(0) = 0,$$

$$\frac{du_{02}(\delta)}{d\delta} + u_{02}(\delta) = 2e^{-2\delta} - 3e^{-3\delta}, u_{02}(\delta)e^{\delta} = -2e^{-\delta} + \frac{3e^{-2\delta}}{2} + c, c = \frac{1}{2},$$

$$u_{02}(\delta) = \frac{e^{-\delta}}{2} - 2e^{-2\delta} + \frac{3e^{-3\delta}}{2}, \quad (2.15)$$

Equating the coefficients of ϵ_2^2:

$$\frac{du_{20}(\delta)}{d\delta} + 4u_{10}(\delta)\left(u_{00}(\delta)\right)^3 + u_{20}(\delta) = 0, u_{20}(0) = 0,$$

$$\frac{du_{20}(\delta)}{d\delta} + \frac{4}{3}\left(e^{-7\delta} - e^{-4\delta}\right) + u_{20}(\delta) = 0,$$

$$\frac{du_{20}(\delta)}{d\delta} + u_{20}(\delta) = -\frac{4}{3}\left(e^{-7\delta} - e^{-4\delta}\right),$$

$$\frac{du_{20}(\delta)}{d\delta} + u_{20}(\delta) = -\frac{4}{3}(e^{-7\delta} - e^{-4\delta}),$$

$$u_{20}(\delta)e^{\delta} = \frac{4}{3}\left(\frac{e^{-2\delta}}{6} - \frac{e^{-3\delta}}{3}\right) + c,$$

$$u_{20}(\delta) = \frac{2e^{-\delta}}{9} - \frac{4e^{-4\delta}}{9} + \frac{2e^{-7\delta}}{9}, \quad (2.16)$$

Equating the coefficients of $\epsilon_1\epsilon_2$:

$$\frac{du_{11}(\delta)}{d\delta}+u_{10}(\delta)\frac{du_{00}(\delta)}{d\delta}+u_{00}(\delta)\frac{du_{10}(\delta)}{d\delta}+u_{11}(\delta)+4u_{01}(\delta)\big(u_{00}(\delta)\big)^3=0,$$

$$\frac{du_{11}(\delta)}{d\delta}+u_{11}(\delta)-\frac{1}{3}(e^{-5\delta}-e^{-2\delta})+\frac{1}{3}(-4e^{-5\delta}+e^{-2\delta})+(4e^{-4\delta}-4e^{-5\delta})=0,$$

$$\frac{du_{11}(\delta)}{d\delta}+u_{11}(\delta)-\frac{17e^{-5\delta}}{3}+\frac{2e^{-2\delta}}{3}+4e^{-4\delta}=0,$$

$$u_{11}(\delta)e^{\delta}=\frac{2e^{-\delta}}{3}+\frac{4}{3}e^{-3\delta}-\frac{17e^{-4\delta}}{12}+\text{c},\frac{2}{3}+\frac{4}{3}-\frac{17}{12}+\text{c}=0,\text{c}=-\frac{7}{12},$$

$$u_{11}(\delta)=\frac{1}{12}[-7e^{-\delta}+8e^{-2\delta}+16e^{-4\delta}-17e^{-5\delta}]. \tag{2.17}$$

Substituting equations (2.12−2.17) into equation (2.11) so the perturbation solution is obtained as;

$$u(\delta)=u_{00}(\delta)+\epsilon_1u_{01}(\delta)+\epsilon_2u_{10}(\delta)+\epsilon_1^2u_{02}(\delta)+\epsilon_1\epsilon_2u_{11}(\delta)+\epsilon_2^2u_{20}(\delta),$$

$$u(\delta)=e^{-\delta}+\epsilon_1(e^{-\delta}-e^{-2\delta})+\epsilon_2(\frac{e^{-4\delta}}{3}-\frac{e^{-\delta}}{3})+\epsilon_1^2(\frac{e^{-\delta}}{2}-2e^{-2\delta}+\frac{e^{-3\delta}}{2})+$$
$$\epsilon_1\epsilon_2[-7e^{-\delta}+8e^{-2\delta}+16e^{-4\delta}-17e^{-5\delta}]+\epsilon_2^2\left(\frac{2e^{-\delta}}{9}-\frac{4e^{-4\delta}}{9}+\frac{2e^{-7\delta}}{9}\right),$$

$$u(\delta)=e^{-\delta}\left(+\epsilon_1(e^{-\delta}-e^{-2\delta})\right)+\epsilon_2\left(\frac{e^{-4\delta}}{3}-\frac{e^{-\delta}}{3}\right)+\epsilon_1^2\left(\frac{e^{-\delta}}{2}-2e^{-2\delta}+\frac{3e^{-3\delta}}{2}\right)+\frac{\epsilon_1\epsilon_2}{12}[-7e^{-\delta}+8e^{-2\delta}+16e^{-4\delta}-$$
$$17e^{-5\delta}]+\epsilon_2^2\left(\frac{2e^{-\delta}}{9}-\frac{4e^{-4\delta}}{9}+\frac{2e^{-7\delta}}{9}\right) \tag{2.18}$$

2.4 Homotropy Perturbation Method:

In equation (2.9), $u_a=u_s=0$, the resultant equation become,

$[\frac{du}{d\delta}+u]+[\epsilon_1u\frac{du}{d\delta}+\epsilon_2u^4]=0,$	(2.19)

Now applying homotopy perturbation to equation (2.19),

$$L(u)-L(u_0)+qL(u_0)+qN(u)=0. \tag{2.20}$$

Where $L(u)=\frac{du}{d\delta}+u, L(u_0)=\frac{du_0}{d\delta}+u_0, N(u)=\epsilon_1u\frac{du}{d\delta}+\epsilon_2u^4,$

Here, q is a homotopy parameter. The approximate solution of equation (2.20) can be written as a series of powers of q by using the homotopy perturbation method, i.e.

$$u=v_0+qv_1+q^2v_2 \tag{2.21}$$

This series represents an iterative process where each term represents an approximation of the solution at a specific order. Where, $v_0=u_0=e^{-\delta}$ are the initial approximations.

Putting equation (2.21) into equation(2.20) get the following

$$\frac{d}{d\delta}(v_0+qv_1+q^2v_2)+v_0+qv_1+q^2v_2)-(\frac{du_0}{d\delta}+u_0)+q(\frac{du_0}{d\delta}+u_0)+q(\epsilon_1(v_0+qv_1+q^2v_2)\frac{d}{d\delta}(v_0+qv_1+$$
$$q^2v_2)+\epsilon_2(v_0+qv_1+q^2v_2)^4)=0,$$

Arranging it using coefficient of, $1, q$ and q^2, equating the coefficients of 1:

$$\frac{dv_0}{d\delta}+v_0-\frac{du_0}{d\delta}-u_0=0.$$

Following initial approximation satisfied the above equation.

$$v_0=e^{-\delta}, \tag{2.22}$$

Equating the coefficient q:

$$\frac{dv_1}{d\delta}+v_1+\frac{du_0}{d\delta}+u_0+\epsilon_1v_0\frac{dv_0}{d\delta}+\epsilon_2v_0^4=0, v_1(0)=0,$$

$$\frac{dv_1}{d\delta}+v_1-e^{-\delta}+e^{-\delta}-e^{-2\delta}\epsilon_1+\epsilon_2e^{-4\delta}=0,\frac{dv_1}{d\delta}+v_1-e^{-2\delta}\epsilon_1+\epsilon_2e^{-4\delta}=0,$$

$$\frac{dv_1}{d\delta}+v_1-e^{-2\delta}\epsilon_1+\epsilon_2e^{-4\delta}=0,\frac{dv_1}{d\delta}+v_1=e^{-2\delta}\epsilon_1-\epsilon_2e^{-4\delta},$$

$$d(e^{\delta}v_1)=(e^{-\delta}\epsilon_1-\epsilon_2e^{-3\delta})d\delta, e^{\delta}v_1=\int(e^{-\delta}\epsilon_1-\epsilon_2e^{-3\delta})d\delta+c,$$

$$e^{\delta}v_1=-e^{-\delta}\epsilon_1+\frac{1}{3}\epsilon_2e^{-3\delta}+c, c=\epsilon_1-\frac{1}{3}\epsilon_2,$$

$e^{\delta}v_1 = -e^{-\delta}\epsilon_1 + \frac{1}{3}\epsilon_2 e^{-3\delta} + \epsilon_1 - \frac{1}{3}\epsilon_2, v_1 = -e^{-2\delta}\epsilon_1 + \frac{1}{3}\epsilon_2 e^{-4\delta} + e^{-\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right),$

$$v_1 = e^{-\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - e^{-2\delta}\epsilon_1 + \frac{1}{3}\epsilon_2 e^{-4\delta}, \quad (2.23)$$

Equating the coefficient q^2:

$\frac{dv_2}{d\delta} + v_2 + \epsilon_1 v_0 \frac{dv_1}{d\delta} + \epsilon_1 v_1 \frac{dv_0}{d\delta} + 4\epsilon_2 v_1 v_0^4 = 0, v_2(0) = 0,$

$\frac{dv_2}{d\delta} + v_2 - \epsilon_1 e^{-2\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + 2\epsilon_1^2 e^{-3\delta} - \frac{4}{3}\epsilon_1\epsilon_2 e^{-5\delta} - \epsilon_1 e^{-2\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + e^{-3\delta}\epsilon_1^2 - \frac{1}{3}\epsilon_1\epsilon_2 e^{-5\delta} + 4\epsilon_2 e^{-4\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - 4e^{-5\delta}\epsilon_1\epsilon_2 + \frac{4}{3}\epsilon_2^2 e^{-7\delta} = 0,$

$\frac{dv_2}{d\delta} + v_2 = 2\epsilon_1 e^{-2\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - 3\epsilon_1^2 e^{-3\delta} + \frac{5}{3}\epsilon_1\epsilon_2 - 4\epsilon_2 e^{-4\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + 4e^{-5\delta}\epsilon_1\epsilon_2 - \frac{4}{3}\epsilon_2^2 e^{-7\delta},$

$\int d\left(e^{\delta}v_2\right) = \int\left(2\epsilon_1 e^{-\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - 3\epsilon_1^2 e^{-2\delta} + \frac{5}{3}\epsilon_1\epsilon_2 e^{-4\delta} - 4\epsilon_2 e^{-3\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + 4e^{-4\delta}\epsilon_1\epsilon_2 - \frac{4}{3}\epsilon_2^2 e^{-6\delta}\right)d\delta + c,$

$e^{\delta}v_2 = -2\epsilon_1 e^{-\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \frac{3}{2}\epsilon_1^2 e^{-2\delta} - \frac{5}{12}\epsilon_1\epsilon_2 e^{-4\delta} + \frac{4}{3}\epsilon_2 e^{-3\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - e^{-4\delta}\epsilon_1\epsilon_2 + \frac{4}{18}\epsilon_2^2 e^{-6\delta} + c,$

$0 = -2\epsilon_1\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \frac{3}{2}\epsilon_1^2 - \frac{5}{12}\epsilon_1\epsilon_2 + \frac{4}{3}\epsilon_2\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - \epsilon_1\epsilon_2 + \frac{4}{18}\epsilon_2^2 + c,$

$c = 2\epsilon_1\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - \frac{3}{2}\epsilon_1^2 + \frac{5}{12}\epsilon_1\epsilon_2 - \frac{4}{3}\epsilon_2\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \epsilon_1\epsilon_2 - \frac{4}{18}\epsilon_2^2,$

$e^{\delta}v_2 = -2\epsilon_1 e^{-\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \frac{3}{2}\epsilon_1^2 e^{-2\delta} - \frac{5}{12}\epsilon_1\epsilon_2 e^{-4\delta} + \frac{4}{3}\epsilon_2 e^{-3\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - e^{-4\delta}\epsilon_1\epsilon_2 + \frac{4}{18}\epsilon_2^2 e^{-6\delta} + 2\epsilon_1\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - \frac{3}{2}\epsilon_1^2 + \frac{5}{12}\epsilon_1\epsilon_2 - \frac{4}{3}\epsilon_2\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \epsilon_1\epsilon_2 - \frac{4}{18}\epsilon_2^2,$

$$v_2 = -2\epsilon_1 e^{-2\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \frac{3}{2}\epsilon_1^2 e^{-3\delta} - \frac{5}{12}\epsilon_1\epsilon_2 e^{-5\delta} + \frac{4}{3}\epsilon_2 e^{-4\delta}\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - e^{-5\delta}\epsilon_1\epsilon_2 + \frac{2}{9}\epsilon_2^2 e^{-7\delta} + e^{-\delta}(2\epsilon_1\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) - \frac{3}{2}\epsilon_1^2 + \frac{5}{12}\epsilon_1\epsilon_2 - \frac{4}{3}\epsilon_2\left(\epsilon_1 - \frac{1}{3}\epsilon_2\right) + \epsilon_1\epsilon_2 - \frac{4}{18}\epsilon_2^2), \quad (2.24)$$

Putting v_0, v_1 and v_2 in above series given in equation (2.21) and q approach to 1, get the following form

$$u(\delta) = e^{-\delta}(1 + \epsilon_1(e^{-\delta} - e^{-2\delta})) + \epsilon_2\left(\frac{e^{-4\delta}}{3} - \frac{e^{-\delta}}{3}\right) + \epsilon_1^2\left(\frac{e^{-\delta}}{2} - 2e^{-2\delta} + \frac{3e^{-3\delta}}{2}\right) + \frac{\epsilon_1\epsilon_2}{12}[-7e^{-\delta} + 8e^{-2\delta} + 16e^{-4\delta} - 17e^{-5\delta}] + \epsilon_2^2\left(\frac{2e^{-\delta}}{9} - \frac{4e^{-4\delta}}{9} + \frac{2e^{-7\delta}}{9}\right) \quad (2.25)$$

This is the required solution of equation(2.10) gives the behavior of heat transfer. That is the same as obtained by Ganji [8] in his problem.

2.5 Homotopy Analysis Method

The answer can be described by a collection of base functions using the unstable nonlinear convection-radiation equation and the initial condition given in equation (2.9).

$\{e^{-n\delta} \,|\, n = 1,2,3,\ldots.\},$

such as

$$u(\delta) = \sum_{n=1}^{\infty} a_n e^{-n\delta}, \quad (2.26)$$

where a_n is the unknown coefficients to be found out later. It indicates that the solution of equation (2.10) must be expressed similarly to equation (2.26) and their alternative forms such as $\delta^m e^{-n\delta}$ must be ignored. Using the HAM [7], the auxiliary linear form is written as

$$\mathcal{L}[(\delta;\, p)] = \frac{\partial\phi(\delta;\, p)}{\partial\delta} + \phi(\delta;\, p), \quad (2.27)$$

with property

$$\mathcal{L}\,[c_1 e^{-\delta}] = 0, \quad (2.28)$$

and defining the non-linear operator as

$$\mathbb{N}[(\delta; p)] = \left(1 + \epsilon_1\phi(\delta; p)\right)\frac{\partial\phi(\delta;\, p)}{\partial\delta} + (\delta;\, p) + \epsilon_2\phi^4(\delta;\, p), \quad (2.29)$$

Choose $u_0(\delta) = e^{-\delta}$ is the simplicity initial approximation of $u(\delta)$. Clearly, $u_0(\delta) = e^{-\delta}$ automatically satisfies the boundary condition given in equation (2.9). Therefore, the zero-order deformation of general equation (2.2) and the related limit conditions also expressed as:

$\phi(0;\, p) = 1,$

From equation (2.4) and (2.29) following is obtained

$$R_m(u_{m-1}(\delta) = u'_{m-1}(\delta)+\epsilon_1 \sum_{n=0}^{m-1} u_n(\delta)\, u'_{m-1-n}(\delta)+u_{m-1}(\delta) + \epsilon_2 \sum_{n=0}^{m-1}(\sum_{i=0}^{n} u_i(\delta)\, u_{m-i}(\delta))(\sum_{j=0}^{m-1-n} u_j(\delta)\, u_{m-1-n-j}(\delta)), \quad (2.30)$$

where, prime represents the differentiation with respect to δ. The solution of the mth-order deformation equation (2.3) for $m \geq 1$ becomes:

$$u_m(\delta) = \chi_m u_{m-1}(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\delta} H(\xi) R_m(u_{m-1}(\xi)\, d\xi + c_1 e^{-\delta}, \quad (2.31)$$

Integration constant c_1 can be determined by using initial condition given in equation (2.9), $H(\delta) = e^{-k\delta}$ is the auxiliary parameter and k representsthe integer. For $k = 0$, the auxiliary parameter $H(\delta) = 1$, according to solution expression in equation (2.3).

For m = 1, the equation (2.31) becomes:

$$u_m(\delta) = \chi_m\, u_{m-1}(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\xi} H(\xi) R_m(u_{m-1}(\xi))\, d\xi + c_1 e^{-\delta},$$

$$u_1(\delta) = (0)\, u_{1-1}(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\xi} (1) R_1(u_{1-1}(\xi)\, d\xi + c_1 e^{-\delta},$$

$$u_1(\delta) = \hbar e^{-\delta} \int_0^{\delta} e^{\xi} (u'_0(\xi) + \epsilon_1 u_0(\xi) u'_0(\xi) + u_0(\xi) + \epsilon_2 u_0^4(\xi))\, d\xi + c_1 e^{-\delta},$$

$$u_1(\delta) = \hbar e^{-\delta} \int_0^{\delta} e^{\xi} (-e^{-\xi} + \epsilon_1 (-e^{-2\xi}) + e^{-\xi} + \epsilon_2 e^{-4\xi})\, d\xi + c_1 e^{-\delta},$$

$$u_1(\delta) = \hbar e^{-\delta} \int_0^{\delta} (-\epsilon_1 e^{-\xi} + \epsilon_2 e^{-3\xi})\, d\xi + c_1 e^{-\delta},$$

$$u_1(\delta) = \hbar e^{-\delta} \left(\epsilon_1 e^{-\delta} - \frac{\epsilon_2 e^{-3\delta}}{3} - (\epsilon_1 - \frac{\epsilon_2}{3}) \right) + c_1 e^{-\delta},\ u_1(0) = 1,$$

$$u_1(\delta) = \hbar e^{-\delta} \left(\epsilon_1 e^{-\delta} - \frac{\epsilon_2 e^{-3\delta}}{3} - (\epsilon_1 - \frac{\epsilon_2}{3}) \right) + e^{-\delta},$$

$$u_1(\delta) = (\hbar\epsilon_1) e^{-2\delta} - \frac{\hbar\epsilon_2 e^{-4\delta}}{3} - \hbar e^{-\delta} (\epsilon_1 - \frac{\epsilon_2}{3}) + e^{-\delta},$$

$$u_1(\delta) = e^{-\delta}(1 - \hbar (\epsilon_1 - \frac{\epsilon_2}{3})) + e^{-2\delta}((\hbar\epsilon_1) - e^{-4\delta}(\frac{\hbar\epsilon_2}{3}), \quad (2.32)$$

and m = 2, get the equation (2.24) in following form

$$u_2(\delta) = (1)\, u_{2-1}(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\xi} (1) R_2(u_{2-1}(\xi))\, d\xi + c_1 e^{-\delta},$$

$$u_2(\delta) = (1)\, u_1(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\xi} R_2(u_1(\xi)))\, d\xi + c_1 e^{-\delta},$$

$$u_2(\delta) = (1)\, u_1(\delta) + \hbar e^{-\delta} \int_0^{\delta} e^{\xi} R_2(u_1(\xi)))\, d\xi + c_1 e^{-\delta},$$

$$u_2(\delta) = e^{-\delta}(1-\hbar)(2+\hbar)(\epsilon_1 - \frac{\epsilon_2}{3}) + \hbar^2(\frac{\epsilon_1^2}{2} - \frac{7}{12}\epsilon_1\epsilon_2 + \frac{2}{9}\epsilon_2^2)) + e^{-2\delta}(\epsilon_1 (2\hbar + \hbar^2) - 2\hbar^2 \epsilon_1(\epsilon_1 - \frac{\epsilon_2}{3})) + e^{-3\delta}(\frac{3}{2}\epsilon_1^2\hbar^2) + e^{-4\delta}(\frac{\epsilon_2}{3}(-2\hbar - \hbar^2) + \frac{4}{3}\epsilon_2\hbar^2(\epsilon_1 - \frac{\epsilon_2}{3})) - e^{-5\delta}(\frac{17}{12}\epsilon_1\epsilon_2\hbar^2) + e^{-7\delta}(\frac{2}{9}\epsilon_2^2\hbar^2), \quad (2.33)$$

putting $\hbar = -1$ into equation (2.33) obtained following relation

$$u(\delta) = e^{-\delta}(1 + \epsilon_1(e^{-\delta} - e^{-2\delta})) + \epsilon_2\left(\frac{e^{-4\delta}}{3} - \frac{e^{-\delta}}{3}\right) + \epsilon_1^2\left(\frac{e^{-\delta}}{2} - 2e^{-2\delta} + \frac{3e^{-3\delta}}{2}\right) + \frac{\epsilon_1\epsilon_2}{12}[-7e^{-\delta} + 8e^{-2\delta} + 16e^{-4\delta} - 17e^{-5\delta}] + \epsilon_2^2\left(\frac{2e^{-\delta}}{9} - \frac{4e^{-4\delta}}{9} + \frac{2e^{-7\delta}}{9}\right) \quad (2.34)$$

The similar solution given in equation (2.34) has also been yielded by using the Homotopy perturbation method and the perturbation method. Hence, the mth-order approximation $u(\delta)$ can be generally expressed by

$$u(\delta) = \sum_{n=1}^{3m+1} a_{m,n}(\hbar) e^{-n\delta}, \quad (2.35)$$

Where $a_{m,n}$ is a coefficient that depends on ħ. Equation (2.35) is a family of solution expression in the auxiliary parameter ħ. The homotopy analysis method includes an auxiliary parameter ħ that provides an easy way to adjust and control the convergence region of the solution series. First we plot the ħ − curves $u'(0)$ and $u''(0)$ as shown in Figure 1. The valid range of ħ can be easily explored for different values of ϵ_1 and ϵ_2.

Here, equation (2.9) is solved using Mathematica software and is also used to calculate the error. We can see that the optimal value of ħ is not −1, but depends on ϵ_1 and ϵ_2. In Figure 2, we can see that it is better to use other values for ħ than −1, especially when ϵ_1 and ϵ_2 are large. In Figure 3, we compared the approximate solution of order 5 with the exact solution in $\delta = 1$ for different values of ħ where $\epsilon_1 = \epsilon_2 = \epsilon$. Briefly speaking Homotopy Analysis Method's ability to modify and manage the convergence zone and rate of approximation series makes it a valuable technique in handling nonlinear problems, providing more control and flexibility compared to some other traditional analytical and numerical methods.

Clearly, $u'(0) = -\frac{1+\epsilon_2}{1-\epsilon_1}$, the homotopy analysis method solution using exactly this solution shows accuracy for large values of ϵ_1 and ϵ_2, see Figure 4 when $2\epsilon_1 = \epsilon_2 = \epsilon$. The perturbation method and the homotopy perturbation method are invalid for large values of ϵ_1 and ϵ_2. The perturbation method and the homotopy perturbation method are both asymptotic methods, which means that they are only valid for small values of the perturbation parameters. For large values of the perturbation parameters, the asymptotic expansions become inaccurate and the methods break down. Figure 5 and 6 show the error at $\delta = 1$ and $\delta = 2$, respectively, with respect to ħ for various ϵ_1 and ϵ_2.

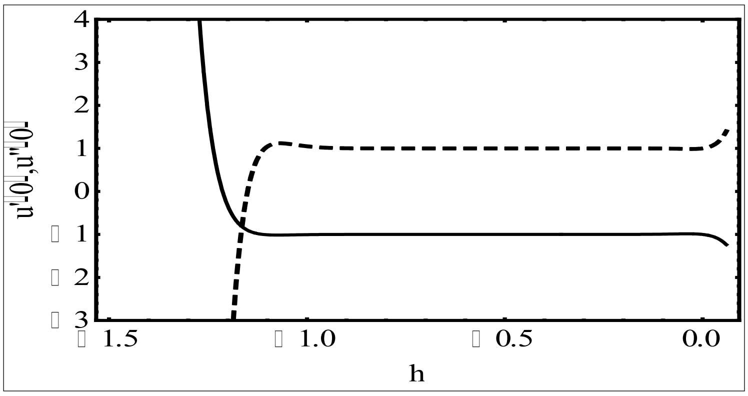

Figure 2.1: ħ- curve obtain by using equation (2.28) for m=15,δ=0,ϵ_1= 0.7 and ϵ_2=0.3. The dotted line represents the 15th order approximation of u''(0) and the solid line represents the 15th order approximation of u^' (0)

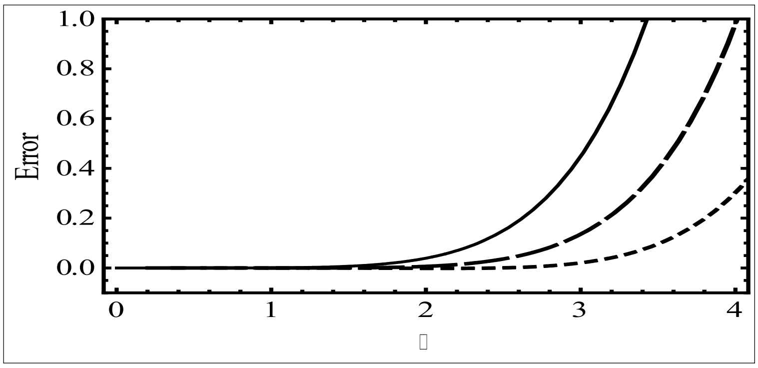

Figure 2.2: 5th-order approximation of solution error of homotopy analysis method at $\delta = 1$, $\epsilon_1 = \epsilon_2 = \epsilon$ against different value of ħ, solid line plotted at ħ $= -1$, dotted line at ħ $= -0.8$ and dashed line at ħ $= -0.9$

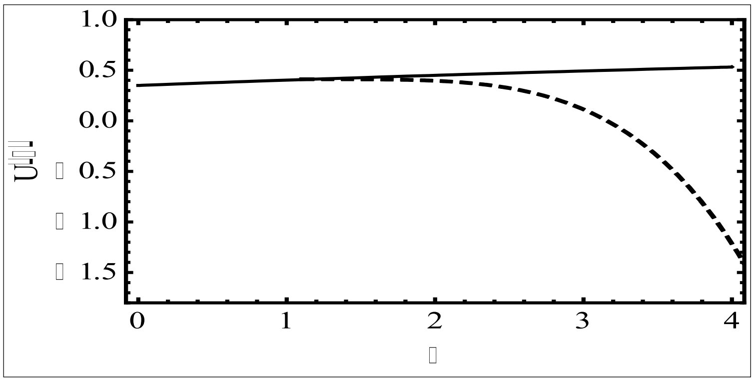

Figure 2.3: The obtained solution by HAM, solid line: ħ $= -0.7$; dashed line: 5th order HPM solution

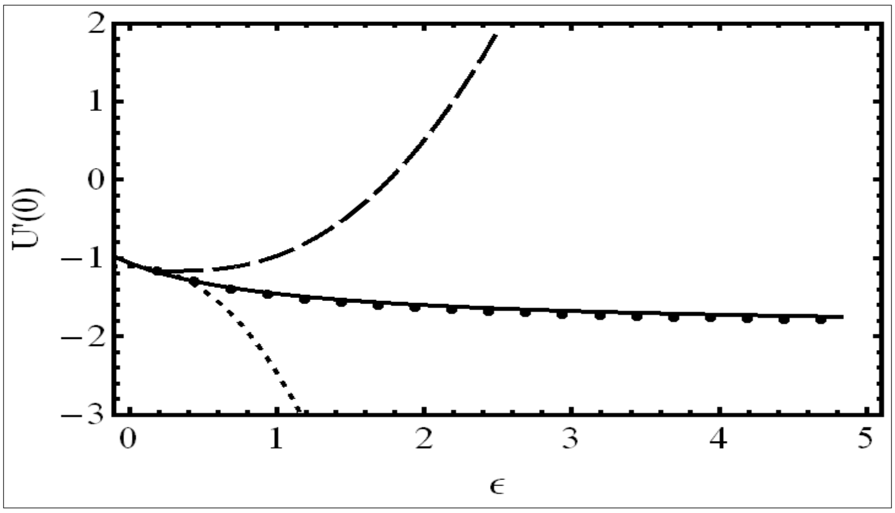

Figure 2.4: The graph of $u'(0)$ with respect to ϵ obtained for $\hbar = \frac{-1}{1+\epsilon}$, bold circle:exact values; solid line: 1st–order homotopy analysis method; dotted line: second order homotopy perturbation method; dashed line: 5th order homotopy perturbation method

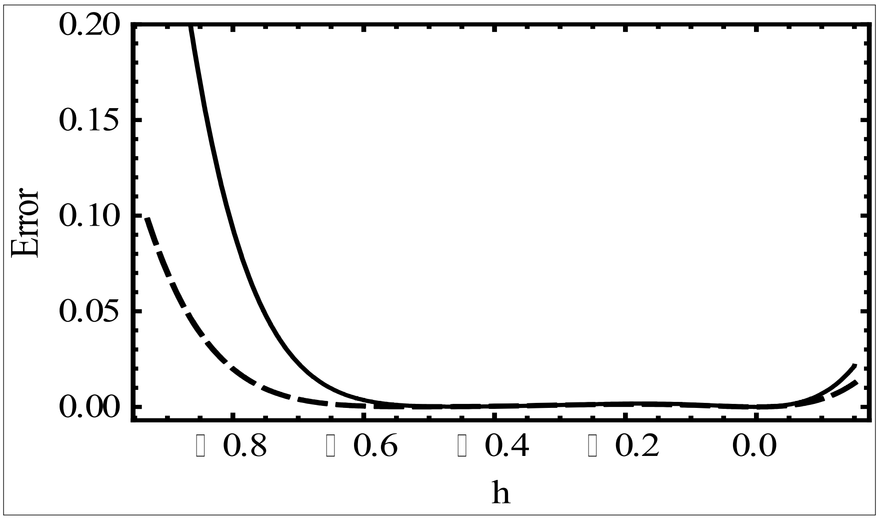

Figure 2.5: Error of 5th-order homotopy analysis method solution at $\delta = 1$, solid line; $\epsilon_1 = 1$ and $\epsilon_2 = 2$; dashed line: $\epsilon_1 = 0.8$ and $\epsilon_2 = 1.6$

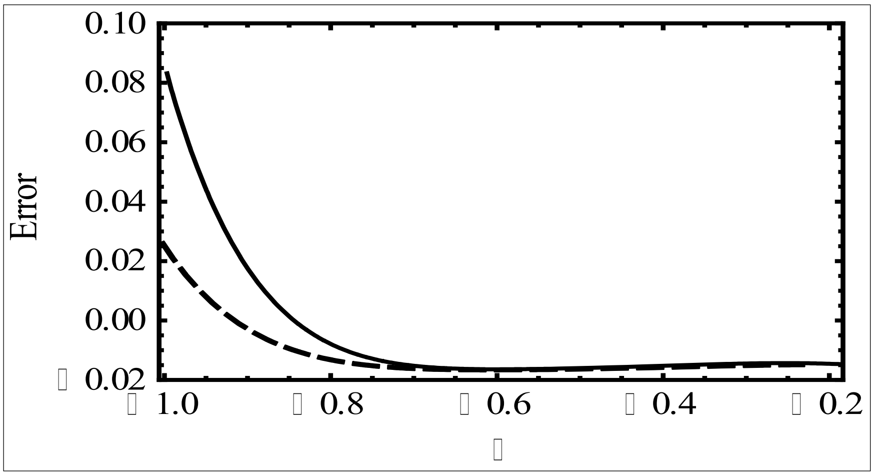

Figure 2.6: Error of 5th-order homotopy analysis method solution at $\delta = 2$, solid line; $\epsilon_1 = 1$ and $\epsilon_2 = 2$; dashed line: $\epsilon_1 = 0.8$ and $\epsilon_2 = 1.6$

2.6 Conclusions

The homotopy analysis method HAM [11], is used in this paper to solve the heat radiation equations. HAM has been successfully applied to solve various types of nonlinear problems, including ordinary differential equations and partial differential equations. With the homotopy analysis method, we can quickly change how approximation series converge; the homotopy analysis method and other approaches have a basic qualitative difference in analysis. It has also been demonstrated that the homotopy perturbation technique and the perturbation method are only applicable for small values, but the homotopy analysis approach allows us to select h in a suitable way. The outcomes demonstrate the usefulness and great potential of the HAM for nonlinear problems in science and engineering.

3. A New Application of the Homotopy Analysis Method: Solving the Sturm-Liouville Problems

The purpose of this chapter is to create a numerical technique for determining the eigenvalue and corresponding eigenfunction of Sturm-Liouville Problems [11]. Equations of this nature commonly arise in engineering applications, particularly in vibrating elastic stability problems. The Sturm-Liouville problem is a fundamental topic in mathematical physics that arises in a variety of scientific and engineering fields. It is concerned with the investigation of eigenvalue problems for second-order ordinary differential equations with specific boundary conditions. To address such problems, the Homotopy Analysis Method can be extended to compute eigenvalues and eigenfunction associated with second-order and fourth-order Sturm-Liouville problems. Many straight plateaus come about; each one corresponds to a Sturm-Liouville problem of eigenvalue. This extension involves utilizing a modified linear operator L and an initial guess to facilitate the calculation of these eigenvalues. These eigenvalues are unique; they are multiple solutions. The HAM uses the auxiliary parameter ħ to adjust how quickly the approximation series solution converges. The results show that this method is more valid, has higher accuracy, and requires less iteration.

3.1 Basic Concept of the Homotopy Analysis Method

3.1.1 Zero-Order Deformation Equation

This chapter investigated the following nonlinear systems to investigate the concept of the homotopy analysis method:

$\mathbb{N}[\mathrm{v}(\delta)] = 0,$	(3.1)

ℕ Stands for the nonlinear operator, $\mathrm{v}(\delta)$ is the undefined function and δ is the independent variable. To broaden the scope of the conventional HAM, Shijun Liao invented the idea of the zero-order deformation equation, is formulated as:

$(1 - p)\mathcal{L}[\psi(\delta; p) - \mathrm{v}_0(\delta)] = p\hbar \mathrm{H}(\delta)\mathbb{N}[\psi(\delta; p)],$	(3.2)

3.1.2 Mth-Order Deformation Equation

By differentiating the zero-order deformation equation m times with respect to p at $p = 0$, and then dividing it by m!, the mth-order deformation equation can be derived from the zero-order deformation equation. As a result, the following occurs:

$$\mathcal{L}\,[\mathrm{v_m}(\delta) - \mathrm{v_{m-1}}(\delta)] = \hbar \mathrm{R_m}(\mathrm{v_{m-1}}), \tag{3.3}$$

where

$\mathrm{R_m}(\vec{\mathrm{v}}_{\mathrm{m}-1}) = \dfrac{1}{(\mathrm{m}-1)!} \dfrac{\partial^{\mathrm{m}-1}\mathbb{N}[\psi(\delta;\, p)\,]}{\partial p^{\mathrm{m}-1}} \Big\vert_{p=0},$	(3.4)

3.2 Second order Sturm-Liouville Problems

The general form of a second order Sturm-Liouville problem [10] can be written in the following equation:

$$\frac{d}{dx}\left\{f(x)\frac{dy}{dx}\right\} + g(x)y = \lambda \mathrm{v}(x)\mathrm{y}.$$

$f(x), g(x), f'(x)$ and $\mathrm{v}(x)$ are positive and continue on the close interval [a, b], λ is the parameter it does not depend on x.

Example 3.2.1 Regular eigenvalue problem [11]

$$\mathrm{y}''(x) + \lambda \mathrm{y}(x) = 0.\, x \in (0,1), \tag{3.5}$$

having the boundary conditions

$y'(0) = 0,$	(3.6a)
$\mathrm{y}(1) = 0.$	(3.6b)

Suppose that the solution of the above equation (3.5) may be represented by a collection of base functions given as

$\mathrm{v}(x) = \displaystyle\sum_{\mathrm{i}=0}^{\infty} \mathrm{d_i}\, x^{2\mathrm{i}},$	(3.7a)

where, d_i are the unknown coefficients that will be find later. By using HAM, choose auxiliary linear operator such as:

$$\mathcal{L}[\phi(x;\, p)] \;=\; \phi''(x;\, p),$$

and defining the non-linear operator such as:

$\mathcal{N}\; [\phi(x;\, p)] \;=\; \phi''(\, x;\, p), +\lambda\phi(x;\, p).$	(3.7b)

Choose $v_0(x) = 1$ for the simplicity initial approximation of $v(x)$. Clearly, $v_0(x) \; = 1$ automatically satisfies the boundary condition given in equation ((3.6) i.e. $v'(0) = \; 0)$. The general equation (3.2) and related limit conditions also expressed as:

$(1 \,-\, p)\mathcal{L}[\phi(x;\, p)) - v_0(x)] \;=\; p\,\hbar\mathcal{N}[\phi(x;\, p)],$	
$\phi'(0;\, p) \;=\; 0.$	(3.8)

From equations (3.4) and (3.7b) get the following:

$$R_m(v_{m-1}(x)) \;=\; v''_{m-1}(x) \,+\, \lambda(v_{m-1}(x)), \tag{3.9}$$

where, ($''$) prime represents differentiation with respect to x. The mth-order deformation equation (3.3) for $m \geq 1$ is becomes:

$$v_m(x) = \; \chi_m v_{m-1}(x) \,+\, \hbar\int_0^x\int_0^\eta R_m\,(v_{m-1}(\tau))d\tau d\eta. \tag{3.10}$$

Putting $m = 1$ from above equation (3.9) and (3.10) and setting $\lambda = \; a$:

$$v_1(x) = \chi_1(v_{1-1}(x)) + \;\hbar\int_0^x\int_0^\eta R_1\,(v_{1-1}(\tau))\, d\tau d\eta,$$

$$v_1(x) = (0)(v_0(x)) + \;\hbar\int_0^x\int_0^\eta R_1\,(v_0(\tau))\, d\tau d\eta,$$

$$v_1(x) = \;\hbar\int_0^x\int_0^\eta R_1\,(v_0(\tau))\, d\tau d\eta,$$

$$v_1(x) \;=\; \hbar\int_0^x\int_0^\eta (v''_0(\tau) + \; a(v_0(\tau)))\, d\tau d\eta,$$

$$v_1(x) \;=\; \hbar\int_0^x\int_0^\eta \; a(v_0(\tau))\, d\tau d\eta, v_1(x) = \; a\hbar\int_0^x\int_0^\eta \;(1)\, d\tau d\eta,$$

$$v_1(x) = \hbar a\int_0^x \eta\, d\eta, v_1(x) = \hbar a\left(\frac{x^2}{2}\right),$$

$$v_1(x) \;=\; \frac{1}{2}a\hbar x^2, \tag{3.11}$$

For $m \;=\; 2$

$$v_2(x) = \chi_2(v_{2-1}(x)) + \;\hbar\int_0^x\int_0^\eta R_2\,(v_{2-1}(\tau))\, d\tau d\eta,$$

$$v_2(x) = (1)(v_1(x)) + \;\hbar\int_0^x\int_0^\eta R_2\,(v_1(\tau))\, d\tau d\eta,$$

$$v_2(x) = \frac{1}{2}a\,\hbar x^2 \;+\; \hbar\int_0^x\int_0^\eta (v''_1(\tau) + a(v_1(\tau)))\, d\tau d\eta,$$

$$v_2(x) = \;\frac{1}{2}a\,\hbar x^2 \;+\hbar\int_0^x\int_0^\eta \left(a\hbar + a\left(\frac{1}{2}a\hbar\tau^2\right)\right)d\tau d\eta,$$

$$v_2(x) \;=\; \frac{1}{2}a\hbar x^2 \;+ a\hbar^2\int_0^x\int_0^\eta (1 \;+\; a(\frac{1}{2}\tau^2))\, d\tau d\eta,$$

$$v_2(x) \;=\; \frac{1}{2}a\,\hbar x^2 + \frac{1}{2}a\hbar^2 x^2 \;+\; \frac{1}{24}a^2\hbar^2 x^4, \tag{3.12}$$

For $m \;=\; 3$

$$v_3(x) = \chi_3(v_{3-1}(x)) + \;\hbar\int_0^x\int_0^\eta R_3\,(v_{3-1}(\tau))\, d\tau d\eta,$$

$$v_3(x) = (1)(v_2(x)) + \;\hbar\int_0^x\int_0^\eta R_3\,(v_2(\tau))\, d\tau d\eta,$$

$$v_3(x) = v_2(x) + \int_0^x\int_0^\eta (a\hbar + \; a\hbar^2 + \frac{1}{2}a^2\hbar^2\tau^2 + \; a((\frac{1}{2}a\hbar \;+\; \frac{1}{2}a\hbar^2)\;\tau^2 + \frac{1}{24}a^2\hbar^2\tau^4)d\tau d\eta,$$

$$v_3(x) = (\frac{a\hbar^3}{2} + \; a\hbar^2 + \frac{a\hbar}{2})x^2 + (\frac{1}{12}a^2\hbar^2 + \; \frac{1}{12}a^2\hbar^2)x^4 + \frac{1}{720}a^3\hbar^3 x^6, \tag{3.13}$$

The given below series solution of equation (3.10) from m = 1 until the order m = 4

$v_1(x) = \frac{1}{2}a\hbar x^2,$

$v_2(x) = \frac{1}{2}a\,\hbar\, x^2 + \frac{1}{2}a\hbar^2 x^2 + \frac{1}{24}a^2\hbar^2 x^4,$

$v_3(x) = (\frac{a\hbar^3}{2} + a\hbar^2 + \frac{a\hbar}{2})x^2 + (\frac{1}{12}a^2\hbar^2 + \frac{1}{12}a^2\hbar^2)x^4 + \frac{1}{720}a^3\hbar^3 x^6,$

$v_4(x) = \frac{1}{2}a\hbar x^2 + \frac{3}{2}a\hbar^2 x^2 + \frac{3}{2}a\hbar^3 x^2 + \frac{1}{2}a\hbar^4 x^2 + \frac{1}{8}a^2\hbar^2 x^4 + \frac{1}{4}a^2\hbar^3 x^4 + \frac{1}{8}a^2\hbar^4 x^4 + \frac{1}{240}a^3\hbar^3 x^6 + \frac{1}{240}a^3\hbar^4 x^6 + \frac{a^4\hbar^4 x^8}{40320},$

As a result, the series solution of the mth-order approximation homotopy analysis method $u_m(x)$, takes the following form:

$$u_m(x) = \sum_{i=0}^{m} v_i(x). \quad (3.14)$$

To the mth-order approximation solution (3.14), which still depends on the eigenvalue λ and auxiliary parameter ħ, condition (3,6b) reads

$$u_m(1) \approx U_m(1; \lambda, \hbar) = 1. \quad (3.15)$$

The relationship between the λ and ħ is shown in equation (3.15). Equation (3.15) is still dependent on the auxiliary variable ħ and the eigenvalue λ. Figure 3.2.1(a) shows the profile of λ against ħ with range [-2, 0] plotted in accordance with the equation (3.15) for m = 25. Eigenvalue λ can be determined by using equation (3.15) for m = 25 by setting $\hbar = -0.9$ given in table(3.2.1).

Table No 3.2.1:

k	λ_k
1	2.4674011
2	22.2066099
3	61.6850275
4	120.9026680
5	199.8563351
6	298.3136496

Figures 3.2.1(b − d) show the first three distinct λ-plateaus. Figure 3.2.1(e) shows the estimated eigenfunction for the $\lambda_3 = 61.6850275$, which was obtained against $\hbar = -0.9$, m = 25 according to equation (3.5). Figure3.2.1(f) shows the eigenfunction error corresponding to the third eigenvalue $\lambda_3 = 61.6850275$, m = 25.

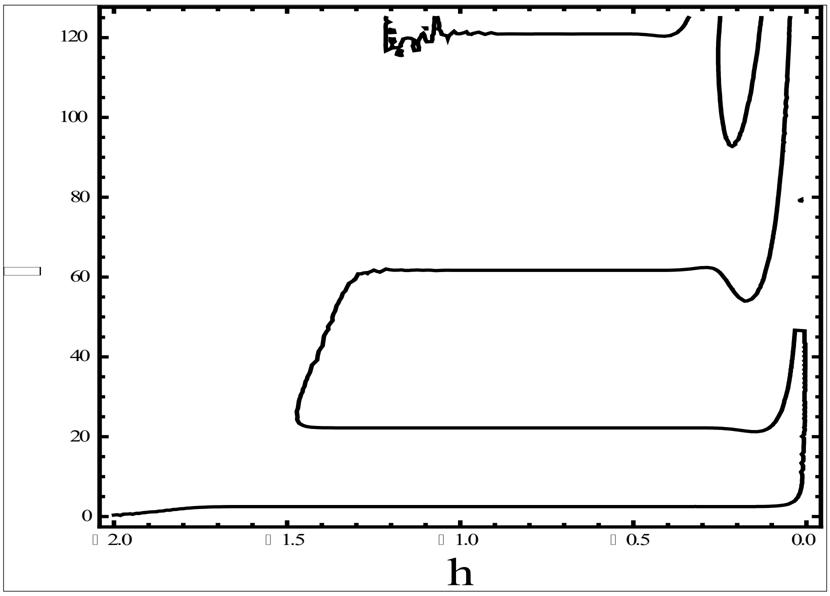

Figure 3.2.1(a): According to Equation (3.15), the ħ -curve is constructed with a range of λ values from 2 to 125 and a range of ħ values from −2 to 0. This construction is specifically for the case when m is set to 25

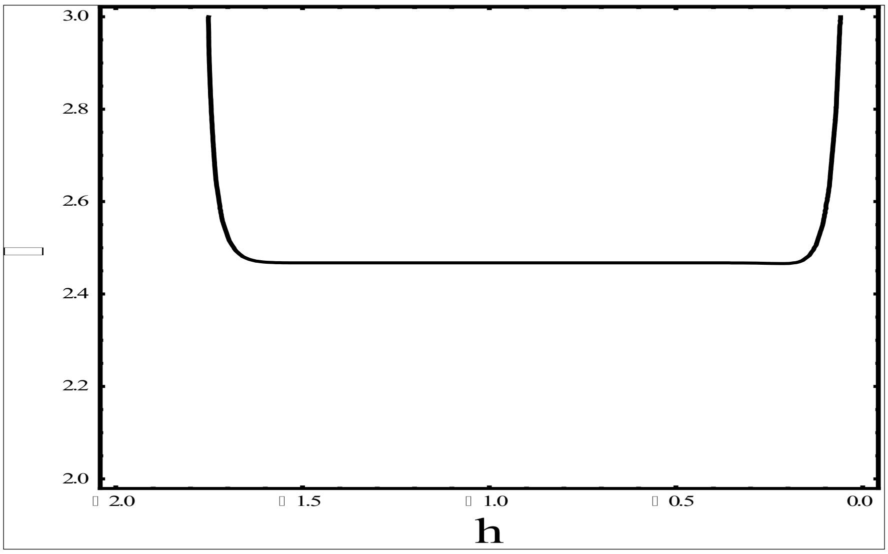

Figure 3.2.1(b): λ –plateau corresponding to λ_1= 2.4674

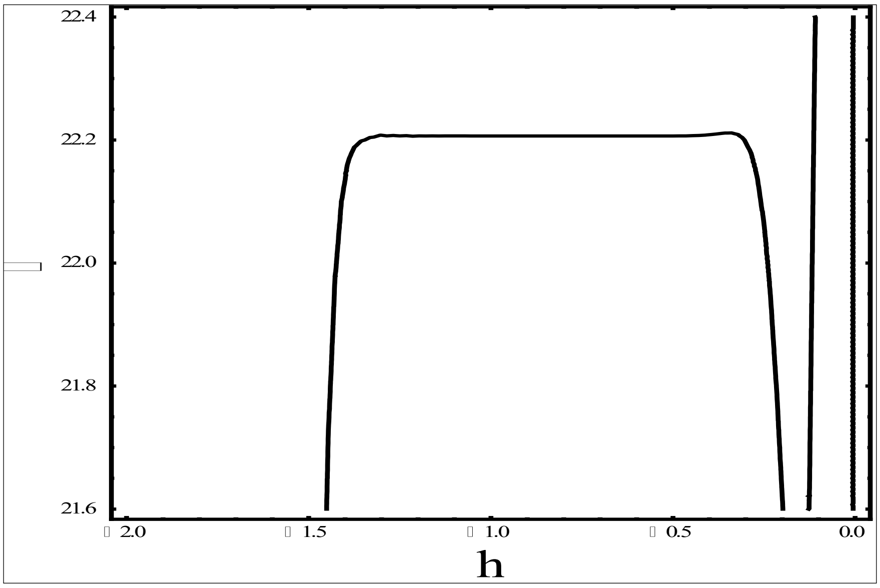

Figure 3.2.1(c): λ –plateau corresponding to λ_2= 22.2066

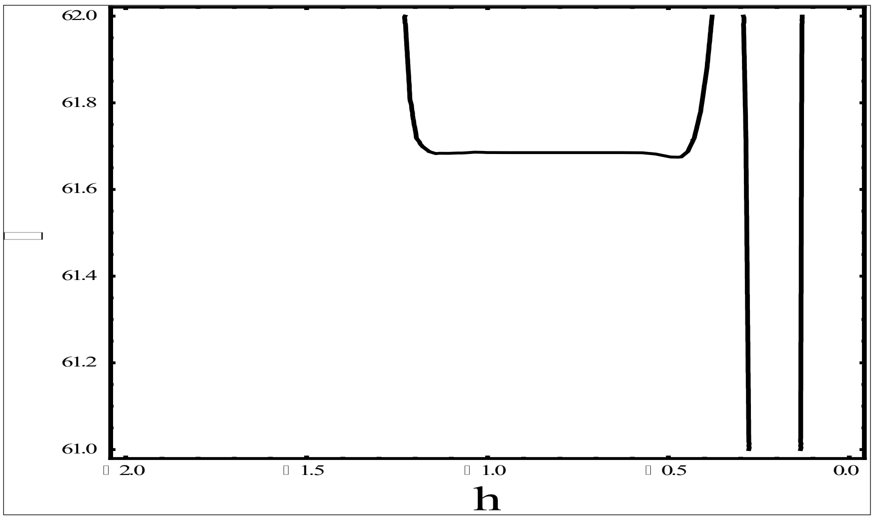

Figure 3.2.1(d): λ –plateau corresponding to λ_3= 61.6850

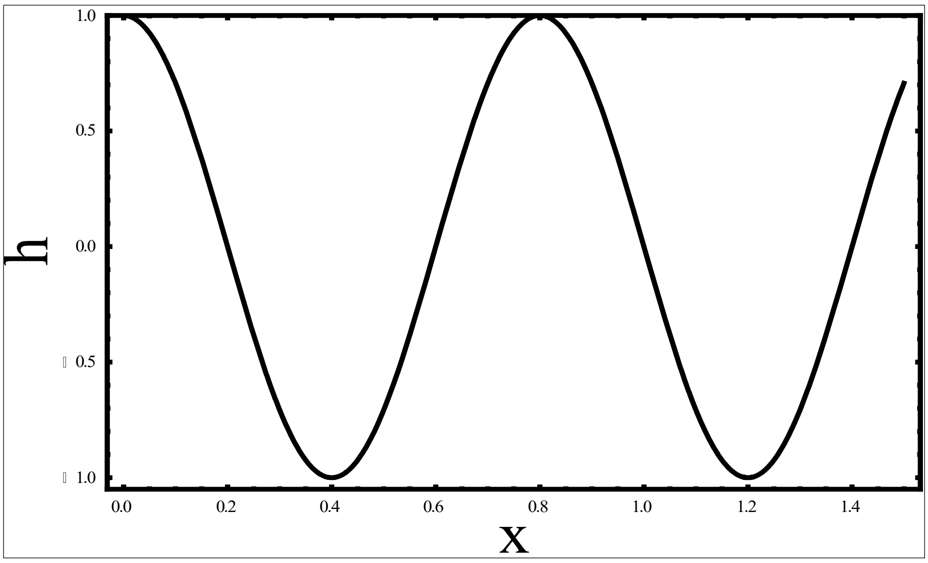

Figure 3.2.1(e): Eigenfunction corresponding to λ_3= 61.6850

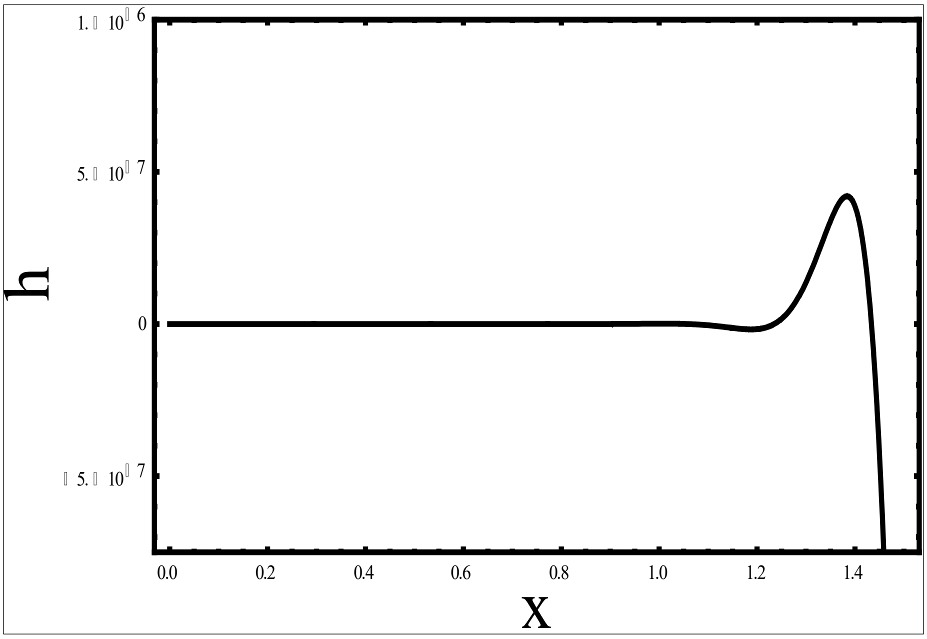

Figure 3.2.1(f): Eigenfunction error corresponding to λ_3= 61.6850

Example 3.2.2 Singular eigenvalue problem [11].

$$y''(x) + (\frac{1}{x} + \lambda)y(x) = 0, x \in (0,1), \quad (3.16)$$

The boundary conditions are

$$y(0) = 0, \quad (3.17)$$

$$y(1) = 0. \quad (3.18)$$

Suppose that the solution of the above equation (3.16) may be represented by a collection of base functions.

$$v(x) = \sum_{i=0}^{\infty} d_i x^{i}, \quad (3.19)$$

Where d_i are the unknown coefficients to be found out later. By using HAM technique we select auxiliary linear operator as following

$$\mathcal{L}[\phi(x;p)] = \phi''(x;p).$$

With the non-linear operator being defined as follows

$N[\phi(x;p)] = \phi''(x;p) + (\frac{1}{x} + \lambda)\phi(x;p),$	(3.20)

Choose $v_0(x) = x$ for the simplicity of the initial approximation of $v(x)$. Clearly, $v'_0(x) = 1$ automatically satisfies boundary condition given in equation (3.17). The general equation (3.2) and the related limit conditions, also expressed as:

$$(1-p)\mathcal{L}[\phi(x;p) - v_0(x)] = p\hbar N[\phi(x;p)], \quad (3.21)$$

$$\phi(0;p) = 0.$$

Putting equation (3.20) into equation (3.2) the residual function is written as

$R_m(v_{m-1}(x)) = v''_{m-1}(x) + (\frac{1}{x} + \lambda)(v_{m-1}(x)),$	(3.22)

where, the prime $('')$indicates the differentiation with respect to x, the mth-order deformation equation (3.3) for $m \geq 1$ now becomes

$v_m(x) = \chi_m(v_{m-1}(x)) + \hbar\int_0^x\int_0^\eta R_m(v_{m-1}(\tau))d\tau d\eta.$	(3.23)

Putting m =1 from above equation (3.22) and equation (3.23) and using $\lambda = a$:

$$v_1(x) = \chi_1(v_{1-1}(x)) + \hbar\int_0^x\int_0^\eta R_1(v_{1-1}(\tau))d\tau d\eta,$$

$$v_1(x) = (0)(v_0(x)) + \hbar\int_0^x\int_0^\eta R_1(v_0(\tau))d\tau d\eta,$$

$$v_1(x) = \hbar\int_0^x\int_0^\eta (v''_0(\tau) + (\frac{1}{\tau} + a)(v_0(\tau)))\, d\tau d\eta,$$

$$v_1(x) = \hbar\int_0^x\int_0^\eta (0 + (\frac{1}{\tau} + a)(\tau))d\tau d\eta,$$

$$v_1(x) = \hbar\int_0^x\int_0^\eta (1 + a\tau)d\tau d\eta,\ v_1(x) = \frac{\hbar x^2}{2} + \frac{a\hbar x^3}{3},$$

For $m = 2$

$$v_2(x) = \chi_2(v_{2-1}(x)) + \hbar\int_0^x\int_0^\eta R_2(v_{2-1}(\tau))d\tau d\eta,$$

$$v_2(x) = (1)(v_1(x)) + \hbar\int_0^x\int_0^\eta R_2(v_1(\tau))d\tau d\eta,$$

$$v_2(x) = \frac{\hbar x^2}{2} + \frac{a\hbar x^3}{3} + \hbar\int_0^x\int_0^\eta (v''_1(\tau) + (\frac{1}{\tau} + a)(v_1(\tau)))d\tau d\eta,$$

$$v_2(x) = \frac{\hbar x^2}{2} + \frac{\hbar^2 x^2}{2} + \frac{a\hbar x^3}{6} + \frac{a\hbar^2 x^3}{6} + \frac{\hbar^2 x^3}{6} + \frac{a\hbar^2 x^4}{18} + \frac{a^2\hbar^2 x^5}{120},$$

For $m = 3$

$$v_3(x) = \chi_3(v_{3-1}(x)) + \hbar\int_0^x\int_0^\eta R_3(v_{3-1}(\tau))d\tau d\eta,$$

$$v_3(x) = (1)(v_2(x)) + \hbar\int_0^x\int_0^\eta R_3(v_2(\tau))d\tau d\eta,$$

$$v_3(x) = \frac{\hbar x^2}{2} + \hbar^2 x^2 + \frac{\hbar^3 x^2}{2} + \frac{1}{6}a\hbar x^3 + \frac{\hbar^2 x^3}{6} + \frac{1}{3}a\hbar^2 x^3 + \frac{\hbar^3 x^3}{6} + \frac{1}{6}a\hbar^3 x^3 + \frac{1}{9}a\hbar^2 x^4 + \frac{\hbar^3 x^4}{144} + \frac{1}{9}a\hbar^3 x^4 + \frac{1}{60}a^2\hbar^2 x^5 + \frac{1}{144}a\hbar^3 x^5 + \frac{1}{60}a^2\hbar^3 x^5 + \frac{23a^2\hbar^3 x^6}{10800} + \frac{a^3\hbar^3 x^7}{5040},$$

The below series solution of the equation (3.23) for m = 1, 2, 3.

$$v_1(x) = \frac{\hbar x^2}{2} + \frac{a\hbar x^3}{3},$$

$$v_2(x) = \frac{\hbar x^2}{2} + \frac{\hbar^2 x^2}{2} + \frac{a\hbar x^3}{6} + \frac{a\hbar^2 x^3}{6} + \frac{\hbar^2 x^3}{6} + \frac{a\hbar^2 x^4}{18} + \frac{a^2\hbar^2 x^5}{120},$$

$$v_3(x) = \frac{\hbar x^2}{2} + \hbar^2 x^2 + \frac{\hbar^3 x^2}{2} + \frac{1}{6}a\hbar x^3 + \frac{\hbar^2 x^3}{6} + \frac{1}{3}a\hbar^2 x^3 + \frac{\hbar^3 x^3}{6} + \frac{1}{6}a\hbar^3 x^3 + \frac{1}{9}a\hbar^2 x^4 + \frac{\hbar^3 x^4}{144} + \frac{1}{9}a\hbar^3 x^4 + \frac{1}{60}a^2\hbar^2 x^5 + \frac{1}{144}a\hbar^3 x^5 + \frac{1}{60}a^2\hbar^3 x^5 + \frac{23a^2\hbar^3 x^6}{10800} + \frac{a^3\hbar^3 x^7}{5040},$$

To the mth-order approximation solution (3.24), which still depends on the eigenvalue λ and auxiliary parameter h, condition (3,18) reads

$u_m(1) \approx U_m(1; \lambda, \hbar) = 1.$	(3.25)

The equation (3.25) shows that the λ is a function of ħ. There are multiple straight plateaus, all of which correspond to a specific Sturm-Liouville eigenvalue problem. The equation (3.25) still depends on the auxiliary parameter ħ and eigenvalue λ. In Figure 3.2.2(a) show that the λ is plotted according to equation (3.25) for $m = 25$. Against the specific eigenvalue and setting $\hbar = -0.8$ in the solution obtained by using HAM possible to obtain the corresponding approximate eigenfunction. Figures3.2.2$(b - d)$ show the first three distinct λ-plateaus. Eigenvalue λ can be determined by using equation (3.25) for $m = 25$ by setting $\hbar = -0.8$ given in table(3.2.2).

Table 3.2.2: Shows the first six eigenvalues

k	λ_k
1	7.37398501
2	36.33601851
3	85.29251075
4	154.10192997
5	352.47100980
6	568.16321224

In each figure, these plateaus represent the ranges of eigenvalues where the behavior of the system remains relatively stable and distinct from other ranges. Figure 3.2.2(e) presents an approximate eigenfunction for the second eigenvalue (i.e.$\lambda_2 = 36.33601851$), calculated at $\hbar = -0.8$. Figure 3.2.2(f) shows the inaccuracy of the eigenfunction corresponding to the second eigenvalue is obtained by using equation (3.16).

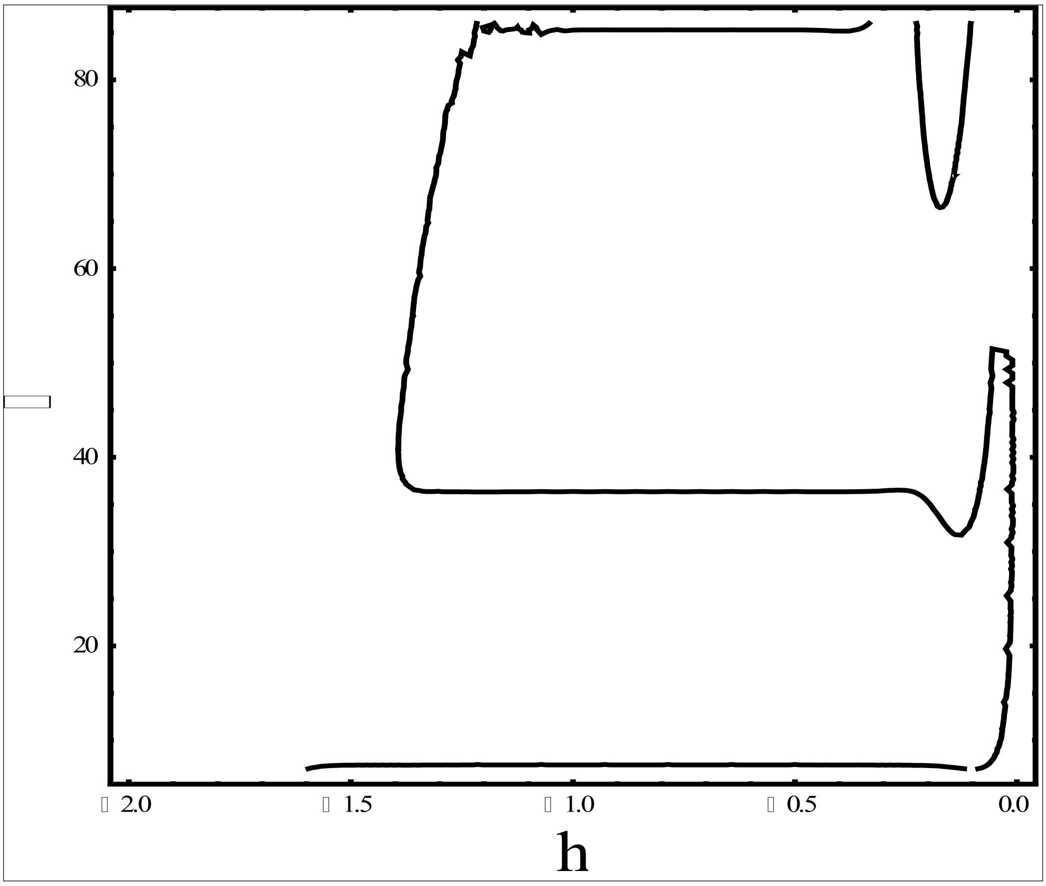

Figure 3.2.2(a): ħ -curve according to equation (3.25) for m = 25

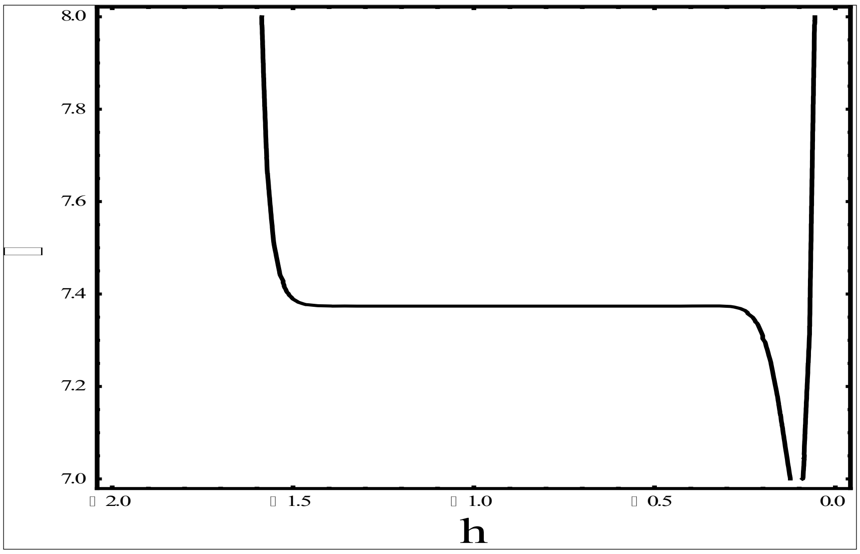

Figure 3.2.2(b): λ –plateau corresponding to λ_1= 7.3739

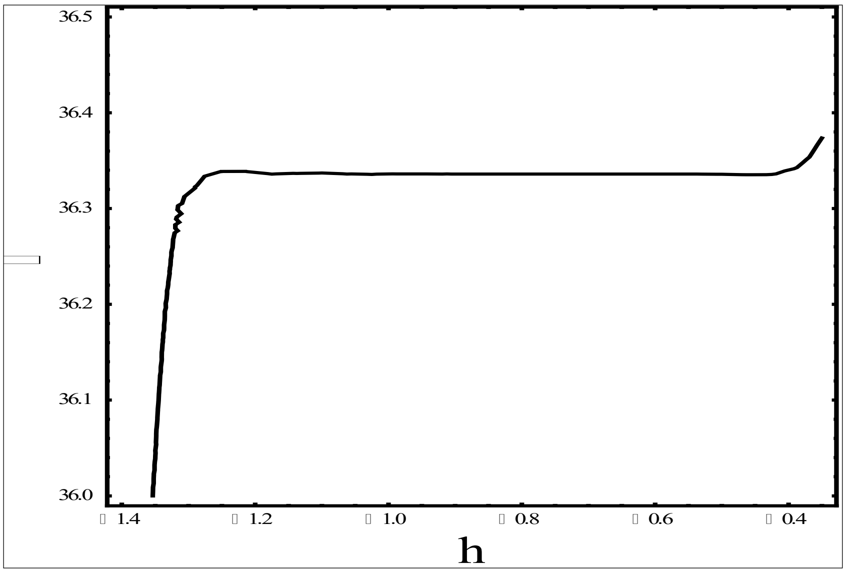

Figure 3.2.2(c): λ –plateau corresponding to λ_2= 36.3360

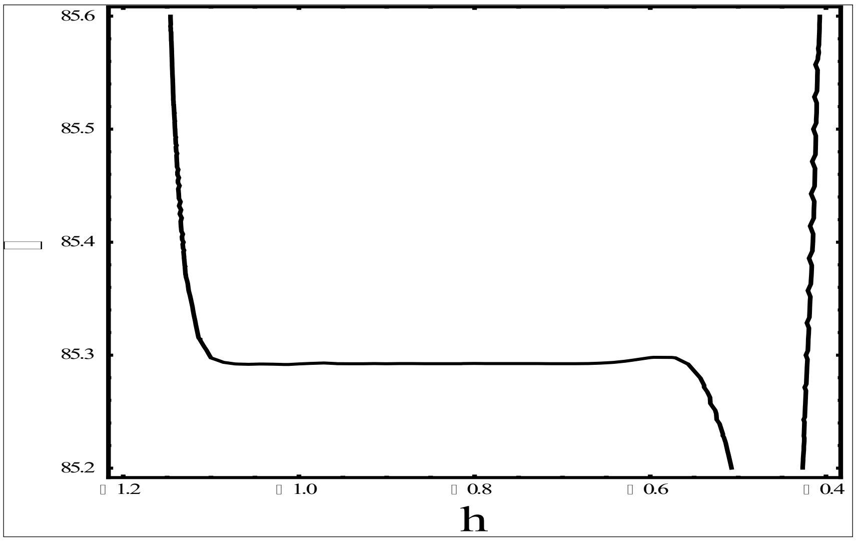

Figure 3.2.2(d): λ –plateau corresponding to λ_3= 85.2925

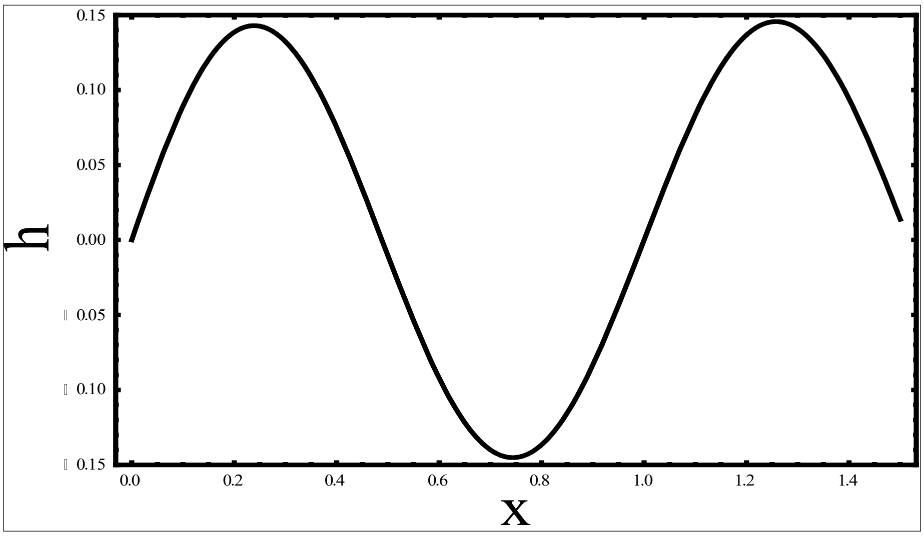

Figure 3.2.2(e): Eigenfunction corresponding to λ_2= 36.33601851

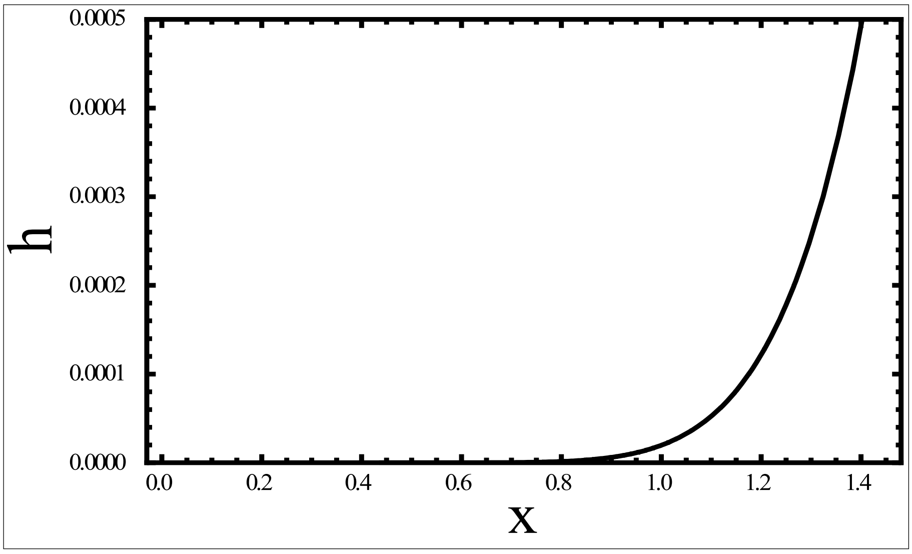

Figure 3.2.2(f): Error of eigenfunction corresponding to λ_2= 36.33601851

3.3 Fourth-order Sturm-Liouville Problems

The numerical procedure to calculate the eigenvalues for the fourth-order nonsingular Sturm-Liouville problem is given as

$(q_0(x)y''(x))'' = (q_1(x)(y'(x))' + (\lambda w(x) - q_2(x))y(x), x \in (0\,,1),$	(3.26)

Subject to some four points specified conditions at the boundary of the domain (two conditions at initial point a and two other conditions at point b). $q_0(x), q_1(x), q_2(x)$ and $w(x)$ are piecewise continuous functions with $q_0(x), w(x) \geq 0.$ Ordinary differential equations with boundary value problems explain the many different physical, biological, and chemical phenomena that are relevant to theory and application. Variational Iteration Methods [17], are an easy and efficient method for approximating solutions to nonlinear differential equations. In this technique, utilize VIM as a basis for solving Sturm-Liouville problems, aiming to accurately determine eigenvalues and eigenfunctions. The initial approximation v_0 is chosen to satisfy two initial conditions at $x = a$ and includes two parameters, namely c and $d.$ when carrying out the numerical details and two conditions are specified initially at $x = a$, the solution obtained will be a two parameter series solution that has the form.

$y_{\mathrm{m}}(x, \mathrm{h}, \lambda) = cf_{\mathrm{m}}(x, \hbar, \lambda) + dg_{\mathrm{m}}(x, \hbar, \lambda).\ n > 0,$

To fulfill the other requirements, such as:

$y(k, \hbar, \lambda) = y'(k, \hbar, \lambda) = 0,$

Obtained the following system

$$cf_{\mathrm{m}}(k, \hbar, \lambda) + dg_{\mathrm{m}}(k, \hbar, \lambda) = 0,$$
$$cf'_{\mathrm{m}}(k, \hbar, \lambda) + dg'_{\mathrm{m}}(k, \hbar, \lambda) = 0,$$

the nonzero solution is possible if

$$W_n(\lambda) = \begin{vmatrix} f_m(k,\hbar,\lambda) & g_m(k,\hbar,\lambda) \\ f''_m(k,\hbar,\lambda) & g''_m(k,\hbar,\lambda) \end{vmatrix} = 0. \quad (3.27)$$

As a result, the eigenvalues are the roots of $W_m(\lambda)$. The following algorithm summarizes the entire method [16].

Algorithm:

Step 1: Select the initial approximation of the form $v_0(x)$.

Step 2: Use iteration formula:

$$v_m(x) = \chi_m(v_{m-1}(x)) + \hbar\int_0^x\int_0^{\eta_4}\int_0^{\eta_3}\int_0^{\eta_2} R_m(v_{m-1}(\eta_1))d\eta_1 d\eta_2 d\eta_3 d\eta_4.$$

Step 3: Calculate $W_m(\lambda) = 0$.

Example 3.3.1 Eigenvalue problem

$$y''''(x) - \lambda y(x) = 0, x \in (0,1), \quad (3.28)$$

the boundary conditions are:

$$y(0) = 0, y'(0) = 0, \quad (3.29)$$

$$y(1) = 0, y''(0) = 0, \quad (3.30)$$

Suppose that the solution of the above equation (3.28) may be represented by a collection of base functions are written as

$$v(x) = \sum_{i=0}^{\infty} f x^{4i+2} + \sum_{i=0}^{\infty} g x^{4i+3}, \quad (3.31)$$

here, f and g are the unknown coefficients to be found out later. By using HAM technique, the auxiliary linear operator and nonlinear operator chosen for our analysis are as follows:

$$\mathcal{L}[\psi(x;p)] = \psi''''(x;p).$$

$$N[\psi(x;p)] = \psi''''(x;p) - \lambda\psi(x;p). \quad (3.32)$$

Choose $v_0(x) = \frac{cx^2}{2} + \frac{dx^3}{6}$ for the simplicity of the initial approximation of v(x). Clearly,$v_0(x) = \frac{cx^2}{2} + \frac{dx^3}{6}$ automatically satisfies boundary condition given in equation (3.29), (i.e. $v(0) = 0, v'(0) = 0$). Therefore, the general equation of zero-order deformation and the related limit conditions are also expressed as:

$$(1-p)\mathcal{L}[\psi(x;p) - v_0(x)] = p\hbar N[\psi(x;p)], \quad (3.33)$$

From equation (3.4) and (3.32) obtained the following

$$R_m(v_{m-1}(x)) = v''''_{m-1}(x) + (\lambda)(v_{m-1}(x)). \quad (3.34)$$

In the above relation, prime($'$)represents differentiation with respect to x. Now, the solution of the mth-order deformation equation(3.3) for $m \geq 1$ becomes:

$$v_m(x) = \chi_m(v_{m-1}(x)) + \hbar\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1} R_m(v_{m-1}(\eta_1))d\eta_1 d\eta_2 d\eta_3 d\eta_4. \quad (3.35)$$

For m =1, the equation (3.34) and (3.35) are becomes:

$$v_1(x) = \chi_m(v_{1-1}(x)) + \hbar\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1} R_1(v_{1-1}(\eta))d\eta d\eta_1 d\eta_2 d\eta_3,$$

$$v_1(x) = \hbar\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1} (v''''_{m-1}(\eta) + (\lambda)(v_{m-1}(\eta)))d\eta d\eta_1 d\eta_2 d\eta_3,$$

$$v_1(x) = \hbar\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1} (\lambda)(\frac{c\eta^2}{2} + \frac{d\eta^3}{6})d\eta d\eta_1 d\eta_2 d\eta_3,$$

$$v_1(x) = -\frac{1}{720}\lambda c\hbar x^6 - \frac{1}{5040}\lambda d\hbar x^7,$$

For m = 2

$$v_2(x) = \chi_2(v_1(x)) + \hbar\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1} R_2(v_{2-1}(\eta))d\eta d\eta_1 d\eta_2 d\eta_3,$$

$$v_2(x) = -(\frac{1}{720}\lambda c\hbar + \frac{1}{720}\lambda c\hbar^2)x^6 - \left(\frac{1}{5040}\lambda d\hbar + \frac{1}{5040}\lambda d\hbar^2\right)x^7 + \frac{1}{3628800}\lambda^2 c\hbar^2 x^{10} + \frac{1}{39916800}\lambda^2 d\hbar^2 x^{11},$$

As a result, a few terms from the homotopy analysis technique series are listed below:

$$v_1(x) = -\frac{1}{720}\lambda c\hbar x^6 - \frac{1}{5040}\lambda d\hbar x^7,$$

$$v_2(x) = -(\frac{1}{720}\lambda c\hbar + \frac{1}{720}\lambda c\hbar^2)x^6 - \left(\frac{1}{5040}\lambda d\hbar + \frac{1}{5040}\lambda d\hbar^2\right)x^7 + \frac{1}{3628800}\lambda^2 c\hbar^2 x^{10} + \frac{1}{39916800}\lambda^2 d\hbar^2 x^{11},$$

$$v_3(x) = -c\left(\frac{1}{720}a\hbar x^6 + \frac{1}{360}a\hbar^2 x^6 + \frac{1}{720}a\hbar^3 x^6\right) - d\left(\frac{ad\hbar x^7}{5040} + \frac{ad\hbar^2 x^7}{2520} + \frac{a\hbar^3 x^7}{5040}\right) + c\left(\frac{1}{1814400}a^2\hbar^2 x^{10} + \frac{1}{1814400}a^2\hbar^3 x^{10}\right) + d(\frac{1}{19958400}a^2\hbar^2 x^{11} + \frac{1}{19958400}a^2\hbar^3 x^{11}) - c(\frac{1}{87178291200}a^3\hbar^3 x^{14}) - d(\frac{1}{1307674368000}a^3\hbar^3 x^{15}),$$

is generalized as:

$$v_m(x;\ \lambda,\hbar) = cf_m(x;\ \lambda,\hbar) + dg_m(x;\ \lambda,\hbar)\ ,\ m > 0,$$

here, f_m and g_m are constant. $W_m(x)$ has the following form according to the HAM's mth-order approximate solution:

$$W_m(x) = \sum_{i=0}^{m} v_i(x) = cf_m(x;\ \lambda,\hbar) + dg_m(x;\ \lambda,\hbar). \qquad (3.36)$$

The mth-order approximation solution is depends on parameters c, d, and ħ. The following two equations were obtained utilizing the boundary conditions:

$$c\, f_m(1;\ \lambda,\hbar) + dg_m(1;\ \lambda,\hbar) = 0.$$
$$c\, f''_m(1;\ \lambda,\hbar) + d\, g''_m(1;\ \lambda,\hbar) = 0.$$

To obtain a nontrivial eigenfunction solution for a specific eigenvalue, we need to solve the equation (3.36) for $m = 25$ as

$$W_{25}(\lambda) = \begin{vmatrix} f_{25}(1,\hbar,\lambda) & g_{25}(1,\hbar,\lambda) \\ f''_{25}(1,\hbar,\lambda) & g''_{25}(1,\hbar,\lambda) \end{vmatrix} = 0. \qquad (3.37)$$

The equation (3.37) represents the relation between the eigenvalue λ and the auxiliary parameter ħ. There are multiple straight plateaus, all of which correspond to a specific Sturm-Liouville eigenvalue problem. Consider equation (3.37) as an example of the HAM uniqueness criteria, which is still dependent on the auxiliary parameters ħ and eigenvalue λ. Eigenvalue λ can be determined by using equation (3.37) for $m = 25$ by setting $\hbar = -1.02$ given in table(3.3.1). Against the specific eigenvalue and setting $\hbar = -1.02$ in the solution obtained by using HAM possible to obtain the corresponding approximate eigenfunction.

Table: 3.3.1

k	λ_k
1	237.72106751
2	2496.48743785
3	10867.58221697
4	31780.09645427
5	74000.84934655
6	148634.47747229

Figures 3.3.1(a − c) show the three distinct λ-plateaus Figure 3.3.1(d) shows the estimated eigenfunction for the sixth eigenvalue $\lambda_6 = 148634.47747229$ against $\hbar = -1.02$. Figure 3.3.1(e) shows the error profiles in eigenfunction according to the sixth eigenvalue as determined by equation (3.28).

$$W_{25}(\lambda) = f_{25}(x) - \left(\frac{f_{25}(1)}{g_{25}(1)}\right) g_{25}(x).$$

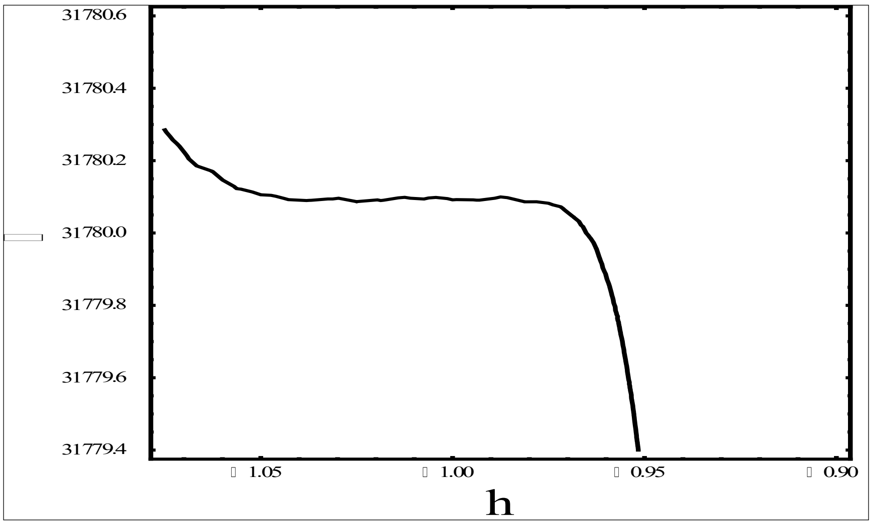

Figure 3.3.1(a): λ-plateau corresponding to λ_4

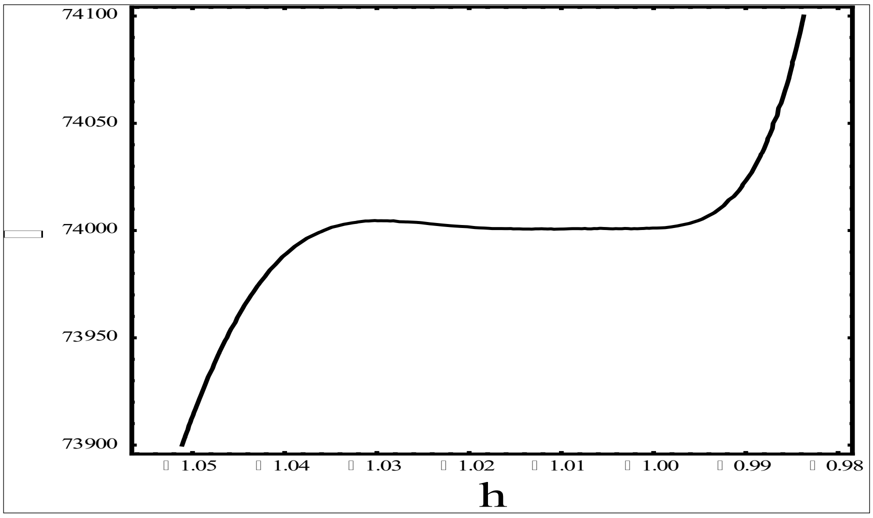

Figure 3.3.1(b): λ-plateau corresponding to λ_5

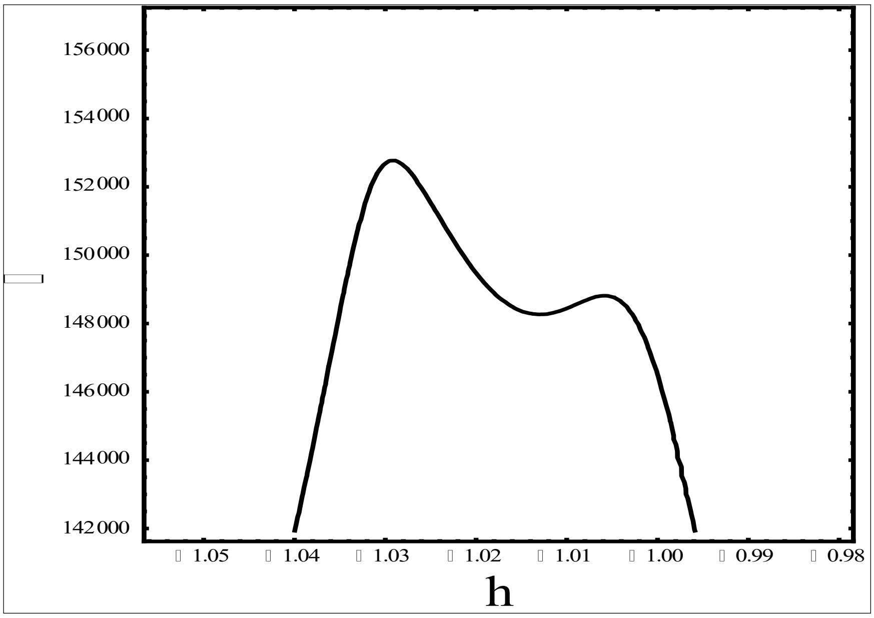

Figure 3.3.1(c): λ-plateau corresponding to λ_6

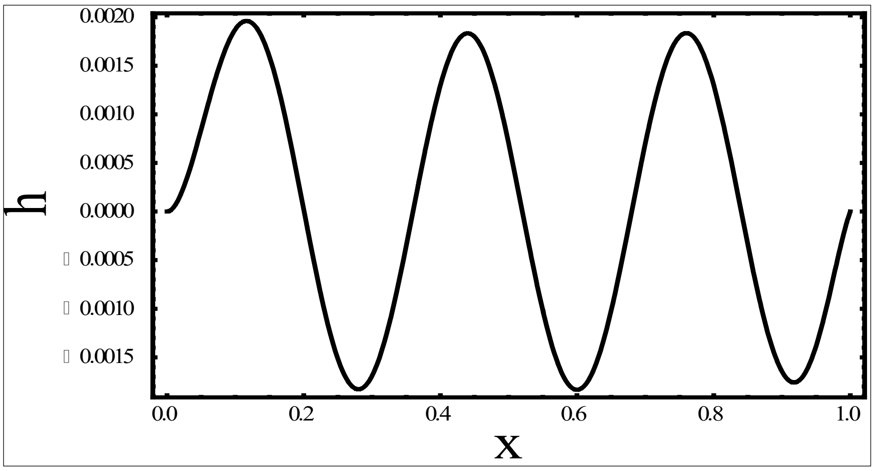

Figure 3.3.1(d): Eigenfunction corresponding to λ₆

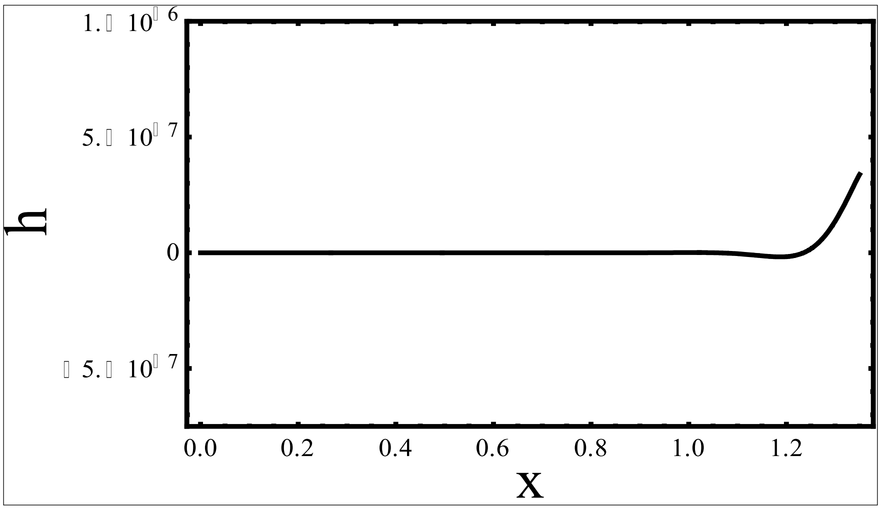

Figure 3.3.1(e): Eigenfunction error corresponding to λ₆

Suppose that the solution of the above equation (3.38) can be represented by a collection of base functions are given as:

$$v(x) = \sum_{i=0}^{\infty} d_i x^{2i+1}, \quad (3.41)$$

where, d_i are the unknown coefficients to be found out later. By using HAM technique by selecting $\mathcal{L}$ is the auxiliary linear operator and N is the non-linear operator are written as:

$$\mathcal{L}[\varphi(x;\, p)] = \varphi''''(x;\, p).$$

$$N[\varphi(x;\, p)] = \varphi''''(x;\, p) - \lambda\varphi(x;\, p) - 0.02x^2\varphi''(x;\, p) + 0.04x\varphi'(x;\, p) - (0.0001x^4 - 0.02)\varphi(x;\, p). \quad (3.42)$$

Choose $v_0(x) = cx + dx^3$ for the simplicity of the initial approximation $v(x)$. Clearly $v_0(x) = cx + dx^3$, automatically satisfies the boundary condition equation (3.39), (i.e. $v(0) = 0, v'(0) = 0$). Therefore, the general equation of zero-order deformation and the related limit conditions expressed as:

$$(1-p)\mathcal{L}[\varphi(x;\, p) - v_0(x)] = p\,\hbar\, N[\varphi(x;\, p)], \quad (3.43)$$

From equation (3.4) and (3.42), get the following form.

$$R_m(v_{m-1}(x)) = v''''_{m-1}(x) - (\lambda)(v_{m-1}(x)) - 0.02x^2 v''_{m-1}(x) + 0.04x v'_{m-1}(x) - (0.0001x^4 - 0.02)v_{m-1}(x) \quad (3.44)$$

The solution to the mth-order deformation equation (3.3) for $m \geq 1$ becomes:

$$v_m(x) = \chi_m v_{m-1}(x)) + \hbar \int_0^x \int_0^{\eta_3} \int_0^{\eta_2} \int_0^{\eta_1} R_m(v_{m-1}(\eta_1))\, d\eta_1 d\eta_2 d\eta_3 d\eta_4. \quad (3.45)$$

Putting m =1 from above equation (3.44) and (3.45) are expressed as

$v_1(x) = \chi_m(v_{1-1}(x)) + ħ\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1}(v_0''''(\eta) - (\lambda)(v_0(\eta)) - 0.02\eta^2 v_0''(\eta) + 0.04\eta v_0'(\eta) - (0.0001\eta^4 - 0.02)v_0(\eta))d\eta d\eta_1 d\eta_2 d\eta_3,$

$v_1(x) = ħ\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1}(v_0''''(\eta) - (\lambda)(v_0(\eta)) - 0.02\eta^2 v_0''(\eta) + 0.04\eta v_0'(\eta) - (0.0001\eta^4 - 0.02)v_0(\eta))d\eta d\eta_1 d\eta_2 d\eta_3,$

$v_1(x) = 0.0005cħx^5 - 0.00833acħx^5 + 0.000023dħx^7 + 0.000023dħx^7 - .306 \times cħx^9 - 1.2626 \times 10^{-8}dħx^{11},$

for m = 2

$$v_2(x) = \chi_2(v_{2-1}(x)) + ħ\int_0^x\int_0^{\eta_3}\int_0^{\eta_2}\int_0^{\eta_1}(v_1''''(\eta) - (\lambda)(v_1(\eta)) - 0.02\eta^2 v_1''(\eta)\ 0.04\eta v_1'(\eta) - (0.0001\eta^4 - 0.02)v_1(\eta))d\eta d\eta_1 d\eta_2 d\eta_3,$$

$v_2(x) = 0.0004cħx^5 - 0.0083acħx^5 + 0.0004cħ^2x^5 - 0.0083acħ^2x^5 + 0.00002dħx^7 - 0.0011adħx^7 + 0.000023dħ^2x^7 - 0.0011adħ^2x^7 - 3.306 \times 10^{-8}cħx^9 - 6.283 \times 10^{-8}cħ^2x^9 + 3.306 \times 10^{-7}acħ^2x^9 - 1.262 \times 10^{-8}dħx^{11} - 1.424 \times 10^{-8}dħ^2x^{11} + 7.816 \times 10^{-8}adħ^2x^{11} + 1.503 \times 10^{-7}a^2dħ^2x^{11} - 8.710 \times 10^{-13}cħ^2x^{13} + 5.048 \times 10^{-11}acħ^2x^{13} + 5.979 \times 10^{-13}dħ^2x^{15} + 4.019 \times 10^{-12}adħ^2x^{15} + 5.789 \times 10^{-17}cħ^2x^{17} + 1.357 \times 10^{-17}dħ^2x^{19},$

From the above relation get the following series:

$v_1(x) = 0.0005cħx^5 - 0.00833acħx^5 + 0.000023dħx^7 + 0.000023dħx^7 - .306 \times cħx^9 - 1.2626 \times 10^{-8}dħx^{11},$

$v_2(x) = 0.0004cħx^5 - 0.0083acħx^5 + 0.0004cħ^2x^5 - 0.0083acħ^2x^5 + 0.00002dħx^7 - 0.0011adħx^7 + 0.000023dħ^2x^7 - 0.0011adħ^2x^7 - 3.306 \times 10^{-8}cħx^9 - 6.283 \times 10^{-8}cħ^2x^9 + 3.306 \times 10^{-7}acħ^2x^9 - 1.262 \times 10^{-8}dħx^{11} - 1.424 \times 10^{-8}dħ^2x^{11} + 7.816 \times 10^{-8}adħ^2x^{11} + 1.503 \times 10^{-7}a^2dħ^2x^{11} - 8.710 \times 10^{-13}cħ^2x^{13} + 5.048 \times 10^{-11}acħ^2x^{13} + 5.979 \times 10^{-13}dħ^2x^{15} + 4.019 \times 10^{-12}adħ^2x^{15} + 5.789 \times 10^{-17}cħ^2x^{17} + 1.357 \times 10^{-17}dħ^2x^{19},$

is simplified as

$v_m(x;\ \lambda, ħ) = cf_m(x;\ \lambda, ħ) + dg_m(x;\ \lambda, ħ),\ m > 0.$

The mth-order approximate solution by using HAM is represented $W_m(x)$ in the following form:

$$W_m(x) = \sum_{i=0}^{m} v_i(x) = cf_m(x;\ \lambda, ħ) + dg_m(x;\ \lambda, ħ). \quad (3.46)$$

The mth-order approximation solution is depends on parameters c, d, and ħ. The following two equations were obtained utilizing the boundary conditions:

$cf_m(5;\ \lambda, ħ) + dg_m(5;\ \lambda, ħ) = 0.$

$c\, f_m''(5;\ \lambda, ħ) + dg_m''(5;\ \lambda, ħ) = 0.$

$$W_{25}(\lambda) = \begin{vmatrix} f_{20}(5, ħ, \lambda) & g_{20}(5, ħ, \lambda) \\ f''_{20}(5, ħ, \lambda) & g''_{20}(5, ħ, \lambda) \end{vmatrix} = 0. \quad (3.47)$$

Equation (3.47) represents the relation between the eigenvalue λ and the auxiliary parameter ħ. There are multiple straight plateaus of eigenvalue λ, each of which corresponds to a Sturm-Liouville Problem. The relation given in equation (3.47) is still dependent on the auxiliary parameters ħ and eigenvalue λ. Against the specific eigenvalue and setting $ħ = -0.9$ in the solution obtained by using HAM possible to obtain the corresponding approximate eigenfunction. Eigenvalue λ can be determined by using equation (3.47) by setting $ħ = -0.9$ and $m = 20$. Figure 3.3.2(a) shows the profile of λ against ħ with range [-2, 0] plotted in accordance with the equation (3.47) for m *= 20*. Eigenvalue λ can be determined by using equation (3.47) for $m = 20$ by setting ħ = - 0.9 given in table(3.3.2). Figures3.3.2$(b - d)$ show the first three distinct λ-plateaus.

Table 3.3.2: Represents the first six eigenvalues

k	λ_k
1	0.21505086
2	2.75480888
3	13.21539518
4	40.94821490
5	99.40710013
6	192.96356062

Figure 3.3.2(e) shows the estimated eigenfunction for the third eigenvalue $\lambda_3 = 13.21539518$ against $\hbar = -0.9$. Figure 3.3.2(f) shows the error profiles in eigenfunction according to the third eigenvalue as determined by equation (3.38). $W_{20}(\lambda) = f_{20}(x) - \left(\frac{f_{20}(5)}{g_{20}(5)}\right) g_{20}(x)$.

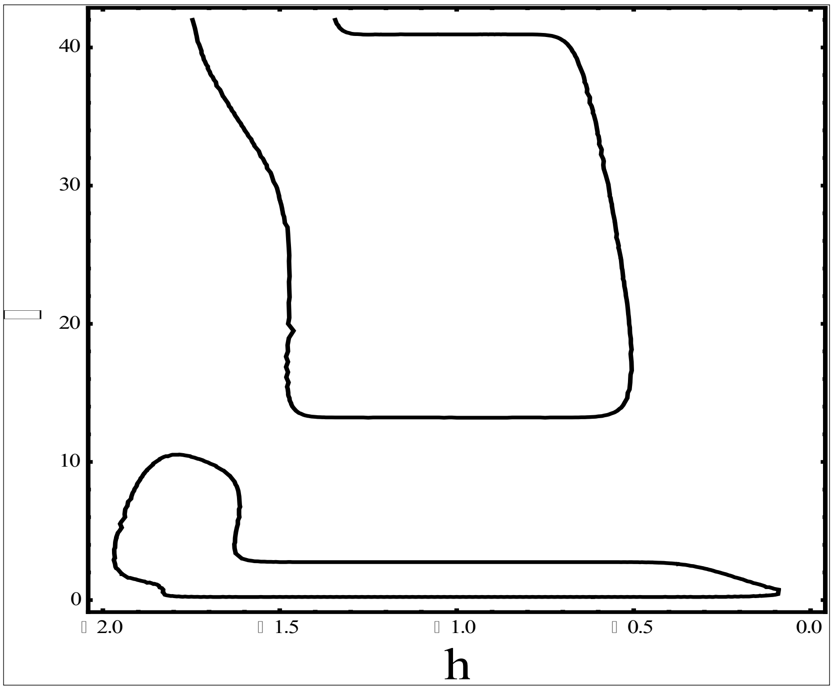

Figure 3.3.2(a): ħ -curve according to equation (3.47)

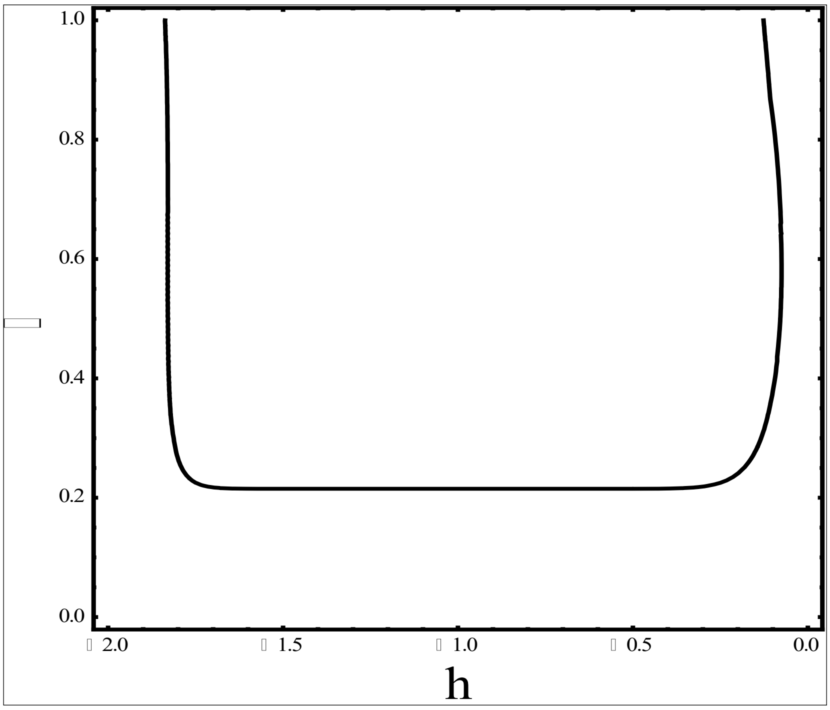

Figure 3.3.2(b): λ-plateau corresponding to λ_1

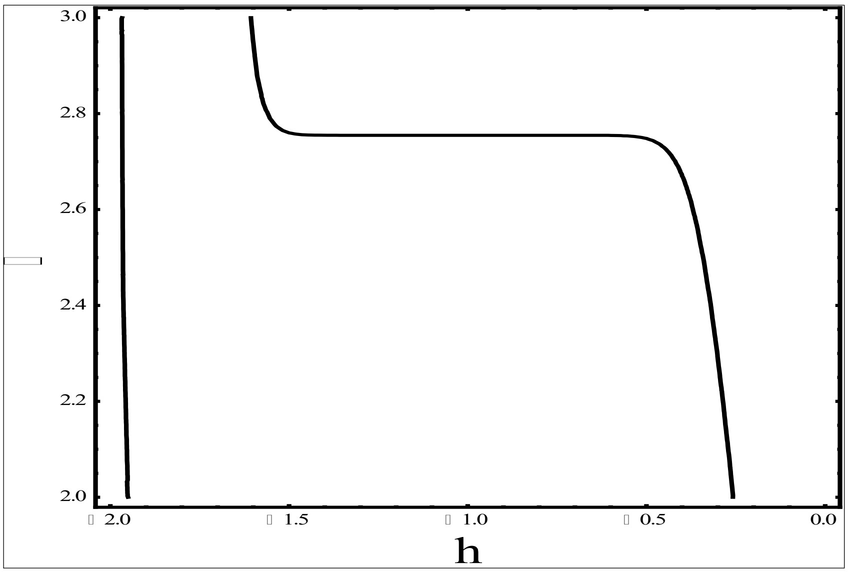

Figure 3.3.2(c): λ-plateau corresponding to λ_2

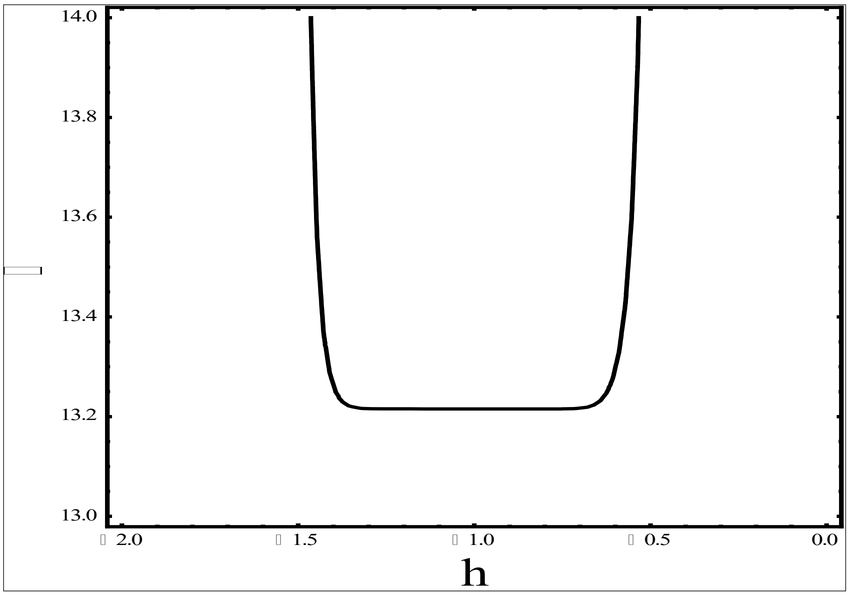

Figure 3.3.2(d): λ-plateau corresponding λ_3

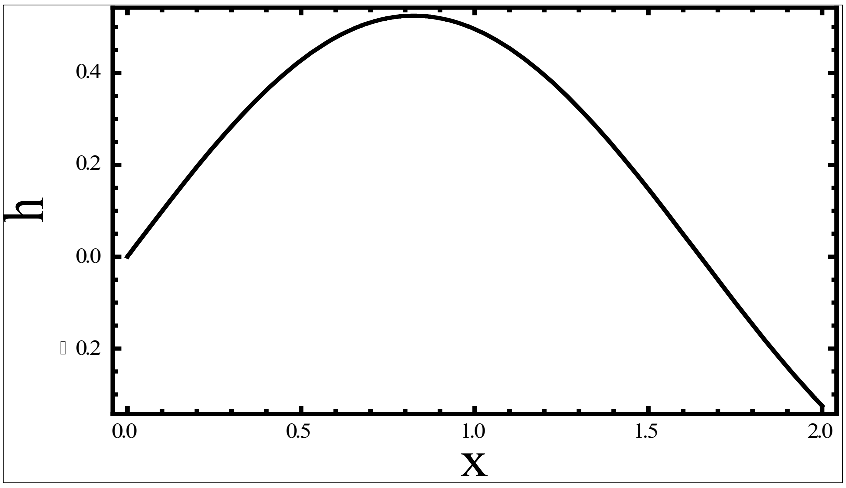

Figure 3.3.2(e): Eigenfunction corresponding to λ_3

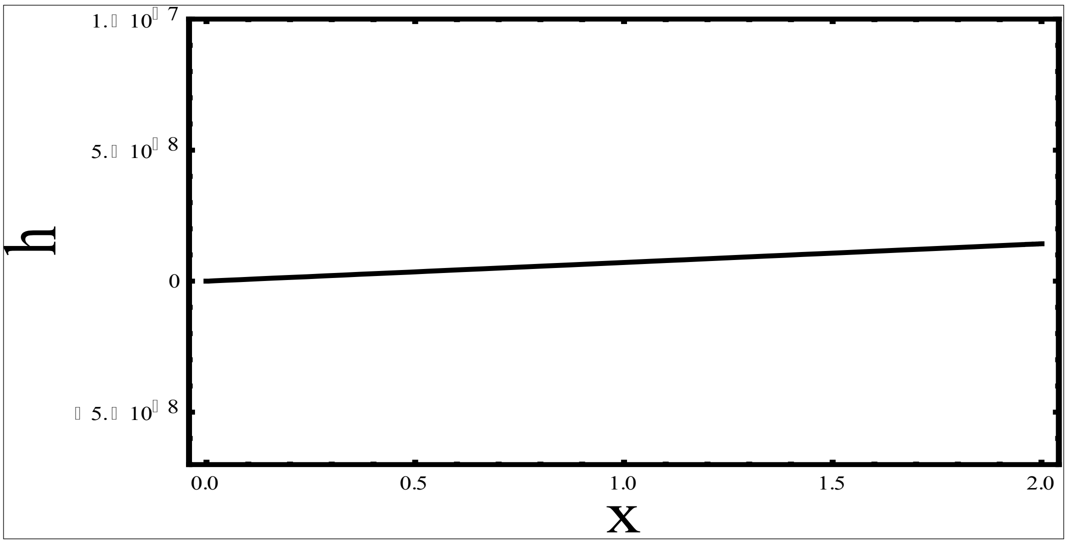

Figure 3.3.2(f): Error in Eigenfunction corresponding to λ_3

3.4 CONCLUSIONS

The eigenvalues of the Sturm-Liouville problems have been numerically estimated using the HAM technique in the present investigation. The eigenvalues cannot be all identical. There are a number of horizontal plateaus in the plot of λ as a function of h, which suggests that there are multiple solutions. By utilizing the Homotopy Analysis Method (HAM) with the same initial estimate and linear operators $\mathcal{L}$, it becomes possible to identify multiple solutions or eigenvalues. In this research, the auxiliary parameter ħ, which governs the convergence of the HAM approximation series solutions, was found to have an additional important role. This significant application makes numerous solution predictions by taking into account the number of plateaus that appear in the ħ -curve. This approach's fundamental concepts are expected to be applied to additional challenges in the future.

Nomenclature

A	Area (m^2)
c	Specific heat (J/kg K)
c_a	Specific heat at temperature T^a (J/kg K)
V	Volume
u_m	Mth-order approximation
u	Reduced stream function
T_s	Effective skin temperature (K)
T_I	Initial temperature (K)
T_a	Environment temperature (K)
T	Temperature (K)
p	Embedding parameter
$\mathcal{N}$	Nonlinear operator
$\mathcal{L}$	Auxiliary linear operator
ħ	Auxiliary parameter
H(δ)	Auxiliary function
h	Coefficient of natural convection (W/m^2 K)

Greek symbols

β	Constant, volumetric thermal expension coefficient (1/K)
ϵ	Small parameter (-)
σ	Stefan- Boltzman constant (-)
χ_m	Two value function (-)

Subscripts

a	Air
m	Order of approximation
s	surface

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