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Metric Stability and Jacobi-Gauss Periods

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Abstract **Review Article**

A spinor definition of matter is extended to simplest cycles in interval [0,1] and to permutations of quartic-cubic roots in elliptic curves. Minkowski spacetime and Mandelstam plane are linked with wave vectors due to rotation on real interval. Metrical geometry is discussed by rational triangles which are connected with modular invariants. An adiabatic solution in terms of cyclotomic units for constant elliptic invariants is derived and extended to iterated invariants. The pseudo-congruences allow to formulate coupling constants. Iterated invariants connect a scale factor with diagram expansions in different eras in bifurcating k-components.

Keyword: Complex Lagrangian, quadratic Newton root finding, Minkowski spacetime, Green's function singularity, bifurcation, elliptic curves, spinor definition.

1. INTRODUCTION

Metrical geometry can be formulated by rational triangles (Menger, 1931). Complex quadratic Newtonian root finding $N_q(z)$ depends on angles $\varphi_q = \ln\frac{z_{k+1}-z_{k+2}}{z_k-z_{k+3}}$ of points z in a planar triangle $T(z_q)$ (Schröder, 1870). $N_q(z)$ iterates for conformal maps $\gamma^{(n)} \circ z$ also yield one of two roots of q(z) (Schröder, 1870). This uncertainty even for quadratic q(z) is discussed within unique factorization domains (UFD) (Ziep, 2026). It yields a multisheet Riemann surface $\mathbb{R}_L$ of complex cubic algebraic units $L(w,z) = \ln(w-z)$ which are related to regulator indices $R(\mathbb{K})$ (Ziep, 2026). A cubic number field $\mathbb{K}$ proves to be an optimal (best) approximation of the linear vicinity of simple nontrivial zeros of any entire transcendent function $\phi^{(\infty)}(z)$. Circles define complex time on a multisheet complex plane. Bifurcating k-components in Mandelbrot set galleries obey an optimal representation by cubic parts which are parts of $T(z_q)$. Triangles are capable resolving roots by circles and not by secants with higher computational precision. The corresponding triangulated surface $\mathbb{R}_L\left(T(z_q)\right)$ obeys infinite degrees of freedom by rotations of $T(z_q)$. This ball of discrete angles of triangles constitutes Minkowski spacetime $\mathcal{M}$ in (Ziep, 2026). The present paper extends invariant two-step cycles to unified fields in $\mathcal{M}$ with coupling constants differing by up to hundreds of orders of magnitude. A two-step conformal invariance is capable to formulate massless Lorentz-invariant spinors and half-order differentials (Dick, 1995) (Schiffer, 1966). For complex time z a Schwarzian derivative corresponds to conformal stress-energy (Schiffer, 1966) (Polchinski, 2005) (Ziep, 2026). The present paper links a complex thermodynamic potential Ω to δ- sources caused by theta functions as heat sources. The logarithmic surface $\mathbb{R}_L(T(z_q))$ is connected with Abelian integrals of the third kind. Under $\gamma^{(n)}$ a Newton fractal defines two complex points, two curvatures and two masses in each spacetime points of $\mathcal{M}$. Half-order differentials with first derivative in a Cayley-Newton iteration is not conformally invariant. First a chain k,k+1,k+2 of squared differentials with Schwarzian derivatives is metrical one-dimensional stress-energy (Schiffer, 1966). Point-like sources arise from angles φ_q on one-dimensional real interval [0,1]. The one-periodic binary ball of bits $\{\varphi_q\}$ constitutes Green's functions in $\mathcal{M}$. A doubly-periodic complex Ω operates with invariant Laplacian on Poincaré upper $\mathbb{H}^2$half plane $\Delta_h = y^2\,\partial_z\,\partial_{\bar{z}} = y^2(\partial_x^2 + \partial_y^2)$ with complex Green's functions $L(w,z) = \ln(w-z)$. Dimensionless fields are capable to capture coupling constants of many orders of magnitude as candidates for unified interactions (strong, weak, electromagnetic, gravitational, dark) (Ziep, 2024) (Ziep, 2025). A doubly-periodic period-doubling is viewed as a self-consistent spacetime confinement. Vacuum energy Ω of confined space (Casimir effect) is proportional to Riemann zeta functions ζ(z) at arguments z=-3 (Ruggiero & others, 1977). For sets of open systems Ω- minima is predicted for iterated nontrivial zeros $\zeta(z_{nt})=0$ (Ziep, 2025) (Ziep, 2025). Section 2 describes an optimal scan algorithm. Section 3 reduces cycles to quadruples. Section 5 defines wave vectors in Mandelstam plane followed by complex Lagrangians in Section 6. Triangles are related to doubly-periodic functions in $\mathcal{M}$. Section discusses a unified picture for Feynman diagrams and Section 8 explains coupling constants by Jacobi- Gauss periods.

2. SCANNING A SPLIT HYPERELLIPTIC SURFACE

Two-step conformal invariance in Newton root finding of $q(z) = (z-z_1)(z - z_2)$ (Schiffer, 1966)

$$z_{k+1} \leftarrow N_q(z) = z - \frac{q(z)}{\partial_z q(z)}|_{z_k} \leftarrow \gamma^{(3)}(w) \circ z_k \quad (1)$$

is subjected a Hermite-Tschirnhausen process by fractional substitutions (Weber, 1895)

$$z \leftarrow F^{(n)}(w,z) = \frac{\phi^{(n)}(w)}{w-z} - ⅓\,\partial_w \phi^{(n)}(w) = \gamma^{(n)} ⍰ z = \begin{vmatrix} a & b \\ -1 & w \end{vmatrix} ⍰ z \quad (2)$$

where $a = \frac{1}{n}\partial_w \phi^{(n)}$ and $b = \phi^{(n)} - \frac{w}{n}\partial_w \phi^{(n)}$. Transvectants with $z \wedge w$ yield for n=3

$$\phi^{(n)}(z) = \sum_{i=0,\dots,n} a_{n-i}\, z^i|_{n=3} \rightarrow \phi^{(3)}(z) = z^3 - g_2 z - g_3 \quad (3)$$

elliptic invariants g_2 and g_3. A quadratic $\gamma^{(3)}$ is simply a permutation of quartic roots $x_i = x_q = x_s$ in a quadruple q where permutations of two roots constitute $\mathcal{M}$

$$q: 1, \delta_k, \delta_k\delta_k, \delta_k\delta_k\delta_k \simeq k+3 \in k, k+1, k+2 \simeq \pm\infty, \pm i\infty, e_i \quad (4)$$

Invariants (3) enter a Lagrangian $\mathcal{L}[\psi_i]$ of four-component $\psi_i = x_{i1} + ix_{i2}$ with nearly constant

$$a + g_2/3 = z^2 = (x_i x_j)(x_i x_k) \simeq -¼\psi_i \wedge \bar\psi_j \psi_i \wedge \bar\psi_k \simeq \bar\psi_i \bar\psi_j \Gamma_{ijkl} \psi_k \psi_l \quad (5)$$

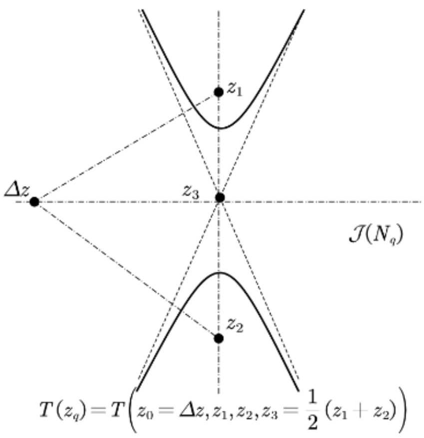

Fig.1: the planar triangle $T_q(z)$- configuration for deciding whether root finding (1) tends to z_1 or z_2. The Fatou set is the range of the hyperbola (Schröder, 1870)

A constant b=0 in (2) expands into a rational vertex Γ_{ijkl}

$$z - e_i = \gamma^{(3)} \circ x = (x_i x_k)(x_i x_l) M_{j,i}(x) \rightarrow \bar\psi_i \bar\psi_j \Gamma_{ijkl} \psi_k \psi_l \quad (6)$$

Γ_{ijkl} ensures that any expansion of the difference $z - e_i$ into pairs of different quartic roots depends on the symmetrized Moebius map $M_{1,2}(z) = \frac{z-z_1}{z-z_2}$. A solution of (3) is $z = f^8(\omega)$ with

$$f(\omega) = e^{\frac{-i\pi\omega}{24}} \prod_{n=1}^{\infty}\left(1 + e^{i\pi\omega(2n-1)}\right) \simeq \zeta^{(12)} e^{\frac{-i\pi\omega}{24}} \quad (7)$$

The 'spinor' ψ_i ix expanded by cyclotomic bases $\zeta^{(2)}$, $\zeta^{(3)}$ and $\zeta^{(4)}$. Whereas $\zeta^{(3)}$ yield non-periodic non-Abelian ternary CF cyclotomic roots $\zeta^{(2)}$ and $\zeta^{(4)}$ constitute Abelian functions within the KWT. Map (2) implies a cubic base $\{w\} = \{w_0 = 1, w_1, w_2\}$ by $w^k \rightarrow w_k$ surrounded by $\zeta^{(4)}$ which splits into the Mandelbrot set $z_{k+1} \leftarrow 4z_k^2 + c, c = \frac{2}{3}g_2$ for base vector w_0 and linear map $z_{k+1} \leftarrow z_k$ for base vector w_1 for n=3 (Ziep, 2026). $\mathcal{L}[\psi_i\,]$ is supported by writing $D_{\mu\nu} = 2\Re z_k$ and $I\mu\nu = 2\Im z_k$, $\mathcal{F} = ½\Re(c - z_{k+1}), \mathcal{G} = ½\Im(c - z_{k+1})$ which rewrites map (2) as the invariant $D_{\mu\nu}^4 + 2\mathcal{F}D_{\mu\nu}^2 - \mathcal{G}^2 = 0$ or $\mathcal{F} + i\mathcal{G} = D_{\mu\nu}^2 + 2iD_{\mu\nu}I_{\mu\nu} - I_{\mu\nu}^2$ of field energy density and current (Schwinger, 1951). The invariant $g_2 = 3(\mathcal{F} + i\mathcal{G}) + \frac{3}{2}z_{k+1}$ in (5) is like an energy density. For a quadratic $q(z) = (z - z_1)(z - z_2)$ the two-step algorithm (1) combines convergence with the information uncertainty bit $z_1 \vee z_2$ on the hyperbolic border between the Julia set and the Fatou set $\mathcal{J}(N_q) - \mathcal{F}(N_q)$. A regular chaotic distribution of rotations of $L(w,z) = \ln|w - z|$ by $\varphi_q = \pm\pi$ of triangles $T(z_q) = T(\Delta z = z_0, z_1, z_2, z_3 = ½(z_1 + z_2))$ in Fig.1. yields Green's functions

$$G^{(\pm)}(\varphi_q) = S(\varphi_q)\theta(\pi - \varphi_q) + \left(1 \pm S(\varphi_q)\right)\theta(\varphi_q - \pi) \quad (8)$$

for a one-dimensional source

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dp\cos\left(p(\varphi_q - \varphi_{q'})\right) \simeq \delta(\varphi_q - \varphi_{q'}) \quad (9)$$

in [0,1]. Here $S(\varphi_q) = \sum d\varphi_q \rightarrow \zeta(\mathcal{L}, \varphi_q)$ are geometric zeta functions being occupation numbers where ± correspond to a period-3 cycle π+π+π=3π. Triangulated surface elements of satisfy $\Delta z \leftarrow e^D \Delta z$. $D = D_{\mu\nu}[\gamma^\mu, \gamma^\nu]_-$ is a four-component representation $D_{\mu\nu} = [k^\mu, k^\nu]_-$ of Lorentz transformations $\exp(k_\mu \sigma^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix}$ of homogeneous angles (φ_1, φ_2) with Pauli matrices σ_μ. Wave vectors k_μ are 12-component strings φ_q in units of inverse length in Vol($\mathcal{M}$). Period-3 fixpoints in period-n fixpoints z(z(z))=z(z) or z(z(z))=z or z(z)=1 or period-3 multiplicative inverses f(f(f(z))=z or f(f(z))=z yield the $\frac{1}{n!}$ factor in $D^3 = X^2 D, D^4 = X^4, D^5 = X^4 D$ where $X = |\mathbf{X}|$ of the matrix exponential of the tensor $D_{\mu\nu} + D_{\mu\nu}^* = \mathbf{X} = \sqrt{(\mathcal{F} + i\mathcal{G})} \simeq \mathbf{E} + i\mathbf{B}$. The Weber invariant f(ω) enters a heat equation with unit diffusion coefficient $\frac{(du)^2}{d\omega}$. A two-step-invariance defines a sum of $(\Delta z)^2$.

Periods ω of the general Riemann surface $\mathbb{R}_L$ satisfy $\delta_k\omega \simeq \delta_k\delta_k\omega$ of because (2) splits ω. $\frac{(du)^2}{d\omega} \to \frac{(du)^2}{(d\omega)^2}$ yields the tangent in Fig.1 as the velocity of light. The Riemann surface $\mathbb{R}_L$ contains two $\binom{d\varphi_1}{d\varphi_2}$- balls formed by logarithmic sheets $L(w,z) = \ln(w-z)$ of roots (4). The permutation (4) is a genus 3 reduced to genus 1 surface which is hyperelliptic. A rationalizable quartic surface $K(X(f))$ and $W(Y(f))$ in projective space are points on Kummer and Weddle surface with vanishing $4 \cdot 4$ determinant $|K(X)| = |W(Y)| = 0$ where (Baker, 1907) (Ziep, 2024) (Ziep, 2024)

$$C_{tw}: X(f) = \wp_{\pm\pm}, 1) = (1, -f, f^2, 1), Y(f) = \wp_{\pm\pm\pm} = (1, -f, f^2, -f^3). \quad (10)$$

The hyperelliptic ℘- function on a twisted cubic C_{tw} is seen as a complex potential parametrized by the Weber invariant f(ω). Points on K(X) are zeros of the hyperelliptic sigma function $\sigma(u_\pm)$ or zeros of

$$(1, -f_k, f_k^2, 1) i\sigma_3 \otimes \sigma_2 (1, -f_{k+1}, f_{k+1}^2, 1) \to 0 \quad (11)$$

which is a f(ω)- parametrized hyperelliptic addition law (Baker, 1907). The division algebra (1) is regular for a unique factorization domain (UFD) and complex multiplication (CM) of periods ω for an imaginary quadratic field $\mathbb{Q}[\Delta]$ of a cubic field $\mathbb{K}[\Delta]$ (Heegner, 1952) (Weber, 1908). Transforms $X(\gamma_3 \circ f)$ are SE(3)- like steps on $K(X(f))$ where quartic roots $x_i(f)$ mix cyclotomic roots $\zeta^{(12)}$ in a quadruple $f_q(\omega) \in \zeta^{(12)} e^{\frac{-i\pi\omega}{24}}$. With norm $N(f_q(\omega), \mathbb{K}) = ff'f'' = 2$ of three-component complex conjugates $f', f'' \simeq \sum_{c=1,2,3} a_c 1^{\frac{c-1}{3}} \partial^{c-1}$ e.g. with $\partial = 2^{1/3}$ only substitutions of two f-components of are Abelain. Two components $\binom{d\varphi_1}{d\varphi_2}$ of f_q on complex plane in $\mathbb{R}_L$ connect the real interval [0,1] via $\exp(\sigma_\mu x^\mu)\binom{d\varphi_1}{d\varphi_2}$ to $\mathcal{M}$ (Sharkovsky, November 2019). $f_q(\omega)$ can be substituted by generators $\zeta^{(12)}$ in three distinct circles $\mathbb{S}^1(b)$, $\mathbb{S}^1(w)$, $\mathbb{S}^1(s)$ for a $\hat{B} = \{b_0, b_1\}[\{w_0 = 1, w_1, w_2\} \otimes \{\zeta^{(4)}\}]$ by binary, cubic and quartic bases with congruences mod 2,3 and 4. Some properties of root finding (1) are shortly described. The invariant complex process (1) obeys a φ_q time scale Δt on a Vol($\mathcal{M}$). Cyclotomic expansions in quadruples $\psi_i[\zeta^{(12)}]$ are Δt-symmetric. q(z)-roots depend on the real condition $\varphi_q = \pm\pi$ in an underconstrained complex potential Ω

$$q(z) = e^{\int_0^z \frac{dz}{z_{k+1}-z}} = e^{-\int_0^z G dz} = e^{-\int_0^{\Omega_k} d\Omega} \simeq G_0^{-1}(z) G_0^{-1}(z) \quad (12)$$

which solve as well (11) on K(X) because $z \simeq f_k$. Simple nontrivial zeros $z_1, z_2 \simeq z_{nt}, \bar{z}_{nt}$ setting $q(z) = \zeta(z)\bar{\zeta}(\bar{z})$ should be detectable by orbits of laps as fixed points of (2). Diagrams of (11) is given by Σ_{linear} a linear in Γ and a quadratic in Γ exchange term Σ_{ex} of self-energy $G^{-1} = G_0^{-1} - \Sigma$. Quantum statistics (QS) with Δt→∞ is not capable to distinguish between Σ_{linear} and Σ_{ex} (double-slit experiment). Gravitation and dark matter with Δt→0 is capable to resolves a bell-shaped maximum Σ_{linear} and a double-well twin peak Σ_{ex}. Green's function (8) depends on a quadruple $\psi_i \simeq f(\omega_q)$ of permuted quartic roots for $\cos\varphi = -g_3(3/g_2)^{3/2}$ in units of $\sqrt{(g_2/3)}$ (Brizard, 2009)

$$e_i(\varphi) = \left(\cos((\varphi-\pi)/3), \cos((\varphi+\pi)/), \cos(\varphi/3)\right) \quad (13)$$

for complex φ where $\Delta = g_2^3 \sin^2\varphi$ and $\lambda = \frac{\pm\sqrt{3}\sin(\varphi/3)}{\sin(\varphi/3 \pm \pi/6)}$. A quadratic map (2) with ω→2ω yields $\lambda \to 0$ for $k \to \infty$. The invariant equation (3) gets highly-nonlinear where $g_2 \to \infty$ and $\sin(i\varphi) \to 0$ for $k \to \infty$ yields the classical one-periodic behavior. The algorithm has the lowest complexity for iterating over discrete number of roots $\in \zeta^{(12)}$ for given invariants $g_2 \simeq \gamma_2 \in \mathbb{Z}$ and $g_3 \simeq \gamma_3 \in \mathbb{Q}(\sqrt{\Delta})$ for class number one fields $h_\Delta = 1$ in $\mathbb{K}[\Delta]$. This case covers QS. Here nearly unit coupling constants ensure optimality $\varphi_q^2 \simeq e^{\varphi_q}$ and $\varphi_q \simeq f(\omega_q)$ for sequential steps of mod 4, mod 3, and mod 2 congruences of $\zeta^{(12)}$ in the approximation (7). Varying g_2 and g_3 the number of steps k increases which leads to optimality $\varphi_q^2 \simeq e^{\Pi\varphi_q}$ for low coupling constants which covers classical fields. The variable z on Riemann surface sheets $\mathbb{R}_L$ is proportional to the number of vertices Γ_{ijkl} in (6) as the number of permutations of quartic roots. One has a linear behavior $\partial_z N_q(z) = \partial_z F^{(3)}(w,z) = 2w_0a_0z + w_1a_0 + w_0a_1 = 8w_0z + 4w_1$ which justifies to set $\partial_z F^{(3)}(w,z) \simeq \Gamma_{ijkl}$. Formed φ_q circles are time cycles equivalent to astronomical time with years and months. The Green' function (8)

$$G^{(\pm)}(\varphi_q) \simeq \bar{\psi}_q\psi_q \simeq \bar{\varphi}_q\varphi_q \simeq S(\varphi_q)G_a(f(\omega)) + (1 \pm S(\varphi_q))G_r(f(\omega)) \quad (14)$$

is a causal tent map which splits into an advanced $G_a(f(\omega))$ and retarded part $G_r(f(\omega))$. Here $L(w,z) \simeq \varphi_q$ is time and $\partial_z L(w,z) = \partial_{\sigma_\mu x_\mu} L(w,z)$ is curvature/stress-energy (Ziep, 2026). The paper mainly discusses the $\partial_z F^{(3)}(w,z) \simeq \Gamma_{ijkl}$ behavior for unified interactions. A Feigenbaum renormalized quartic polynomial $\phi^{(4)}(f)$ in (11) is supposed with constant α_F $F^{(3)}(z) = -\alpha_F F^{(3)}(F^{(3)}(-z/\alpha_F))$. The quartic $\phi^{(4)}(f)$ allows in $\partial_z F^{(3)}(w,z) = \prod_k \partial_{z_k} F^{(3)}(w,z_k)$ a quadratic expansion into Γ_{ijkl} which places $\partial_z F^{(3)}(w,z)$ in the universe radius R(τ) as a universe potential of the boson function $D_{\mu\nu}$. A quadratic equation for $\partial_z F^{(3)}(w,z)$ includes γ-fixed points which are a product of periodic CF $\begin{vmatrix} \mathcal{L} & \zeta^{(4)} \\ 0 & 1 \end{vmatrix}$ with $\begin{vmatrix} 1 & 0 \\ -\Sigma & 1 \end{vmatrix}$. Whereas the former product yields string $\mathcal{L}$ geometric zeta functions $\zeta(\mathcal{L}, \ln z)$ as occupation numbers the latter yield the Dyson equation for (8) and a Bethe- Salpeter equation for Γ_{ijkl} for self-energy Σ replaced by GG. This approach supposes that complex variables z are Fourier components within the base $\widehat{B}$. The generality of the approach will be demonstrated by setting $G^{-1} \simeq z - e_i$ and $G(w,z) = \partial_z L(w,z) = \frac{1}{z_{k+1} - z_k}$ (Ziep, 2026). This suggests a relation to the thermodynamic Green' function

$$G \simeq f^{-1}\theta(\Delta f) \simeq f' f''\theta(\Delta f) \simeq \mathrm{tr}\,(e^{\Omega/k_B T} f'_q f''_q \theta(\Delta f)) \quad (15)$$

by cubic conjugates f′ and f′′ and explains the known expression of the real thermodynamic potential (Abrikosov, Gorkov, & Dzyaloshinski, 1965)

$$\delta\Omega \simeq \sum \Delta z \Delta f\{F,z\} \to \sum \int \frac{d\tau}{\tau} G_0^{-1}(G - G_0) \quad (16)$$

A Schwarzian derivative of frayed paths w→z yields $\gamma^\nu \partial_\mu G^{(\pm)}_{q,q'}$ in (Schiffer, 1966). With γ-invariant Cayley quotient which is a Bezout matrix in base $\widehat{B}$

$$B(f,g) = \big(f(z)g(w) - f(w)g(z)\big)/(z - w) \quad (17)$$

the Schwarzian derivative is written as (Schiffer, 1966)

$$\{F^{(3)}(w,z), z\} = -6\,\partial_z \partial_w \ln B(F^{(3)}(w,z), 1)\big|_{w \to z} \quad (18)$$

Abbreviating $\Gamma = \partial_z F^{(3)}(w,z)$, $G = \frac{1}{F^{(3)}(w,z) - F^{(3)}(w,w)}$, $G_0 = \partial_z L(w,z) = \frac{1}{z-w}$ one gets

$$\{F^{(3)}(w,z), z\} = -6\,\partial_z(G\Gamma - G_0) \simeq -6\,\partial_z G^{-1} \quad (19)$$

Here $G\Gamma \to \overline{\psi}_i \overline{\psi}_j \Gamma_{ijkl} \psi_k \psi_l \simeq (x_i x_j)\Gamma_{ijkl}(x_k x_l)$ is viewed as a curvature form in $\mathcal{M}$ by exterior calculus of one-forms $(x_i x_j)$. A Green's function is differentiable in homogeneous angles (φ_1, φ_2) in $\mathcal{M}$- coordinates $\exp(\sigma_\mu x^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix}$ by transforming from discrete angles φ_q. $\mathcal{M}$ arises form one-periodic permutations of quartic roots on complex plane (Ziep, 2026).

3. SIMPLEST CYCLES

Sharkovskii ordering of periods ν_{Sh} concerns the real interval [0,1] with >,<,= signs defined. Ordering enters for embedding cubic roots which can be written as (Brizard, 2009)

$$\Delta e_i(\varphi) = (\sin(\varphi/3 - \pi/6), -\sin(\varphi/3 + \pi/6), -\sqrt{3}\sin(\varphi/3)) \quad (20)$$

by rotations $\varphi_q = \pm\pi$, which projects the 12-component string φ_q of ⅓ angles of quartic roots to interval [0,1]. The Schwarzian derivative of the map (2)

$$\{F^{(3)}(w,z), z\} = \frac{\dddot{F}}{\dot{F}} - \frac{3}{2}\left(\frac{\ddot{F}}{\dot{F}}\right)^2 = -\frac{3}{2}\left(\frac{2w_0 a_0}{w_1 a_0 + 2w_0 a_0 z + w_0 a_1}\right)^2 < 0$$

abbreviating $\dot{F} = \partial_z F^{(3)}(w,z)$ is negative which is a sufficient condition for a map to be chaotic. Simplest cycles in [0,1] $f_0 = f_n$ of odd period n is of two types (Sharkovsky, November 2019)

$$f_{n-1} < \{f_{n-(2i+1)}\}|_{i=1,\dots,(n-1)/2} < f_0 = f_n < f_{n-2i}\,|_{i=(n-1)/2,\dots,1} \quad (21)$$

or

$$f_{n-2} < \{f_{n-2i}\}|_{i=2,\dots,(n-1)/2} < f_0 = f_n < f_{n-2i+1}\,|_{i=(n-1)/2+1,\dots,1} \quad (22)$$

which is

$$f_{even} < f_0 = f_n < f_{odd} \quad (23\text{-}1)$$

or

$$f_{odd} < f_0 = f_n < f_{even} \quad (24\text{-}2)$$

4. CM

The iterated quadratic form q(z) in (1) creates correspondences $z \to \omega \to f(\omega)$ with iterated periods $\gamma^{(3)} \circ \omega \to \omega$. Root finding with iterates $q(z_k)$ is seen as a statistical distribution of interacting energies with measurable $\Re q(z)$ and dark $\Im q(z)$ component. In dependence on γ processing phantom energies $\Re q(z) < 0$ are achievable even for equivalent detγ=1 as discussed in Section 6. The CM multiplicator $M = \left(\frac{\eta(\omega/2)}{\eta(\omega)}\right)^{24} = f_1^{24}(\omega)$ leads to the Weber invariant $f(\omega)$ with $f f_1 f_2 = \sqrt{2}$ (Weber, 1908)

$$\varepsilon\omega\bar{\varepsilon}\bar{\omega} \sim \prod_i \frac{\eta^2(\omega_i)}{\eta^2(\omega_{i-1})} \overline{\prod_l \frac{\eta^2(\omega_l)}{\eta^2(\omega_{l-1})}} \sim f_1 f/f_2^2 \quad (25)$$

with conjugated units $\varepsilon \sim f_1/f_2$, $\bar{\varepsilon} \sim f/f_2$ of a cubic normal field $\mathbb{K}\mathbb{K}'\mathbb{K}''$ of discriminant Δ and Dedekind eta function η(ω). The algorithm (1) iterates variable $z = \wp(u,\omega)$ on elliptic curves $y^2 = \phi^{(3)}(z)$ for a discrete set of points in universal covering space $u = a\omega, a \in \mathbb{Q}^2$. $\wp(\omega/2,\omega) = e_i, y = \partial_u \wp(u = \omega/2, \omega) = 0$ are invariant for iterates $\gamma^{(3)} \circ \omega$ of periods. Equivalent substitutions $\gamma \circ e_i = \gamma \circ \wp$ in elliptic invariant $j(\omega) = j(\gamma \circ \omega)$ lead to a polynomial $\phi^{(6)}(\lambda)$ for the transformed Legendre module $\lambda = \delta e_{ij}/\delta e_{il}$. It is assumed that equivalent substitutions change vacuum energy which should be of great importance. The 1:2 relation (6) between the Weierstrass ℘ and theta $\vartheta^2_{[gh]}$ – function of characteristics is written γ-invariant by modular units $x_i \simeq \vartheta_{[gh]} \simeq g(a_i\omega_i)$. Elliptic curves are self-similar because the square of η(ω) in (25) contains itself modular congruences $g(a_i\omega_i)$ (Lang & Kubert, 1977)

$$\eta^2(\omega) = \prod_i \frac{g(a_i\omega_i)}{g(a_{i-1}\omega_{i-1})} \quad (26)$$

This Gordian knot is approximately solved by permutations of cyclotomic units $\zeta^{(m)}$ in base $\widehat{B}$ where $\varepsilon\bar{\varepsilon} = ½ f^3(\sqrt{\Delta}) \simeq \zeta^{(m)} e^{\frac{-\pi\sqrt{|\Delta|}}{8} - h_t}$ with topological entropy $h_t = \ln 2$.

5. CUBIC POINTS IN MANDELSTAM PLANE

Nontrivial zeros $z_{nt} = ½ + i m_n$ of the Riemann zeta function ζ(z) are associated to Mandelstam plane s,t,u with masses m_n in QS (Remmen, 2021). The found $z - \sqrt{z}$ relation between s,t,u and the argument of ζ(z) is seen as a subsequent quadratic map (6) in (Ziep, 2024). Starting from interval [0,1] one is led to define the Mandelstam plane by squares of φ_q^2

$$(s,t,u) = ((k_1 + k_2)^2, (k_1 + k_3)^2, (k_1 + k_4)^2) \quad (27)$$

For optimal points $\varphi_q^2 \simeq e^{\varphi_q}$ and $\varphi_q^2 \simeq f(\omega_q)$. A squared quadruple φ_q^2 implies invariance of $(d\varphi_1 \,|d\varphi_2)\exp(k_\mu\sigma^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix}$. This quadratic form is represented as a determinant of a 2·2 matrix exponential $\exp(d\varphi_1\, d\varphi_1\, k_\mu\sigma^\mu)$ which leads to $\exp(\mathrm{vec}(d\varphi_1)\cdot(I\otimes(k_\mu\sigma^\mu))\cdot\mathrm{vec}(d\varphi_1))$. It is assumed that the exponent can be transformed to a line element $ds^2 = dx_\mu g_{\mu\nu} dx_\nu$. Squared quartic roots $(x_i x_j) \simeq \varphi_q^2 \simeq e_i$ are proportional to cubic roots e_i. Therefore, variables on Mandelstam plane stu=as+bt+ct yield a third order polynomial

$$F^{(3)}(z) = \sum_{i=0,\dots,3} a_{3-i}\, z^i = \sum_{i=0,\dots,3} \sigma_{3-i}[s,t,u]\, z^i \quad (28)$$

with elementary-symmetric polynomials σ_0, $\sigma_1 = s+t+u$, $\sigma_2, \sigma_3 = abcstu$ where $\sigma_1 = 0$ which compares to the Weierstrass form (3). The Mandelstam plane stu=as+bt+ct is a third order polynomial (Remmen, 2021). Fixpoints for a quadruple (4) q≃μ≃s is understood as four-momentum $\Pi_\mu = k_\mu + A_\mu$ on Mandelstam plane with inverse Green function $F(w,z) \simeq G^{-1} = \gamma^\nu \Pi_{\nu w} - m = G_0^{-1} - \Sigma$ where self-energy Σ (Ziep, 2025).

6. COMPLEX 'LAGRANGIAN' AND STRESS- ENERGY

Invariant discriminant Δ and Lagrangian $\mathcal{L}[\psi_i]$ related regulator indices $R(\mathbb{K}) = \ln\varepsilon$ of $\mathbb{K}[\Delta]$ obey entropy source constraints. Δ and $R(\mathbb{K})$ are base $\{w_0 = 1, w_1, w_2\}$ invariant. The Diophantine index form allows only shifts of $\{w_1, w_2\}$. A γ-invariant $R(\mathbb{K})$ on complex plane obeys mod 2 and mod 3 congruences $\zeta^{(2)}$ and $\zeta^{(3)}$. The constraints μ_1, μ_2, μ_3 concern a quadratic form $Q[\ln\varepsilon \to L(w,z)]$. A real unit ε is subjected by oscillations of the complex phase $L(w,z)$ on complex plane $Q[l] = \sum \mu_1 l^2 + \mu_2 l + \mu_3 N(l)$ where $N(l) > M^* \to N(f_q)\zeta(\mathcal{L}, l)$ ensures non-trivial solution with Euclidean norm N(l) (Pohst & Zassenhaus, 1997). The highest density of units with extremal Q[l] occurs for

$$2\mu_1 l + \mu_2 + \mu_3 e^{2l}\zeta(\mathcal{L}, l)) + \mu_3 N(\varepsilon_q)\,\partial_l \zeta(\mathcal{L}, l)) + \mu_4 \varepsilon\, \partial_\varepsilon \phi^{(3)}(\varepsilon) = 0. \quad (29)$$

The meaning density concerns a $\mathrm{Vol}(\mathcal{M})$ with one-periodic boundary conditions for $\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix}$ rotations in [0,1] by cyclotomic approximations $\zeta^{(m)}$ in base $\widehat{B}$. This leads to a r-dimensional vector l on circles in complex plane which leads to a circulant determinant which can be approximated by

$$R(\mathbb{K}) = |l| = \mu_1^{-r} |\mu_2 + \mu_3 e^{2l + \ln\zeta(\mathcal{L}, l)} + \mu_3 e^{\ln N(\varepsilon_q) + \ln\zeta'(\mathcal{L}, l)}| \simeq \left(\frac{\mu_3}{\mu_1}\right)^r |e^{2l+\ln\zeta(\mathcal{L}, l)} + e^{\ln N(\varepsilon_q) + \ln\zeta'(\mathcal{L}, l)}| \quad (30)$$

Terms μ_3 in (30) are viewed as eternal ionization and scattering rates R_{net} and R_{scatt} (Ziep, 2025). Geometric zeta functions $\zeta(\mathcal{L}, l_b) \simeq \partial_z R_{net}$ are Bose-like occupation numbers n_B giving net rates $R_{net} = \int dz\, \zeta(\mathcal{L}, l_b)$. Count rates $N(\varepsilon_q)\zeta'(\mathcal{L}, l_b) = \partial_z R_{scatt}$ yield $R_{scatt} = \int dz\, N(\varepsilon_q)\zeta'(\mathcal{L}, l_b)$. Regarding cyclotomic units $\zeta^{(m)}$ as frequencies ν the result can be summarized by

$$R(\mathbb{K}) \simeq |\ln\varepsilon| \simeq \exp \oint d\nu \ln\varepsilon. \quad (31)$$

In difference to the circulant index in (Fueter, 1910) the cyclotomic units on complex in (29-31) are nilpotent of degree 2. The cubic fundamental unit $R(\mathbb{K}) \simeq \ln\varepsilon$ satisfies $\frac{8}{\sqrt{\Delta}}\ln(2\varepsilon) \simeq \pi$ with very high precision (Meyer, 1970). Next, we discuss γ-invariances on complex plane which must involve three steps k,k+1,k+2 in $d_z L(w,z) d_w L(w,z)$ (Schiffer, 1966). The Newton iteration (1) can be arranged as

$$F^{(3)}(w, z_{k+1}) \leftarrow F^{(3)}(w, z_k) + \delta z_k\, \partial_z F^{(3)}(w, z_k). \quad (32)$$

Two iterates $(1 - dz_{k+1}\,\partial z_{k+1})(1 - dz_k\,\partial z_k) l$ yield $dz dw\, \partial_z\, \partial_w F^{(3)}(w,z)$ giving second derivative $z_{k+2} - 2z_{k+1} + z_k$. Alternatively, γ-invariance of

$$\oint d\nu\, l \simeq \int (1 - dz\,\partial_z - ½(dz)^2\,\partial_z^2) l \quad (33)$$

requires only second derivatives. Here the linear term is set constant as a stationarity condition. For l=L(w,z) the quadratic term can be described by $d_z L(w,z) d_w L(w,z)$ giving

$$\sum_{q,q'} \{z_{q}, z_{q'}\} dz_q dz_{q'} = \sum_{q,q'} \delta(\varphi_q - \varphi_{q'}) d\varphi_q d\varphi_{q'}. \quad (34)$$

Here $dz_q dz_{q'} \simeq d\varphi_q d\varphi_{q'}$ contains the line element $ds^2 = dx_\mu g_{\mu\nu} dx_\nu$. Varying $\frac{\delta}{\delta g_{\mu\nu}}$ yields (19). The solution of (29) covers feasible, optimal states $\varphi_q^2 \simeq e^{\varphi_q}$ which explain one-periodic spacetime $\mathcal{M}$. For steps k,k+1,k+2 the Schwarzian derivative chain is $\{z_{k+2}, z_k\} = (\partial_{z_k} z_{k+1})^2 \{z_{k+2}, z_{k+1}\} + \{z_{k+1}, z_k\}$. One-dimensional conformal theory (Polchinski, 2005)

$$(\partial_{z_k} z_{k+1})^2 T(z_{k+2}, z_{k+1}) = T(z_{k+2}, z_k) - c\{z_{k+1}, z_k\} \quad (35)$$

suggests conformal stress-energy as Schwarzian derivative $T(z_{k+1}, z_k) = (\partial_{z_k} z_{k+1})^2 \{z_{k+1}, z_k\}$.

The peculiarity of the Riemann surface $\mathbb{R}_L$ consists in deriving the derivative $(\partial_{z_k} z_{k+1})$ in terms of Green's functions.

7. RATIONAL METRIC

Irreducible steps (4) contain an involution $2^{¼} f(\omega_k) \to \frac{1}{2^{¼} f(\omega_{k+1})}$ of invariants of degree 2 with $ff_1f_2=\sqrt{2}$ which transform holomorphic f(ω) into meromorphic f_1 or f_2 as a simultaneous drift and diffusion process. The present paper claims that the vertex part

$$\partial_z N_q(z) = \partial_z F^{(3)}(w,z) = 8w_0 z + 4w_1 \simeq \Gamma_{ijkl} \simeq R(\tau) \quad (36)$$

is a 'universe' radius. Involutions yield a drift-diffusion step $c_l(R(t)) \to c_l(R(\tau))/R(\tau)$ in $\mathcal{M}$ with velocity of light c_l. Time τ arises from the 12-component strings φ_q. Spacetime $\mathcal{M}$ is due to shifts dz of planar triangles $T(z_q)$. A metric

$$ds^2 = c_l^2 d\tau^2 - R^2(\tau) dr^2 \quad (37)$$

with $\det\hat{g} = -c_l^2 R^6$ instead of metric $ds^2 = d\tau^2 - \frac{R^2(\tau)}{c_l^2} dr^2$ is used (Friedmann, 1922). The aim is to prove that γ triggers a rationalization process to Schlaefli transformation equations of Weber invariant f_k, f_{k+1} of degree 2. Quadratic maps in Cayley-Menger determinants $CM(a,b,c)$

$$(z_0, z_1, z_2) \to (a,b,c) \to (a^2+b^2, a^2-b^2, 2ab) \quad (38)$$

transform a Pythagorean triangle for $(a,b,c) \to (ds,dt,dr)$ into a quartic polynomial for $\vartheta(u,\omega)$. Quadratic

$$(a,b) \to (a^{*2}, b^{*2}) \to \phi^{(4)}(a^*) \to a^* \simeq \vartheta(u,\omega) \to \wp(u,\omega) \to \quad (39)$$

correspond to $F^{(3)}(w,z)$

$$F^{(3)}(w,z) = w_0 F_1(z) + w_1 F_0(z) \to \wp(u,\omega) \to \phi^{(3)}(f) \quad (40)$$

enveloped by a quartic polynomial $\phi^{(4)}(a^*)$ and CM of periods ω which create Weber invariants a,b,c (Heegner, 1952). Cycles $(ds,dr,d\tau) \to (ds, R(\tau)dr, d\tau) \to (ds, R(\tau)dr, d\tau/R(\tau))$ yield a quartic $ds^2 = c_l^2 d\tau^2 - R(\tau)^2 dr^2$ and $R(\tau)^2 ds^2 = c_l^2 d\tau^2 - R(\tau)^4 dr^2$ for $c_l R(\tau) \to c_l/R(\tau)$. Due to a vertex

$$\partial_z N_q(z) \simeq \Gamma[D_{\mu\nu}] \simeq R(\tau) \quad (41)$$

linear in the boson function $D_{\mu\upsilon} \simeq z$ in (36) of Section 3 the 'universe' radius is a drift-diffusion potential- scale factor $V \simeq R(\tau)$ for alternating diffusion coefficient D and c_l (Ziep, 2025). Rational triangles of determinant $CM(a,b,c) = 0$ would imply equianharmonic curves $g_2 = 0$ followed by $e_1 + e_2 + e_3 = s + t + u = 0$. $g_2 = 0$ implies a vanishing Bezout matrix

$$B(\phi^{(3)}, 1) = e_1^2 + e_2^2 + e_3^2 \simeq s^2 + t^2 + u^2 \simeq a^2 + b^2 + c^2 \to 0.$$

Vanishing invariants $B(\phi^{(3)}, 1)$ in base $\{w\}$ where $F = F^{(3)}$

$$B_{00}B_{11} - B_{01}B_{10} = \begin{bmatrix} 1 & F_0(e_1) & F_1(e_1) \\ 1 & F_0(e_2) & F_1(e_2) \\ 1 & F_0(e_3) & F_1(e_3) \end{bmatrix} = \begin{vmatrix} 8g_2 & 12g_3 \\ 12g_3 & -\tfrac{2}{3}g_2 \end{vmatrix} \to 12g_3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \to 0 \quad (42)$$

are realizable by w base shifts giving a $\zeta^{(2)}$ field. Map (1) is conjugate to $M_{1,2}(z_k) = M_{1,2}(z_{k+N})^{2^N}$ (Babajee, Cordero, & Torregrosa, 2016). The elliptic approximation of entire transcendent $\phi^{(\infty)}(z)$ by two roots $z_1, z_2 \simeq e_2, e_3$ implies that the Moebius transform is a Legendre module $M(z_{k+1}) = \frac{z_{k+1} - e_2}{z_{k+1} - e_3}$ which serves as the optimal, feasible (best) approximation of a bifurcating region. Any computation of (1) is invariant with respect to rotations and orthogonal transformations of rational triangles. This additional degree of freedom is a candidate to explain stability of states $\mathcal{L}[\psi_i]$.

8. METRIC STABILITY

Invariant Newton root finding (1) for a quadratic complex $q(z) = (z - z_1)(z + z_2)$ converges for $|\partial_z N_q(z)| < 2$. According to (43) the metrical stability is as well stability for a drift-diffusion potential V which should hold in the vicinity of simple zeros z_1, z_2 of any holomorphic function. For triangles with Δz in quadruples (4)

$$\Delta z \simeq (\Pi\dot{F})\Delta z \simeq < \frac{\Delta z}{\Delta\tau} > \Delta z \simeq \frac{(\Delta s)^2}{\Delta\tau} \simeq \frac{(\Delta s)^2}{(\Delta\tau)^2}\Delta z \simeq V(z) \simeq I \simeq R(\tau) \quad (43)$$

holomorphic functions ξ(z) should yield a current $I \simeq V(z) \simeq \int dz\xi(z)$ circulating around z_1, z_2 which is equivalent to a quantum Hall topology. Periodicities in universal covering space $u = a\omega, a \in \mathbb{Q}^2$ as rotation angles φ_q declare a time in $\mathcal{M}$ as an elliptic integral over z, i.e. $R(\tau)$ (Friedmann, 1922). Complex radii $R(\tau)$ imply two curvatures in $(x_i x_j)\Gamma_{ijkl}(x_k x_l)[D\mu\nu]$. The existence of a quadratic equation of masses in each point in $\mathcal{M}$ is supported by the presence of anti-matter Moreover, equivalent periods ω in stress-energy (33) enable various vacuum states in dependence on the γ-processing. Variable Δz in (1) transforms as Δz→λΔz due to invariant phase changes in $d_z L(w,z) d_w L(w,z)$ for three steps k,k+1,k+2. This justifies to introduce a current j_m =

$j_m = \bar{\psi}\lambda_m\psi$ in $\lambda(\omega) = ½ + \frac{j_m}{m}$ which obeys elliptic symmetries $i(\lambda), 1/\lambda, 1/i(\lambda), 1 - 1/\lambda, -\lambda/i(\lambda)$ for involution $i(\lambda) = 1 - \lambda$ which are elliptic invariants $j(\omega) = j(\omega')$. In iterates of (33) a vanishing linear term is a stationarity condition. A Lagrangian $\mathcal{L}[\psi_i]$ from $\sum_{q,q'}\{z_{q,}, z_{q'}\}dz_q dz_{q'}$ can be derived if the quadratic term $(dz)^2 \to (\Delta z)^2 \to q(z)$ in (33) converges. In the following the case of a weak time dependence $\Gamma[D_{\mu\nu}] \simeq R(\tau)$ is discussed and classified by diagrams. A metrical stability requires a real scalar density $\mathcal{L}(g, \partial g, \dots, \partial^n g)$ linear in higher-order derivatives $\partial^n g$ which leads only to n^{th} order Euler-Lagrange expressions (Harmanni, 2016). A Lagrangian

$$\mathcal{L}(g, \partial g, \dots, \partial^2 g) = \sqrt{-g}\left(\frac{\tilde{R}}{\kappa_4} + \mathcal{L}_M\right) \quad (44)$$

in the Einstein-Hilbert action $S = \int d\sigma_4 L$ with $\mathcal{M}$ -hypersurface $d\sigma_4$ for metric $\hat{g} = g_{\mu\nu}$ in (37) with $\kappa_w = \frac{8\pi G_w}{c_l^4}$, matter Lagrangian L_M depends on the Ricci scalar $\tilde{R}$ and

$$\tilde{R} = \frac{6}{c_l^2}\left(\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2\right) \quad (45)$$

and yields Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial g} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial(\partial g)} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial(\partial^2 g)} = 0 \quad (46)$$

Linear in second derivatives equations $\widehat{R} - \hat{g}\widetilde{R} + \Lambda\hat{g} = -\kappa_4\widehat{T}$ for curvature $\widehat{R}$, metric $\hat{g}$ and stress-energy tensor $\widehat{T}$ are stable. In one-periodic $(d\varphi_1, d\varphi_2)\exp(k_\mu\sigma^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix} \simeq (\Delta z)^2$ of $\pm\pi$ rotations in interval [0,1] a metric originates from $(\Delta z)^2$ on $\mathbb{C}^w$ in $\mathbb{R}_L[T(z_q)]$. Next it is shown that a linear $\partial^2 g-$dependence arises for invariant $\phi^{(3)}(f)$ where $\ddot{F} = 0$. Next constant $\frac{6}{c_l^2}\sqrt{-g}$ as well variable $\frac{6}{c_l^2}\sqrt{-g}$ is discussed for constant c_l and $c_l \simeq R^{-1}$ due to diagram (ii) in Fig. 2. Setting $\mathcal{L}[R] \simeq \frac{\sqrt{-g}}{\kappa_4}\{F^{(3)}(w,z), z\}$ and $\partial_z F^{(3)} = R(\tau)$ one would get a matter Lagrangian $\mathcal{L}_M = -\frac{30}{2\kappa_4 c_l^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{\sqrt{-g}}\phi^{(3)}(R)$ as a partial Schwarzian derivative. The extra term $-\frac{5}{2}\left(\frac{\dot{R}}{R}\right)^2$ changes only the prefactor in Euler equations substituting $\widetilde{R} \to \widetilde{R}_s = \frac{6}{c_l^2}\left(\frac{\ddot{R}}{R} - \frac{3}{2}\left(\frac{\dot{R}}{R}\right)^2\right)$. An auxiliary polynomial $\phi^{(3)}(R)$ arises because $\partial_z F^{(3)}$ is linear where (36) and (3) hold. This leads to constant and variable $\frac{6}{c_l^2}\sqrt{-g} \simeq R^4$ or $\frac{6}{c_l^2}\sqrt{-g} \simeq R^3$ and invariances $\zeta^{(4)}$ and $\zeta^{(3)}$. Next the Friedmann solution 8.1-4 describes a fluid-like Ginzburg-Landau functional whereas remaining cases indicate a massive solid state. For constant $R^m = \text{const}$ one gets $\ddot{R} = \frac{(1-m)\dot{R}^2}{R}$ differentiating two times which implies a presence of a generator $\zeta^{(m)}$. Adding $\frac{d}{d\tau}(\dot{R}R^2)$ or $\frac{d}{d\tau}(\dot{R}R^3)$ to $\mathcal{L}$ does not change the Euler-Lagrange equations. Because variation should be over $\hat{g} = g_{\mu\nu}$ on one-periodic hypersurface $d\sigma_4$ a one-parametric variation is questionable. Here τ is understood as a scaling parameter of vertex Γ_{ijkl} in (5).

8.1. Constant scale R, constant c_l

For m=3 one gets $\ddot{R} = \frac{-2\dot{R}^2}{R}$ in $\mathcal{L}$. With $\sqrt{-g_0} = cR^3 = \text{const}$ the Lagrangian

$$\mathcal{L}(R) = \frac{\sqrt{-g}}{\kappa_4}\widetilde{R}_s - \mu\phi_3(R) \quad (47)$$

with condition $\phi_3(R) = 0$ Lagrange parameter μ for reduces to

$$\mathcal{L}(R) = \frac{-21}{c_l^2\kappa_4}\sqrt{-g_0}\frac{\dot{R}^2}{R^2} - \mu\phi^{(3)}(R) \quad (48)$$

with Euler-Lagrange equations

$$\frac{3\cdot 7\cdot 8}{c_l^2\kappa_4}\sqrt{-g_0}\frac{\dot{R}^2}{R^3} = -\mu\,\partial_R\phi^{(3)} \quad (49)$$

where $R \to \zeta^{(3)}R$ and forces $\mu\,\partial_R\phi^{(3)}$ are like z^2 in (5) with a possible negative differential branch.

8.2. Constant scale R, diffusive $c_l \simeq R^{-1}$

For m=4 $\ddot{R} = \frac{-3\dot{R}^2}{R}$ replacing c by c/R in $\widetilde{R}_s$ in $\mathcal{L}$ one gets for (48)

$$\mathcal{L}(R) = \frac{-3^3}{c_l^2\kappa_4}\sqrt{-g_0}\frac{\dot{R}^2}{R^2} - \mu\phi^{(3)}(R) \quad (50)$$

Euler-Lagrange equations with $\ddot{R} = \frac{-3\dot{R}^2}{R}$ read

$$\frac{2\cdot 3^3\cdot 5}{c_l^2\kappa_4}\sqrt{-g_0}\frac{\dot{R}^2}{R^3} = -\mu\,\partial_R\phi^{(3)}(R) \quad (51)$$

on a quadratic lattice $R \to \zeta^{(3)}R$ and forces $\mu\,\partial_R\phi^{(3)}$ are like z^2 in (5) with a possible negative differential branch.

8.3. Variable scale R, constant c_l

$\mathcal{L}(R) = \frac{\sqrt{-g}}{\kappa_4}\widetilde{R}_s - \mu\phi^{(3)}(R)$ would recover both Friedmann equations for an elliptic time-integral $c_l\tau = \int\frac{\sqrt{R}dR}{\sqrt{\phi_3(R)}}$ (Friedmann, 1922). For $\sqrt{-g} = c_l R^3$ adding $\frac{d}{d\tau}(\dot{R}R^2)$

$$\mathcal{L}(R) = \frac{\sqrt{-g}}{\kappa_4}\widetilde{R}_s - \mu\phi^{(3)}(R) = \frac{-6}{c_l\kappa_4}(R^2\ddot{R} - \tfrac{3}{2}\dot{R}^2 R) - \mu\phi^{(3)}(R) \simeq -\frac{3\cdot 7}{c_l\kappa_4}\dot{R}^2 R - \mu\phi^{(3)}(R). \quad (52)$$

The Lagrangian is a particle with mass proportional to R.

8.4. Variable scale R, diffusive $c_l \simeq R^{-1}$

Adding $\frac{d}{d\tau}(\dot{R}R^3)$ one gets a Lagrangian for a particle with mass proportional to R^2.

$$\mathcal{L}(R) = \frac{\sqrt{-g}}{\kappa_4}\widetilde{R}_s - \mu\phi^{(3)}(R) = \frac{-6}{c_l\kappa_4}(R^3\ddot{R} - \tfrac{3}{2}\dot{R}^2R^2) - \mu\phi^{(3)}(R) \simeq -\frac{3^3}{c_l\kappa_4}\dot{R}^2R^2 - \mu\phi^{(3)}(R) \quad (53)$$

Supposing a discrete quadratic map one would get $\mathcal{L}(R^2 \to R) = -\frac{3^3}{c_l\kappa_4}\dot{R}^2R - \phi^{(3)}(\sqrt{R})$

8.5. Constant complex scale factor

If the γ-invariant unit ε is represented as an invariant $B(F^{(3)}(w,z),1)$ one gets (18) in the quadratic term in (33) where $(\Delta z)^2\,\partial_z^2\ln\varepsilon \to (\Delta z)^2\{z_{k+N}, z_k\}$. A $\{\varphi_q\}$ ball $(d\varphi_1, d\varphi_2)\exp(k_\mu\sigma^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix} \simeq (\Delta z)^2$ forms a hypersurface $(\Delta z)^2$ in $\mathbb{R}_L$ which should corresponds to the surface $\sqrt{-g}d\sigma_4$. But cubic bases with $\Delta z\Delta w \leftarrow \dot{F}^2\Delta z\Delta w$

cover involutions F(F)=z which explain a quadratic Γ_{ijkl} dependence. Variable z_k in (5) is the square root of $z^2 = (x_i x_j)(x_i x_k) \simeq \overline{\psi}_i\overline{\psi}_j\Gamma_{ijkl}\psi_k\psi_l$

by (6) where $z - e_i \simeq \Gamma_{ijkl} \simeq \varphi_q^2 \simeq e^{\varphi_q} \simeq R(\tau)$ where $\dot{R} \simeq \partial_z^2 F \simeq 8w_0$ due to (36). A linear dependence of (5) arises in four-component complex space. Therefore, z_k becomes time τ after a sufficient ball of string $\{\varphi_q\}$. A scale factor $\partial_z F^{(3)} = \prod_k F^{(3)}(z_k) = R(\tau)$ factorizes an iterated polynomial $F^{(3)}\left(F^{(3)}(F^{(3)}\dots)\right)$ of order 2^k. It renormalizes into an invariant polynomial $\phi^{(2^k)}(z) \to \phi^{(3)}(f(\omega))$. The cubic invariant $\phi^{(3)}$ reduces $\partial_z^N F$ to third or second order N=2 or 3 which proves diagram techniques by creating a vanishing third derivative $\dddot{z} = \dddot{F} = 0$. In the following N steps yield $\phi^{(2^N)}$ which require to differentiate Euler-Lagrange equations up to fourth order. First $(\Delta z)^2 =$ const is discussed. Denoting $F = F^{(3)}$ one would get a Lagrangian

$$\mathcal{L} = \mathrm{const}\{z_{k+N}, z_k\} - \mu\phi^{(3)}(z) = \mathrm{const}\left(\frac{\dddot{F}}{\dot{F}} - \frac{3}{2}\left(\frac{\ddot{F}}{\dot{F}}\right)^2\right) - \mu\phi^{(3)}(z) \quad (54)$$

from Section 6. Subsequent steps are summarized conformal steps. Steps interchange functions into variables. Differentiation in $\{z_{k+N}, z_k\} = \{F(z_{k+N-1}), z_k\}$ is over z_k.

Iterated $\phi_{2^k}(z)$ yield derivatives up to fourth power in Euler-Lagrange equations (55). Euler-Lagrange equations (55) read with effective coordinate $F_{eff}[F, \dot{F}, \ddot{F}, \dddot{F}]$ where $\partial_z^3 F = \dddot{F}$. Dots or roman numbers denote derivatives $\partial_z F^{(3)}$ in $L(F, \dot{F}, \ddot{F}, \dddot{F})$. Euler-Lagrange equations of (54) read (z=F)

$$\frac{\partial\mathcal{L}}{\partial z} - \frac{d}{d\tau}\left\{\frac{\partial\mathcal{L}}{\partial\dot{F}} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\ddot{F}} + \frac{d^2}{d\tau^2}\frac{\partial\mathcal{L}}{\partial\dddot{F}}\right\} = \frac{\partial\mathcal{L}}{\partial z} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{F}_{eff}} = 0 \quad (55)$$

with

$$\frac{\partial\mathcal{L}}{\partial\dot{F}_{eff}} = \frac{\partial L}{\partial\dot{F}} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\ddot{F}} + \frac{d^2}{d\tau^2}\frac{\partial\mathcal{L}}{\partial\dddot{F}} = \mathrm{const}\left(\frac{\dddot{F}}{\dot{F}^2} - \frac{\ddot{F}^2}{\dot{F}^3}\right) \quad (56)$$

and

$$\frac{\partial\mathcal{L}}{\partial z} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{F}_{eff}} = \frac{\partial\mathcal{L}}{\partial z} - \mathrm{const}\frac{d}{d\tau}\left(\frac{\dddot{F}}{\dot{F}^2} - \frac{\ddot{F}^2}{\dot{F}^3}\right) = \mathrm{const}\left(\frac{F^{(iv)}}{\dot{F}^2} - 4\frac{\ddot{F}\,\dddot{F}}{\dot{F}^3} + 3\frac{\ddot{F}^3}{\dot{F}^4}\right) - \mu\,\partial_z\phi^{(3)}(z) = 0 \quad (57)$$

8.6. Variable complex scale factor

Next $\Delta z\Delta w \simeq \dot{F}^2$ is discussed which yields the Lagrangian

$$L = \dot{F}^2\{z_{k+1}, z_k\} - \mu\phi^{(3)}(F) = \dddot{F}\dot{F} - \frac{3}{2}\ddot{F}^2 - \mu\phi^{(3)}(z). \quad (58)$$

The Euler-Lagrange equation reads

$$\frac{\partial\mathcal{L}}{\partial z} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{F}_{eff}} = F^{(4)} - \mu\,\partial_z\phi^{(3)}(z) = 0 \quad (59)$$

where,

$$\frac{\partial\mathcal{L}}{\partial\dot{F}_{eff}} = \frac{\partial\mathcal{L}}{\partial\dot{F}} - \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\ddot{F}} + \frac{d^2}{d\tau^2}\frac{\partial\mathcal{L}}{\partial\dddot{F}} = -\dddot{F} \quad (60)$$

which is discussed in the next Section in terms of Feynman diagrams.

8.7. Eras in vertex scale factor R(τ)

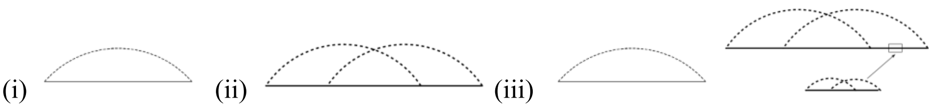

Figure 2: Dominating diagrams in an expansion of q(z) linear and quadratic in $z - e_i \simeq \Gamma_{ijkl}$

It is claimed that invariant root finding (1) is important for all interactions on complex time surface $\mathbb{R}_L\left(T(z_q)\right)$ with underconstrained complex Lagrangian $\mathcal{L}[\psi_i]$. Under certain circumstances a Newton fractal appears on the hyperbolic border between the Julia set and the Fatou set $\mathcal{J}(N_q) - \mathcal{F}(N_q)$. Transcendent roots z_1, z_2 in a quadratic approximation q(z) are orbited by algebraic root-based states with e_2, e_3. Then the invariant form q(z) can be expanded quadratically in terms of the vertex $z - e_i \simeq \Gamma_{ijkl}$ which could be classifiable by Feynman diagrams which should hold for all interactions. A validity check of (8.1-7) could consist in explaining scale factor eras. In early-to-late cosmology (i) radiation- dominated era $R(\tau) \simeq \tau^{1/2}$ (ii) matter-dominated era $R(\tau) \simeq \tau^{2/3}$ (iii) dark matter dominated era $R(\tau) \simeq e^{\tau\,\partial_\tau \ln R(\tau)}$results from Friedmann solutions in (8.3). z_k Identifying time τ with a sufficient large ball of string $\{\varphi_q\}$ variable $z_k \simeq F(w, z)$ becomes time $\tau \simeq [G^{(\pm)}(\varphi_q)]^{-1}$. But Δz_k defines the Greens's functions $G^{(\pm)}(\varphi_q) \simeq 1/\Delta z_k$ in (14) by an interval 3π=π+π+π. Equivalent to the renormalization equation $F^{(3)}(z) = -\alpha_F F^{(3)}(F^{(3)}(-z/\alpha_F))$ eras are explainable by periods (i) F(F(F))=F or $G^{(-)}G^{(-)}G^{(+)} \simeq G^{(+)}$, periods (ii) F(F(F)=F(F) or $G^{(-)}G^{(+)}G^{(-)}G^{(-)}G^{(+)}$ (iii) both periods F(F(F))=F and F(F(F))=F(F). Eras (i-iii) would correspond to expansions

In QS diagrams in Fig.2 is equally weighted yielding an overestimated vacuum density within the cosmological constant problem. Doubly-periodic processing resolves individual k-steps γ as a Newton root finding process to approach points in spacetime. Therefore, QS expands q(z) linear in $[G^{(\pm)}(\varphi_q)]^{-1}$ with self-energy Σ = sum of (i-iii). Matter generation is attributed to quadratic in Γ_{ijkl} q(z) behavior in era (ii) of finite third derivative of the self-energy $\frac{\delta^3\Sigma}{\delta G^3} \simeq \dddot{F} \# 0$. Term (iii) is equivalent to exchange correlation, a correlated classical tidal force or a Bell state as a source of entanglement (Nielsen & Chuang, 2010) (Ziep, 2025).

9. JACOBI-GAUSS PERIODS

Physical fields obey one-periodic (Abelian) bases $\zeta^{(2)}$ and $\zeta^{(4)}$ with possibly non-Abelian cubic irrational ternary CF of base $\zeta^{(3)}$. The present paper searches unified fields by one-periodic $\{\varphi_q\}$ balls of ±π rotations in [0,1] in elliptic curves in $\mathbb{R}_L\left(T(z_q)\right)$ with periodic wave vectors in $(d\varphi_1, d\varphi_2)\exp(k_\mu\sigma^\mu)\begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix} \simeq (\Delta z)^2$. Under interaction w=1,2,3,4,5 a Kronecker product $\zeta^{(m)} \otimes \ldots \otimes \zeta^{(m)}$of w cyclotomic bases $\zeta^{(m)}$ is defined which describes the topological entropy of the $\{\varphi_q\}$ ball. Mathematically a generalized under constrained

Riemann surface requires w<6 or $\binom{w}{2} - 3w + 3 < w$ for quadrics φ_i with w-relations $\varphi_i\varphi_j = \varphi_k^2, (i, j, k \leq w)$ (Weber, 1878). Physically, w nested cyclotomic gyro-twist spheres should be capable to describe a general complex time. Computationally, zoomed Mandelbrot cardioids are independent on w planes. The paper claims that laps of γ-orbits are cyclotomic numbers in very high precision. The algebraic unit in $R(\mathbb{K}) = l = \ln\varepsilon$ depends on cyclotomic bases, e.g. $l = l[a, b] = \sum_{i=0}^{p-2} a^i b^{g^i}$ where $\frac{a^{p-1}-1}{a-1} = 0$ and $\frac{b^{p}-1}{b-1} = 0$ (Jacobi, 1846)

(i) w=5 yields p=11 with $a = g^{p-1-2w} \bmod p$ with primitive root g=2,6,7,8 of p=11 for possible w=1,2,3,4,5 (Jacobi, 1846)

(ii) Gaussian periods, i.e. vanishing periods exist for generator g with $g^{n^2} = 2^{2^k} = g^{2^k}$. Taking g=b=2 one gets $n^2 = 2^g$ and g=2,4,6,8,10. This leads to $n^2 = 2^p$ for $p < 11$ (fuchs, 1863). mod ν^p-1 congruences have roots 10,8,7,6,2 for p=11 (Ondiany & Mude, 2025). This leads to a period matrix $\omega_{ij}(i, j = 1,2, \ldots, 2w)$ in power integral base $(1, \nu, \nu^2, \nu^3, \nu^4)$

(iii) g^{2^p} implies a pseudo- congruences $2^{2^k} \simeq 1$ expected at k≃10. Therefore, Mersenne number transforms $M_{2^t} = \Pi F_t + 1$ should exist which splits into Fermat number transforms with maximal five imaginary units for primes $F_t (t < 5)$ where $2^{2^k} = \prod_{i=0}^{k-1} F_i$+1

The branching of the $d\varphi_1, d\varphi_2)\exp(k_\mu\sigma^\mu)\binom{d\varphi_1}{d\varphi_2} \simeq (\Delta z)^2$ into log-Riemann surface yields an invariance $D_{\mu\upsilon} = 2\Re z_k \rightarrow 2G_w\Re z_k$. The mod 2 field $\zeta^{(2)}$ yields an optimal feasible $\varphi_q^2 \simeq e^{\varphi_q}$ and $\varphi_q \simeq f(\omega_q)$ for sequential steps of mod 4, mod 3, and mod 2 congruences of $\zeta^{(12)}$. The number of ±π rotations $S(\varphi_q) = \sum d\varphi_q \rightarrow \zeta(\mathcal{L}, \varphi_q)$ on w circles should be the product of rotations in each circle $\varphi_q^2 \rightarrow e^{\Pi_w \varphi_q}$. The estimation $\ln G_w = w!\, 2w\ln_3^w 2$ (Ziep, 2025) is very close to the behavior of the Legendre module for ω→2ω where λ→0 reproduces the classical behavior.

10. CONCLUSIONS

Metrical stability is discussed by rational coordinates in triangles $\mathbb{R}_L\left(T(z_q)\right)$. Convergence requires $|\partial_z N_q(z)| < 2$ which is discussed with that in Mandelbrot sets |z|<2. The Lagrangian $\mathcal{L}[\psi_i]$ of iterated z is valid for a logarithmic zoom $G_w\Re z_k$ with coupling constant G_w. The behavior of states ψ_i on $\mathbb{R}_L\left(T(z_q)\right)$ is classifiable by sequential steps k within an adiabatic approximation for constant g_2, g_3 and discrete cyclotomic states $\zeta^{(m)}$ followed by g_2, g_3 variations. For various interactions diagram expansions differ by the scale Δt of time averaging. Whereas QS uses Δt→∞ classical fields at w=4 or w=5 resolve the vertex functional Γ_{ijkl} as an universe radius R(τ) allowing to include a Big Bang $\Gamma_{ijkl} \rightarrow 0$ and Big Rip $\Gamma_{ijkl} \rightarrow \infty$ situation. The present work aims to discuss confinement-dependent vacuum energies for future work which depend on processing correlations.

REFERENCES

- Menger, K. (1931). Bericht über metrische Geometrie. *Jahresbericht der Deutschen Mathematiker-Vereinigung, 40*, 201–218.
- Schröder, E. (1870). Über unendlich viele Algorithmen zur Auflösung der Gleichungen. *Mathematische Annalen, 2*, 317–365.
- Ziep, O. (2026). Quantum Statistics and Zeta Functions. *Scholars Journal of Physics, Mathematics and Statistics, 13(2):* 56-65, https://doi.org/10.36347/sjpms.2026.v13i02.001
- Dick, R. (1995). Half-differentials and fermion propagators. *Reviews in Mathematical Physics, 7*, 689–707, https://doi.org/10.1142/S0129055X9500027X
- Schiffer, M. (1966). Half-order differentials on Riemann surfaces. *SIAM Journal on Applied Mathematics, 14*, 922–934, https://doi.org/10.1007/BF02392208
- Polchinski, J. (2005). *String theory, 978-0-521-63304-8 (ISBN)*
- Ziep, O. (2024). *Fractal zeta universe and cosmic-ray-charge-cloud superfluid* (Bd. 2). (www.epubli.de, Hrsg.) Berlin, https://doi.org/10.5281/zenodo.14193126
- Ziep, O. (2025). Charge Quanta as Zeros of the Zeta Function in Bifurcated Spacetime. *Journal of Modern Physics, 16*, 249–262, https://doi.org/10.4236/jmp.2025.162011
- Ruggiero, J. R., & others. (1977). *Application of the analytical regularization for the Casimir forces.* Tech. rep., Instituto de Fisica Teorica, Sao Paulo (Brazil).
- Ziep, O. (2026). Doubly- Periodic Processing in Particle Acceleratorsand Fusion Reactors. *submitted to Journal of High Energy Physics, Gravitation and Cosmology, 12*, 147-150, https://doi.org/10.4236/jhepgc.2026.121009
- Weber, H. (1895). *Lehrbuch der Algebra: bd. 1* (Bd. 1). F. Vieweg und Sohn.
- Schwinger, J. (1951). On Gauge Invariance and Vacuum Polarization. *Physical Review, 82*, 664, https://doi.org/10.1103/PhysRev.82.664
- Baker, H. F. (1907). *An Introduction to the Theory of Multiply Periodic Functions.* University Press.
- Ziep, O. (2024). *Nucleosynthesis in Thin Layers,* Berlin: www.epubli.de, https://doi.org/10.5281/zenodo.10848593
- Heegner, K. (1952). Diophantische Analysis und Modulfunktionen. *Mathematische Zeitschrift, 56*, 227–253.

- Weber, H. (1908). Lehrbuch der Algebra, Band III. *Elliptische Funktionen und Algebraische Zahlen, F. Vieweg und Sohn. Braunschweig*.
- Sharkovsky, A. N. (November 2019). Sharkovsky ordering and combinatorial dynamics. *Talk*.
- Brizard, A. J. (2009). A Primer on Elliptic Functions with Applications in Classical Mechanics. *European Journal of Physics, 30*, 729, https://doi.org/10.1088/0143-0807/30/4/007
- Abrikosov, A. A., Gorkov, L. P., & Dzyaloshinski, I. E. (1965). Methods of Quantum Field Theory in Statistical Physics [in Russian], GIFML, Moscow (1962); English transl. *Methods of Quantum Field Theory in Statistical Physics [in Russian], GIFML, Moscow (1962); English transl*. Prentice-Hall, Englewood Cliffs, NJ.
- Lang, S., & Kubert, D. S. (1977). Units in the Modular Function Field. IV. The Siegel Functions are Generators. *Mathematische Annalen, 227*, 223–242, https://doi.org/10.1007/BF01361857
- Remmen, G. N. (2021). Amplitudes and the Riemann Zeta Function. *Physical Review Letters, 127.24*, 241602, https://doi.org/10.1103/PhysRevLett.127.241602
- Ziep, O. (2025). Topological Entropy of the Optimal Chaotic Quadratic Map, *Scholars Journal of Physics, Mathematics and Statistics, 12*, 43-58, https://doi.org/10.36347/sjpms.2025.v12i03.002
- Pohst, M., & Zassenhaus, H. (1997). *Algorithmic Algebraic Number Theory* (Bd. 30). Cambridge University Press, *978-0521330602 (IBN)*
- Ziep, O. (2025). Cosmic Rays, Aerosol-Photosynthesis and Vegetational Air Ion. *Journal of Modern Physics, 16*, 1179–1192, https://doi.org/10.4236/jmp.2025.168059
- Fueter, R. (1910). Die Verallgemeinerte Kronecker'sche Grenzformel und ihre Anwendung auf die Berechnung der Klassenzahl. *Rendiconti del Circolo Matematico di Palermo (1884-1940), 29*, 380–395.
- Meyer, C. (1970). Bemerkungen zum Satz von Heegner-Stark über die imaginär-quadratischen Zahlkörper mit der Klassenzahl Eins, *Journal für die reine und angewandte Mathematik*, *242*, 179-214, https://doi.org/10.1515/crll.1970.242.179
- Friedmann, A. (1922). On the Curvature of Space. *Zeitschrift für Physik, 10*, 377, https://doi.org/10.1023/A:1026751225741
- Babajee, D. K., Cordero, A., & Torregrosa, J. R. (2016). Study of iterative methods through the Cayley Quadratic Test. *Journal of computational and applied mathematics, 291*, 358–369, https://doi.org/10.1016/j.cam.2014.09.020
- Harmanni, F. (2016). *Higher Order Lagrangians.* Ph.D. dissertation, Faculty of Science and Engineering.
- Nielsen, M. A., & Chuang, I. L. (2010). *Quantum computation and quantum information.* Cambridge university press, *9781107002173 (ISBN)*
- Ziep, O. (2025). Matter and Quantum Entanglement. *Journal of Applied Mathematics and Physics, 13(4), 1125-1137*, https://doi.org/10.4236/jamp.2025.134059
- Von Weber, H. (1878). Ueber gewisse in der Theorie der Abel'schen Functionen auftretende Ausnahmefälle. *Mathematische Annalen, 13*, 35–48.
- Jacobi, C. G. (1846). Über die Kreistheilung und ihre Anwendung auf die Zahlentheorie. *Journal für die reine und angewandte Mathematik, 33*, 166–182.
- Fuchs, I. (1863). Ueber die Perioden, welche aus den Wurzeln der Gleichung wn= 1 gebildet sind, wenn n eine zusammengesetzte Zahl ist. *Journal für die reine und angewandte Mathematik, 61*, 374–386.
- Ondiany, J. J., & Mude, L. H. (2025). On the Zeros of Linear Factors of Cyclotomic Polynomials Over Galois Fields. *Journal of Advances in Mathematics and Computer Science, 40*, 176–184,https://doi.org/10.9734/jamcs/2025/v40i92053
- Ziep, O. (2025). Fractal Universe and Atoms. *Scholars Journal of Physics, Mathematics and Statistics, 12*, 89-96, https://doi.org/10.36347/sjpms.2025.v12i04.002